INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250-3638

SOME EXTENSIONS OF NERON'S p-DESINGULARIZATION AND APPROXIMATION

by

Mihai CIPU and Dorin POPESCU

PREPRINT SERIES IN MATHEMATICS
No.27/1980

BUCURESTI

SOME EXTENSIONS OF NERON'S p-DESINGULARIZATION AND APPROXIMATION

by

Mihai CIPU*) and Dorin POPESCU**)

May 1980

*) Computing Center, National Institute for Scientific and Technical Creation, Bd. Pacii 220, 77538 Bucharest, ROMANIA

**) Department of Mathematics, National Institute for Scientific and Technical Creation, Bd.Pacii 220, 77538 Bucharest, ROMANIA

SOME EXTENSIONS OF NÉRON'S p-DESINGULARIZATION AND APPROXIMATION IN LOCAL RINGS

Mihai Cipu and Dorin Popescu

§1 Introduction

Let R be a henselian discrete valuation ring and $f=(f_1,\dots,f_m)$ a system of polynomial equations in $Y=(Y_1,\dots,Y_N)$ over R. Suppose f has a "formal" solution $\bar{y}$ in the completion $\hat{R}$ of R such that the jacobian matrix $J:=(\frac{\partial f}{\partial Y})(\bar{y})$ has an invertible m×m-minor. Then using the Implicit Function Theorem we can get a solution of f in R. If J has no invertible m×m-minors, then could $(m,Y,f,\bar{y})$ be enlarged in order to get an invertible m×m-minor in J? By Néron's blowing up, the answer is "yes" if the extension $Q(R)\subset Q(\hat{R})$ is separable (Q(R) denotes the fraction field of R). As consequence we get that excellent henselian discrete valuation rings have the property of approximation (a noetherian local ring A has the property of approximation if every system of polynomial equations over A has a "formal" solution in the completion $\hat{A}$ of A iff it has one in A).This is a weak form of a result of M.J.Greenberg(see [2]). Our work globalizes Néron's blowing up using [9] and [13] and as consequence characterizes the two dimensional factorial rings with the property of approximation. It also shows that all rings with the property of approximation have their formal fibers geometrically normal.

§2 Some extensions of Néron's p-desingularization

Let R,R' be a pair of discrete valuation rings such that R' is an "unramified" extension of R, i.e. a local parameter t of R is still a local parameter in R'.Suppose the field extensions $Q(R)\subset Q(R')$ and $R/(t)\subset R'/tR'$ are separable. Then it is known the following:

(2.0) Theorem (Néron) R' is a filtered inductive limit of finite type smooth sub-R-algebras of R'.

The statement of (2.0) is not exactly the form of Néron's p-desingularization [11] but it is equivalent with it.

A possible extension of (2.0) was done by G.Pfister ([9] ch.VI). Using the same idea, it was given in [13] an improvement of Pfister's result, which says in fact the following:

(2.1) Lemma Let A,A', $A\subset A'$ be noetherian factorial rings such that every prime element p from A remains prime in A' and the extension $k(p):=Q(A/pA)\hookrightarrow k(pA'):=Q(A'/pA')$ is separable. Suppose A' is local and $Q(A)\subset Q(A')$ is a separable extension of infinite transcendence degree. Then every finite presentation sub-A-algebra B of A' can be embedded in a finite presentation sub-A-algebra B' of A'such that B' is a smooth A-algebra in $qA'\cap B'$ for all prime elements $q\in A'$.

(2.1.1) Remark If A' is not local then (2.1) is still true when A' contains an infinite set of invertible elements, which are algebraically independent over A (look at the proof of [13] Lemma 2).Moreover, suppose that A' is not local and contains only a finite set of invertible elements which are algebraically independent over A. Then for every finite presentation sub-A-algebra B of A" there exists a natural number d with the following property:

"Let $A''\supset A'$ be a noetherian factorial ring such that the extension $Q(A'')\supset Q(A)$ is separable. Suppose every prime element p from A' remains prime in A" and the extension $Q(A'/pA')\subset Q(A''/pA'')$ is separable. If $\operatorname{trdeg}_{Q(A)} Q(A'')\geq d$ then B can be embedded in a finite type sub-A-algebra B" of A" such that B" is a smooth A-algebra in $(q)\cap B''$ for all prime elements q of A"."

In fact, for to get B'' it is necessary only a "desingularization" with respect of some prime elements from A'.

(2.2) <u>Corollary</u> Let $u:A \longrightarrow A'$ be an injective morphism of noetherian factorial rings and $\mathcal{P}_{A'}$ the set of all prime ideals of height ≤ 1 from A'. Suppose A' contains an infinite set of invertible elements which are algebraically independent over A and for every prime element from A the ring $(A'/pA')_{red}$ is a domain. Then the following conditions are equivalent:

i) For every $q \in \mathcal{P}_{A'}$ the morphism $u_q : A_{q \cap A} \longrightarrow A'_q$ induced by u is formally smooth

ii) A' is a filtered inductive limit of finite presentation sub-A-algebras B_i of A' such that B_i is a smooth A-algebra in $q \cap B_i$ for every $q \in \mathcal{P}_{A'}$

iii) Every finite presentation sub-A-algebra B of A' can be embedded in a finite presentation sub-A-algebra B' of A' such that B'_q is a smooth A-algebra for all $q \in \mathcal{P}_{A'}$

<u>Proof</u> Clearly we have ii) $\leftrightarrow$ iii). From (2.1) and (2.1.1) we get also i) $\longrightarrow$ iii) and it remains to show ii) $\longrightarrow$ i). Suppose $A' \cong \varinjlim B_i$, B_i being finite type smooth A-algebras in $q \cap B_i$ for every $q \in \mathcal{P}_{A'}$. Then $N_{(B_i)_{q \cap B_i}/A}(A')=0$ and $\Omega_{(B_i)_{q \cap B_i}/A}$, $q \in \mathcal{P}_{A'}$ are projective B_i-modules ($N_{B/A}$ denotes the functor $H_1(A,B,-)$ from [1]). Consequently $N_{A'_q/A} \cong \varinjlim N_{(B_i)_{q \cap B_i}/A}(A')=0$ and $\Omega_{A'_q/A} \cong \varinjlim \Omega_{(B_i)_{q \cap B_i}/A} \otimes_{B_i} A'$ is a flat A'-module and so $u_q : A_{q \cap A} \longrightarrow A'_q$ must be formally smooth for every $q \in \mathcal{P}_{A'}$ (cf. [1]).

(2.3) <u>Theorem</u> Let $u:A \longrightarrow A'$ be an injective morphism of noetherian factorial rings and $\mathcal{P}_{A'}$ the set of all prime ideals of height ≤ 1 from A'. Suppose A' is a local ring and for every prime element of A the ring $(A'/pA')_{red}$ is a domain. Then the following statements are equivalent:

i) For every $q \in \mathcal{P}_{A'}$ the morphism $u_q : A_{q \cap A} \longrightarrow A'_q$ induced by u is formally smooth [illegible] for to get [illegible] it is necessary only [illegible] denominators elements [illegible] more prime elements from A.

(2.4) Definition. Let $u: A \to A'$ be an injective morphism of noetherian [illegible] rings and $\mathcal{P}_1$ the set of all prime ideals of height 1 from A. Suppose A' contains an infinite set of invertible elements [illegible] which are [illegible] and [illegible] over A and for every prime ideal q from A the ring $(A'/qA')_{[illegible]}$ is a domain. Then the following conditions are equivalent:

i) For every $q \in \mathcal{P}_1$ the morphism $u_q: A_q \to A'_q$ induced by u is [illegible].

ii) A' is a filtered inductive limit of [illegible] presentation sub-A-algebras B_i of A' such that $B_i \otimes$ [illegible] is a [illegible] A-algebra in q for every $q \in \mathcal{P}_1$.

iii) Every [illegible] presentation sub-A-algebra B of A' can be embedded in a finite presentation sub-A-algebra B' of A' such that B' is a [illegible] for all $q \in \mathcal{P}_1$.

Proof. [illegible] we have ii)→iii) [illegible] (2.3) and (2.1.1) we get also i)→ii) and it remains to show iii)→i). Suppose $A' =$ $\varinjlim B_i$ [illegible] [illegible] in A'. For every $q \in \mathcal{P}_1$, then [illegible] and Ω [illegible] $\in \mathcal{P}_1$ are projective [illegible] the [illegible] [illegible] see [1]. Consequently [illegible] $\varinjlim$ [illegible] is a flat A'-module [illegible] for every $q \in \mathcal{P}_1$ (see [17]).

(2.5) Theorem. Let $u: A \to A'$ be an injective morphism of [illegible] noetherian [illegible] rings and $\mathcal{P}$ the set of all prime ideals of height 1 from A [illegible] is a local ring and for every prime ideal q of A [illegible] is a domain. Then the following [illegible] are equivalent:

i) For every $q \in \mathcal{P}$ the morphism $u_q: A_q \to A'_q$ induced by u is [illegible].

ii)There exist a filtered inductive set $(B_i)_{i\in I}$ of finite type A-algebras and some morphisms of A-algebras $u_i:B_i\longrightarrow A'$ (not necessarly injective) such that

a)B_i is a smooth A-algebra in $\{u_i^{-1}(q)\,|\,q\in \mathcal{P}_{A'}\}$ for all $i\in I$

b)$(A',(u_i)_i)\cong\varinjlim B_i$

iii)Every finite presentation sub-A-algebra B of A' can be embedded in a finite presentation A-algebra B' and the inclusion $B\hookrightarrow A'$ extends to a morphism of A-algebras $v:B'\longrightarrow A'$ such that B' is a smooth A-algebra in $\{v^{-1}(q)\,|\,q\in \mathcal{P}_{A'}\}$

<u>Proof</u> ii)$\longrightarrow$i) and ii)$\longleftrightarrow$iii) go like in the proof of (2.2). For the implication i)$\longrightarrow$iii) we need the following lemma:

(2.3.1) <u>Lemma</u> Let A[X] be the polynomial ring in some indeterminates $X=(X_1,\dots,X_n)$ over a noetherian local ring $(A,\underline{m})$.Let $\underline{a}$ be an arbitrary proper ideal of A[X] with $ht(\underline{a})\leq \dim A$.Then there exists $x=(x_1,\dots,x_n)$ in $\underline{m}A^n$ such that the morphism $\sigma_x:A[X]\longrightarrow A$ given by $h\leadsto h(x)$ maps $\underline{a}$ in an ideal with height$\geq ht(\underline{a})$.

Let d be the natural number associated to B by (2.1.1). Take $A'':=A'[X]_{(\underline{m}',X)}$,$\underline{m}'$ being the maximal ideal of A' and $X=(X_1,\dots,X_d)$.Clearly,the inclusion $A'\hookrightarrow A''$ satisfies the hypothesis of (2.1.1) and so B can be embedded in a finite presentation sub-A-algebra B' of A'' such that B' is a smooth A-algebra in $q\cap B'$ for all $q\in\mathcal{P}_{A'}$

Now,let $\tau:A[Y]\longrightarrow A'$;$Y=(Y_1,\dots,Y_s)$ be a morphism of A-algebras such that $B'=\operatorname{Im}\tau$.If $g=(g_1,\dots,g_r)$ is a system of r-polynomials from $I:=\operatorname{Ker}\tau$,then we consider the ideal Δ_g generated in A[Y] by all rxr-minors of the jacobian matrix $(\partial g/\partial Y)$ associated to g. Denote $H_I=\sum_g \Delta_g((g):I)$,where the sum is taken over all systems of r-polynomials from I,r being variable and at most s.The closed set $V(H_I+I)\subset \operatorname{Spec}(A[Y]/I)$ corresponds exactly to the set of all prime ideals $p\in\operatorname{Spec}(B')$ for which B'_p is not a smooth A-algebra.Remark that B' is smooth in $q\cap B'$ iff $q\not\supset\tau(H_I)$.So we get $ht(\tau(H_I))\geq 2$. Let $x\in\underline{m}'A'^d$ be given by (2.3.1) for the ideal $\tau(H_I)$.Thus $ht(\sigma_x(\tau(H_I)))\geq 2$.Let σ'_x be the extension of σ_x to A'' and v the

ii) there exists a (filtered) inductive set $(T_i')_{i \in I}$ of finite type A-algebras and some morphisms of A-algebras $u_i : T_i' \longrightarrow A'$ (not necessarily surjective) such that

a) T_i' is a smooth A-algebra in $\left\{ u_i^{-1}(q) \,\middle|\, q \in \mathcal{P}_{A'} \right\}$ for all $i \in I$

b) $(A', (u_i)) = \varinjlim T_i'$

iii) Every finite presentation sub-A-algebra C of A' can be embedded in a finite presentation A-algebra D' and the inclusion $C \subset A'$ extends to a morphism of A-algebras $v : D' \longrightarrow A'$ such that D' is a smooth A-algebra in $\left\{ v^{-1}(q) \,\middle|\, q \in \mathcal{P}_{A'} \right\}$

Proof i) ⟹ ii), and ii) ⟹ iii) go like in the proof of [illegible].
For the implication iii) ⟹ i) we need the following lemma:

(2.3.1) Lemma Let A[X] be the polynomial ring in some indeterminates $X = (X_1, \ldots, X_n)$ over a noetherian local ring (A,m). Let q be an arbitrary prime ideal of A[X] with $ht(q) \leq \dim A$. Then there exists $a = (a_1, \ldots, a_n)$ in A^n such that the morphism $\sigma_a : A[X] \longrightarrow A$ given by $X \longmapsto a$ [illegible] maps q in an ideal with height $\geq ht(q)$.

Let h be the natural number associated to A by [illegible].
Take $\mathcal{P}' := \{$[illegible]$\}$ the maximal ideal of A' and $X = (X_1, \ldots, X_n)$. Clearly, the inclusion $A \subset A'$ satisfies the hypothesis of (2.1.1) and so C can be embedded in a finite presentation sub-A-algebra D of A' such that D' is a smooth A-algebra in $q \cap D'$ for all $q \in \mathcal{P}$.

Now, let $C = A[X] \longrightarrow$ [illegible] $(Y_1, \ldots, Y_n)$ be a morphism of A-algebras such that [illegible] $= (f_1, \ldots, f_r)$ is a system of polynomials [illegible] we consider the ideal Δ_g generated in A[X] by all [illegible] of the Jacobian matrix [illegible] associated to g, [illegible] where the sum is taken over all systems of [illegible] polynomials [illegible] at most h. The [illegible] contains [illegible] the set of all prime [illegible] which [illegible] is not a smooth [illegible] A-algebra [illegible] so we get $ht(C(\ldots)) \geq$ [illegible]
Let [illegible] (2.1.1) for the ideal [illegible] then

[illegible] be the extension of [illegible] to [illegible] extension of $\Gamma'_{x}\mathcal{E}$ to B'. Clearly B' is a smooth A-algebra in $\{v^{-1}(q) \mid q \in \mathcal{P}_A\}$.

Q.E.D.

The following result is an extension of [13] Proposition 3.

(2.4) Corollary Let $u: A \to A'$ be an injective morphism of noetherian factorial rings such that every prime element p from A remains prime in A' and the extension $k(p) \hookrightarrow k(pA')$ is separable. Suppose the extension $Q(A) \hookrightarrow Q(A')$ is separable and let $f = (f_1, \dots, f_r)$ be a system of polynomials in $Y = (Y_1, \dots, Y_n)$ over A and $y \in A'^n$ a solution of f in A'. Then there exist a system of polynomials h in $Y, Z = (Z_1, \dots, Z_t)$ over A and an element $z \in A'^t$ such that $(f) \subset h\, A[Y,Z]$, $h(y,z) = 0$ and $H_{(h)}(y,z)A'$ has height ≥ 2.

Proof of (2.3.1) It is enough to consider the case n=1. Remark that if $\underline{a}' := \underline{a} \cap A$ has height $< ht(\underline{a})$ then $ht(\underline{a}) = ht(\underline{a}') + 1$. If $ht(\underline{a}') = ht(\underline{a})$ we have nothing to prove (take x=0 for example). Otherwise let $q_1, \dots, q_r$ be the minimal prime ideals associated to $\underline{a}' \subset A$ and $f \in \underline{a}$ a polynomial which is not contained in $q_i A[X]$, $i = 1, \dots, r$. As $ht(q_i) < \dim A$, we can choose an element $x \in \underline{m}$ which is not in $\bigcup_{i=1}^{r} q_i$. We claim that there exists a natural number s such that $f(x^s) \notin \bigcup_{i=1}^{r} q_i$ for s sufficiently big. But this is true because f has only a finite number of solutions in every A/q_i and $(x^s)_{s \in \mathbb{N}}$ induces an infinite set of elements in A/q_i.

Q.E.D.

Remark The proof of (2.3.1) is not far from [13] Proposition 4. We prefer our Lemma because it does not ask for A to be a complete local domain as in [13].

§3 Rings with the property of approximation.

A noetherian local ring $(A,\underline{m})$ has the property of approximation (shortly we shall write A is an AP-ring) if every system of polynomial equations with coefficients in A has a solution in its completion $\hat{A}$ at $\underline{m}$ iff the system has a solution in A [12]. The AP-rings are henselian and their formal fibres are geometrically domains, so the AP-rings are universally japanese (see [2], [8])

An important consequence of [4] (4.6.7) and (5.1) is the following:

(3.0) Proposition Let A be an AP-ring. Then A is a normal domain iff $\hat{A}$ is too.

The proof being short enough we shall sketch it here. We shall express the property "A is not integrally closed in Q(A)" by the compatibility and incompatibility of some systems of polynomials equations. Thus A is not normal iff there exist $a_1,\dots,a_n,c,d \in A, n \geqslant 1$ such that $c^n+\sum_{i=1}^{n} a_i c^{n-i} d^i = 0$ and the systems $d=0; c=dX$ are incompatible in A, X being an indeterminate. The proof is a consequence of the following:

(3.0.1) Lemma Let $f,g_1,\dots,g_s$ be some systems of polynomials in some indeterminates $X=(X_1,\dots,X_n)$, $Y=(Y_1,\dots,Y_m)$ over an AP-ring. Suppose f depends only on X. Then f has a solution $\hat{x} \in \hat{A}^n$ which makes the systems $g_j(\hat{x},Y)=0$ incompatible over $\hat{A}$ for every $j=1,\dots,s$ iff f has a solution $x \in A^n$ which makes the systems $g_j(x,Y)=0$ incompatible over A for every $j=1,\dots,s$.

For the proof of (3.0.1) remark that the solutions of f in $\hat{A}$ can be well approximated in the $\underline{m}$-adic topology by the solutions of f in A.

(3.1) Theorem The formal fibres of AP-rings are geometrically normal.

Proof Let q be a prime ideal of $\hat{A}$ and $K \supset Q(A/q\cap A)$ a finite extension. Let A' be the integral closure of A in K. A' is a finite

(3) [illegible] the property of approximation

A noetherian local ring (A,m) has the property of approximation (shortly we shall write A is an AP-ring) if every system of polynomial equations with coefficients in A has a solution in A iff [illegible] the system has a solution in $\hat{A}$ [illegible]. There [illegible] examples and their small [illegible] are [illegible] [illegible] AP-rings are universally japanese [illegible] (3.1)

An important consequence of [6] (2.4.7) and (3.1) is the following:

(3.1) Proposition. Let A be an AP-ring. Then A is a normal domain iff $\hat{A}$ is so.

The proof [illegible] small sketch it [illegible] small [illegible] "A is not integrally closed in Q(A)" [illegible] solvability [illegible] of some system of polynomial equations [illegible] or not normal iff there exist $b_1,\ldots,b_n \in A$ [illegible] and the systems [illegible] in A [illegible] indeterminates. The proof is a consequence of the following:

(3.2) Lemma. Let $f_1,\ldots,f_q$ [illegible] be some systems of polynomial [illegible] $A[X_1,\ldots,X_m]$ [illegible] over A and a solution [illegible] the systems [illegible] inconsistent over A for every $j=1,\ldots,q$ iff there is a solution $x \in A^m$ which makes the systems [illegible] inconsistent over A for every $j=1,\ldots,q$.

For the proof of (3.2) I remark that the solutions of f [illegible] in the m-adic topology by the [illegible]

(3.3) Corollary. The formal fibres of AP-rings are geometrically normal.

Proof. [illegible] prime ideal of A and [illegible] (A/pA) [illegible] the integral closure of [illegible] in $K(A')$ is [illegible]

A-algebra because A is universally japanese. Also A' is a local ring because it is a domain and a finite algebra over a henselian local ring [8]. Thus A' is an AP-ring (a finite local algebra over an AP-ring is still an AP-ring; see for example [9] ch.II(1.2)). The completion $\hat{A}'$ of A' is still an integrally closed domain by (3.0) and then the ring $K\otimes_A \hat{A} = Q(A')\otimes_A \hat{A} \cong Q(A')\otimes_{A'}(A'\otimes_A \hat{A}) \cong$ $\cong Q(A')\otimes_{A'}\hat{A}'$ is an integrally closed domain.

Q.E.D.

(3.1.1) Remark If the formal fibres of a henselian ring are geometrically normal, then they are geometrically integrally closed domains [5].

(3.2) Corollary. Two dimensional AP-rings are excellent henselian.

Indeed, the formal fibres of a two dimensional ring have dimension $\leqslant 1$. Consequently, "geometrically normal formal fibres" is equivalent in this case with "geometrically regular formal fibres".

(3.3) Theorem. A two dimensional factorial local ring is an AP-ring iff it is excellent henselian and $Q(A)\otimes_A \hat{A}$ is a principal ideal domain.

Proof $\hat{A}$ is a factorial ring if A is a factorial AP-ring ([9] ch.V) and so the ring $Q(A)\otimes_A \hat{A}$ is an one dimensional factorial domain. Using (3.2) we get the necessity. **Conversely, if A is excellent henselian and $Q(A)\otimes_A \hat{A}$ is factorial then every prime element from A remains prime in $\hat{A}$ by [5] and $\hat{A}$ is factorial ring (cf. [14]** th. 5, p.31). Thus the hypothesis of (2.4) are fulfilled for the inclusion $A \hookrightarrow \hat{A}$.

Now let $f=(f_1,\dots,f_m)$ be a system of polynomials in $Y=(Y_1,\dots,Y_n)$ over A, and $\hat{y}\in \hat{A}^n$ a solution of f in $\hat{A}$. We may suppose by (2.4) that $ht(H_{(f)}(\hat{y})\hat{A}) \geqslant 2$. By Elkik's Theorem (see [7] or [9](I.6.1)), there exists a function $d: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$, $d(r,c) > c,r$ associated to f and $H_{(f)}$ with the following property:

"If $\bar{y} \in A^n$ satisfies $f(\bar{y}) \equiv 0 \bmod \underline{m}^{d(r,c)}$ and $H_{(f)}(\bar{y}) \supset \underline{m}^r$, then there exists a solution $y \in A^n$ of f in A such that $y \equiv \bar{y} \bmod \underline{m}^c$."

The ideal $H_{(f)}(\hat{y})\hat{A}$ being a m-primary ideal, there exists a natural number r such that $\underline{m}^r \subset H_{(f)}(\hat{y})\hat{A}$. Choose an element $\bar{\bar{y}} \in A^n$ such that $\bar{\bar{y}} \equiv \hat{y} \bmod \underline{m}^{d(r,1)}\hat{A}$. We have $\underline{m}^r\hat{A} \subset H_{(f)}(\bar{y})\hat{A} + \underline{m}^{d(r,1)}\hat{A} \subset H_{(f)}(\bar{y})\hat{A} + \underline{m}^{d(r,1)-r}(H_{(f)}(\bar{y})\hat{A} + \underline{m}^{d(r,1)}\hat{A}) \subset \ldots$ and it results $\underline{m}^r\hat{A} \subset H_{(f)}(\bar{y})\hat{A}$. Thus we obtain $\underline{m}^r \subset H_{(f)}(\bar{y})$. As $f(\bar{y}) \equiv f(\hat{y}) = 0 \bmod \underline{m}^{d(r,1)}$, there exists a solution of f in A.

Q.E.D.

As a corollary we get the main result of [13] :

(3.4) <u>Corollary</u> A two dimensional regular local ring is henselian universally japanese iff it has the property of approximation.

[illegible] and [illegible] by a derivation [illegible] in A such that [illegible]

and [illegible]

Let [illegible] Lemma 1 that, there exists [illegible]

[illegible] We have [illegible]

[illegible] Thus we obtain [illegible]

[illegible] ; hence there exists a derivation [illegible]

Q.E.D.

And [illegible] we get the main result of [illegible]:

(13) [illegible] dimensional regular local ring is [illegible] if it has the property of [illegible].

REFERENCES

[1] M.André, Homologie des algébres commutatives, Springer-Verlag, Berlin, 1974

[2] M.Artin, Algebraic approximation of structures over complete local rings, Publ. Math. IHÈS, 36(1969), p.23-58

[3] M.Artin, Construction techniques for algebraic spaces, Actes Congrès intern. math., tome 1, 1970, 419-423

[4] S.Basarab, V.Nica, D.Popescu, Approximation properties and existential completeness for ring morphisms, INCREST Preprint series in mathematics no.9/1980

[5] A.Brezuleanu, N.Radu, On the morphisms which preserve the prime ideals, to appear in Rev. Roum. Math. Pures et Appl.

[6] A.Grothendieck, J.Dieudonné, E.G.A. IV, Publ. Math. IHES, 20,24(1964)

[7] R.Elkik, Solutions d'équations à coefficients dans un anneau hensélien, Ann.Sc.Ec.Norm.Sup. 4e série, t.6, 1973, p.533-604

[8] H.Kurke, G.Pfister, M.Roczen, Henselsche Ringe und algebraische Geometrie, Berlin, 1974

[9] H.Kurke, T.Mostowski, G.Pfister, D.Popescu, M.Roczen, Die Approximationseingenschaft lokaler Ringe, Lecture Notes in Math., 634, Springer-Verlag, Berlin Heidelberg New-York, 1978

[10] H.Matsumura, Commutative Algebra, Benjamin, New-York, 1970

[11] A.Néron, Modèles minimaux des variétés abéliennes sur les corps locaux et globaux, Publ.Math.IHES, 21(1964)

[12] D.Popescu, Algebraically pure morphisms, Rev.Roum.Math.Pures et Appl., tome XXIV, 6(1979), 947-977

[13] D.Popescu, A remark on two dimensional local rings with the property of approximation,INCREST Preprint series in mathematics no.39/1979

[14] P.Samuel, Anneaux factoriels,Rédaction de A.Micoli,Soc.de Mat. de Sao Paolo,1963