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REMARKS ON IDEALS OF THE CALKIN-ALGEBRA

FOR CERTAIN SINGULAR EXTENSIONS

by

M.PIMSNER, S.POPA and D.VOICULESCU

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May 1980

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by M. Pimsner, S. Popa and D. Voiculescu

One of the classes of extensions which are more general than those of the ideal of compact operators K(H), for which we have the Brown-Douglas-Fillmore theory ([2], [3]), are the extensions of $C_{\Omega}(X) \otimes K(H)$ where X is locally compact. A class of such extensions, the homogeneous ones, for X compact have been studied in ([8],[10]) (see [7] for a more general theory). The opposite case appears to be that of the singular extensions, i.e. those for which the extension is "localised" in a certain sense at infinity, in the Alexandrov compactification of X. Such extensions have been considered by Delaroche ([4]) and in connection with the C*-algebra of the Heisenberg group, by several authors ([9], [7], [11]). The structure of such extensions appears to be rather mysterious. This is due in part to the complicated structure of the "Calkin algebra" corresponding to a singular extension problem. This "Calkin algebra" is far from being simple and the aim of the present note is to classify its clossed two-sided ideals.

We begin with the notations.

Throughout, H. will denote a complex separable infinite-dimensional Hilbert space and L(H), K(H) will denote the set of all bounded operators on H and respectively the ideal of compact operators on H.

Instead of a locally compact space X, it will be more convenient to consider a pointed compact space (Ω , ω), where X corresponds to $\Omega \setminus \{\omega\}$. We shall assume that Ω is metrizable

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and finite-dimensional. By \mathcal{B} (Ω , ω , H) (or simply \mathcal{B}) we shall denote the C*-algebra of bounded norm-continuous functions f: $\Omega \setminus \{\omega\}$ \longrightarrow K(H) and by \mathcal{J} (Ω , ω , H) (or simply \mathcal{J}) the C*-algebra of norm-continuous functions f: Ω \longrightarrow K(H) such that f(ω)=0. Clearly, the restriction to $\Omega \setminus \{\omega\}$ gives an isometric injection of \mathcal{J} into \mathcal{B} , which we shall use to identify \mathcal{J} with a sub-algebra of \mathcal{B} , which is in fact a closed two-sided ideal of \mathcal{B} . The singular extensions will correspond to *-monomorphisms into \mathcal{B}/\mathcal{J} , which is what might be called the "Calkin algebra" for the singular extensions of $\mathrm{C}_{\mathrm{O}}(\Omega \setminus \{\omega\}) \otimes \mathcal{K}(\mathrm{H}) \simeq \mathcal{D}$. The problem we consider is the classification of the closed two-sided ideals of \mathcal{B}/\mathcal{J} or equivalently the classification of the closed two-sided ideals of \mathcal{B} containing \mathcal{J} .

For the sake of completeness we shall record as Lemma 1 a most likely well-known consequence of the finite-dimensionality of Ω .

Lemma 1. Let Ω be a compact metrizable finite-dimensional space. Then there is a number N, depending only on the dimension of Ω such that for every open covering $\mathcal{U} = (U_j)_{j \in J}$ of $\Omega \setminus \{\omega\}$ there is a refinement $\mathcal{V} = (V_i)_{i \in I}$, which is a covering by open sets, with the following property:

whenever p \neq q belong to the same I_k .

For the next proposition we shall introduce some notations. By $E(\sigma;a)$ we shall denote for a positive operator $a\in L(H)$, the spectral projection corresponding to the Borel set $\sigma\in R$. Another notation we shall use, is A_+ for the positive part of a C^* -algebra A.

Proposition 2.Let $M \subset \mathcal{B}_+ = \mathcal{B}_+(\Omega, \omega, H)$ and $x \in \mathcal{B}_+$. Then, the following conditions are equivalent:

⁽i) x is in the closed two-sided $id_{\epsilon a}l$ of $\mathcal B$ generated

and finite-dimensional, by \mathcal{A} (Ω , ω , H) (or simply \mathcal{B}) we shall denote the C*-algebra of bounded norm-continuous functions if $\Omega \setminus i\omega I \longrightarrow I(H)$ and by $\mathcal{J}(\Omega, \omega)$, H) (or simply \mathcal{J}) the C*-algebra of norm-continuous functions if $\Omega \longrightarrow K(H)$ such that $I(\omega) = 0$, distribution for $I(\omega) = 0$, finite restriction to $I(\omega) = 0$, finite and $I(\omega) = 0$, which we shall use to identify metric injection of $I(\omega) = 0$, which is in fact a closed two-gladed deal of $I(\omega) = 0$. The singular extensions will correspond to $I(\omega) = 0$, which is what might be called the "Calkin algebra" for the singular extensions of $I(\omega) = 0$ ($I(\omega) = 0$) $I(\omega) = 0$.

 \simeq J. The problem we consider is the classification of the closed two-sided ideals of β_{ij} or equivalently the classification of the closed two-sided ideals of β_i containing J.

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(i) x is in the closed two-sided is al of X cenerated

by MUJ.

(ii) for every $\varepsilon > 0$ there are $\delta > 0$, $n \in \mathbb{N}$, $y_1, \dots, y_n \in \mathbb{M}$ and $V \subset \Omega$ a neighborhood of ω such that rank $E(\mathcal{E}\varepsilon, \infty); x(t) \leq \frac{n}{\varepsilon}$ rank $E(\mathcal{E}\varepsilon, \infty); y_j(t)$ for all $t \in V \setminus \{\omega\}$. j=1

 $\frac{\text{Proof.We}}{\mathcal{B}}$ = \mathcal{B} + \mathbb{C} e denote the \mathbb{C}^* -algebra obtained by adjoining a unit to \mathcal{B} , then (i) is equivalent with:

for every $\infty > 0$ there are $n \in \mathbb{N}$, $y_1, \dots, y_n \in \mathbb{M}$, $b_1, \dots, b_n \in \mathcal{B}$, $d \in \mathcal{J}_+$ such that

$$\alpha + \sum_{j=1}^{n} b_{j}y_{j}b_{j}^{*} + d \geqslant x.$$

In view of the definition of $\mathcal J$ this gives that (i) is also equivalent to:

(i') for every $\infty > 0$ there are

 $n \in \mathbb{N}, y_1, \dots, y_n \in \mathbb{M}, b_1, \dots, b_n \in \mathcal{B}$, and $V \subset \Omega$ a neighborhood of ω such that

$$\propto I_{H}^{+}$$
 $\sum_{j=1}^{n} b_{j}(t)y_{j}(t)b_{j}^{*}(t) \gg x(t)$

for all $t \in V \setminus \{\omega\}$

With these preparations we can now pass to the proof of the proposition.

This will follow from (i) <=> (i') and some remarks based on consequences of the mini-max principle.

Thus, using results in ch.II, §2 of [6] we have for > 0 the inequality:

(ii) for every $\varepsilon > 0$ there are $\delta > 0$, $n \in \mathbb{N}, y_1, \dots, y_n \in \mathbb{N}$ and $\forall c \leq \Omega$ a neighborhood of ω and that reak $\mathbb{E}([\varepsilon, \infty); x(t)) \leq \sum_{t \in \mathbb{N}} \operatorname{rank} \mathbb{E}([\varepsilon, \infty); y_t(t))$ for all $t \in \mathbb{V} \setminus \{\omega\}$, $j \in \mathbb{N}$

Proof. We shall use in the proof the following fact. bet that 8 + 6 = 8 then can be sainted by adjoining a unit to 8 + 6 = 8.

for every of there are no N , M,, M by the M by the

In view of the definition of 7 this gives that (1) is

(11) for every 0 < >0 there are

 $\Omega\supset V$ by . . . & $\ni_{n}\sigma_{1},\ldots, _{1}\sigma_{n}M$ $\ni_{n}\kappa_{1},\ldots, _{1}\kappa_{n}M$ $\ni \sigma$

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With these proparations we can now pass to the proof of the propertion.

 (\pm) \Leftarrow (\pm)

This will follow from (i) <=> (i') and some remarks bears on consequences of the mini-max principle.

Thus, using results in ch.11, 52 of [6] we have for \$ > 0 the inequality:

rank $E(E_i^*, \infty)$, $b_j(t)y_j(t)b_j^*(t)$ \leq rank $E([\frac{\delta^*}{\|b_j\|^2}, \infty)$, $y_j(t)$)

Further, using Corollary 2.2 in § 2 of ch. II of [6] we have:

rank E([%,
$$\infty$$
), $\sum_{j=1}^{n} b_{j}(t)y_{j}(t)b_{j}^{*}(t)) \leq$

$$\leq \sum_{j=1}^{n} \operatorname{rank} E(\left[\frac{y}{n}, \infty\right], b_{j}(t)y_{j}(t)b_{j}^{*}(t)) \leq$$

$$\leq \sum_{j=1}^{n} \operatorname{rank} E(\left[\frac{\delta}{n \|b_{j}\|^{2}}, \infty\right), y_{j}(t)).$$

Assume now that

$$\propto I_{H^{+}} \sum_{j=1}^{\infty} b_{j}(t)y_{j}(t)b_{j}^{*}(t) \gg x(t),$$

then from the mini-max principle, it follows that for $\delta > \infty > 0$ we have

rank
$$E([x-\alpha, \infty); \sum_{j=1}^{n} b_{j}(t)y_{j}(t)b_{j}(t)) \geqslant$$

$$\gg$$
 rank $E([\%, \infty); x(t))$.

This, together with our previous remarks, gives:

Thus, taking $\alpha = \frac{\varepsilon}{2}$, $\delta = \varepsilon$ we see that (i') implies (ii) with $\delta = \frac{\varepsilon}{2\beta}$ (ii) \Longrightarrow (i)

Let (i") denote condition (i') with M replaced by the closed two-sided ideal generated by M.It will be clearly sufficient

rank $E([x], \infty)$, $b_j(t)y_j(t)b_j''(t)$ $\leq \operatorname{rank} E([\frac{y}{\|b_j\|_{L^2}}, \infty), y_j(t))$ Further, using Corollary 2.2 in § 2 of ob. II of [5] we

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then from the mini-max principle, it follows that for 8 > 0 > 0

rank
$$\mathbb{E}([x-\alpha], \infty)_1 \stackrel{\triangle}{\sum} b_j(t)y_j(t)b_j^*(t)) >$$

Years B([7] oo);x(t)).

This, together with our previous remarks, gives: renk $E(\{v_1, o^{\infty}\}; x(v)) \leq \sum_{j=1}^{\infty} \operatorname{rank} E(\{v_j^{\infty} - \infty\}; y_j(v))$ where $y_j^{\infty} = n(\operatorname{mix} v_j + v_j)$ where $y_j^{\infty} = n(\operatorname{mix} v_j + v_j)$.

Thus, taking $\infty = ^{\mathbb{C}}/2$, $\delta' = \mathbb{C}$ we see that (i') implies (ii) with $\delta = ^{\mathbb{C}}/2$,

 $(11) \iff (11)$

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to prove that (ii) => (i"). Thus, assume (ii) holds. Then for every $t\in V\setminus\{\omega\}$ we can find $b_t^{(h)}\in\mathcal{B}$ (1 $\leq h\leq n$) such that

$$\sum_{h=1}^{n} b_{t}^{(h)}(t)y_{h}(t)b_{t}^{(h)*}(t) + \epsilon I_{H} > x(t),$$

$$\|b_t^{(h)}\| \leq \left(\frac{\|x\|}{S}\right)^{n/2}$$

But then, for every $t\in V\setminus\{\omega\}$ there is an open set $U_t\subset\Omega\setminus\{\omega\}$, $t\in U_t$ such that

$$\sum_{h=1}^{n} b_{t}^{(h)}(s) y_{h}(s) b_{t}^{(h)*}(s) + 2 \epsilon I_{H} \gg x(s)$$

for all s ∈ Ut

Assuming V is compact (which is no loss of generality) we can apply Lemma 1 and find a covering $(V_j)_{j \in J_1} \cup \cdots \cup (V_j)_{j \in J_N}$ by open subsets of $V \setminus \{\omega\}$ (in the relative topology of $V \setminus \{\omega\}$) such that $V_j \subset U_{t(j)}$ and $V_p \cap V_q = \emptyset$ whenever $p \neq q$ belong to the same set J_k . Let further $(g_j)_{j \in J_1} \cup \cdots \cup J_k$ be a partition of unity subordined to this covering of $V \setminus \{\omega\}$. Then we may define bounded continuous K(H)-valued functions $C_k^{(h)}$ on

 $V \setminus \{\omega\}$ $(1 \le k \le N, 1 \le h \le n)$ by

$$c_k^{(h)}(s) = \sum_{j \in J_k} \sqrt{g_j(s)} b_t^{(h)}(s),$$

We have:

$$\sum_{h=1}^{n} \left(\sum_{k=1}^{N} c_{k}^{(h)}(s) y_{h}(s) c_{k}^{(h)}(s) \right) + 2 \in I_{H} = 0$$

to prove that (ii) => (i'). Thus, assume (ii) holds. Then to for every $t \in V - l\omega t$ we can find $b_t^{(h)} \in \mathcal{S}$. $(1 \le h \le h)$ such that

$$\sum_{h=2}^{n} b_{t}^{(h)}(t) y_{h}(t) b_{t}^{(h)*}(t) + \mathcal{E} I_{H} > x(t),$$

But then, for every $t\in V-\{\omega\}$ there is an open set $U_t\subset\Omega\setminus\{\omega\}$, $t\in U_t$ such that

$$(a)x < BI3S+(a)^*(a)^*(a)^*(a)^*(a)^*(a)$$

for all $s \in \mathbb{U}_{b}$

Assuming V is compact (which is no loss of generality) we can apply Lemma 1 and find a covering $(V_j)_{j \in J_1} \cup \ldots \cup (V_j)_{j \in J_2} \cup \ldots \cup (V_j)_{j \in J_3}$ by open subsets of $V \in [\omega]$ (in the relative topology of $V \in [\omega]$) such that $V_j \subset U_1(j)$ and $V_p \cap V_q = \emptyset$ whenever $p \neq q$ belong to the same set J_k . Let further $(g_j)_{j \in J_1} \cup \ldots \cup J_k$ be a partition of unity subordined to this covering of $V \in [\omega]$. Then we may define bounded continuous K(H)-valued functions $O_k^{(h)}$ on

We have:

$$=_{H} I \stackrel{\mathcal{H}}{\to} S * \left((a)^{(a)} \circ (a) \circ (a)^{(a)} \circ (a) \circ (a$$

$$= \sum_{h=1}^{n} \sum_{k=1}^{N} (\sum_{j \in J_{k}} g_{j}(s)b_{t(j)}^{(h)} (s)y_{h}(s)b_{t(j)}^{(h)*}(s))+2 \epsilon I_{H} =$$

$$= \sum_{k=1}^{n} \sum_{j \in J_{k}} g_{j}(s) \left(\sum_{h=1}^{n} b_{t}^{(h)}(s) y_{h}(s) b_{t}^{(h)*}(s) + 2 \in I_{H} \right) \ge$$

$$\Rightarrow$$
 $\sum_{k=1}^{n}$ $\sum_{j \in J_k} g_j(s)x(s)=x(s)$

Remarking that the K(H)-valued functions $c_k^{(h)}$ can be prolonged from $V \setminus \{\omega\}$ to all of $\Omega \setminus \{\omega\}$ we see that we have proved that (ii) \Longrightarrow (i").

Q.E.D.

We turn now to the classification of the closed two-sided ideals of $\mathcal B$ which contain $\mathcal J$. This will be achieved by exibiting a bijection between these ideals and the class of cones $\mathcal C$ of positive continuous functions on $\Omega \setminus \{\omega\}$, satisfying the following "completeness" property:

 $(*) \ \underline{\text{If }} f\colon \Omega \setminus \{\omega\} \longrightarrow [\circ, \infty) \quad \underline{\text{is a continuous function}}$ such that for every \$\xi\$ > 0 there exists a neighborhood \$V_{\xi}\$ of \$\omega\$ and a function \$g_{\xi} \in \xi\$ such that $f(t) \leq g_{\xi}(t) + \xi$ for all $t \in V_{\xi}$$

then f belongs to C.

We pass now to the construction of the correspondence between ideals and cones.

By \mathcal{F} we shall denote the set of continuous functions $\varphi: [0,\infty) \longrightarrow [0,\infty)$ such that supp $\P\subset (0,\infty)$. Let further, for E>0, Ψ_E stand for the following particular function in \mathcal{F} $\Psi_E(t)=\max(t-E,0)$.

For $x \in \mathcal{B}_+$ and $Y \in \mathcal{F}$ we get a continuous function $T_{\varphi}(x)$ and $\Omega \setminus \{\omega\}$ defined by $T_{\varphi}(x)$ are Y(x).

 $= \sum_{h=1}^{n} \sum_{k=1}^{n} (\sum_{j \in J_{k}} g_{j}^{(a)}(a) g_{k(j)}^{(b)}(a) g_{k(j)}$

 $\sum_{k=1}^{N} \sum_{j \in J_K} g_j(a) x(a) = x(a)$

Remarking that the K(H)-valued functions $\sigma_K^{(h)}$ can be prolonged from V < [ω] to all of Ω < [ω] we see that we have proved that (11) \Rightarrow (1").

Q. E. D.

We turn now to the classification of the closed two-sided ideals of \mathcal{S} which contain \mathcal{I} . This will be achieved by exibiting a bijection between these ideals and the class of cones \mathcal{E} of positive continuous functions on $\Omega \setminus \omega_i$, satisfying the following "completeness" property:

(*) If $f: \Omega \setminus \{\omega\} \longrightarrow \{0,\infty\}$ is a continuous function such that for every $\epsilon > 0$ there exists a neighborhood V_{ϵ} of ω and a function $g_{\epsilon} \in \mathcal{E}$ such that $f(t) \leq g_{\epsilon}(t) + \epsilon \cdot for \text{ all } t \in V_{\epsilon}$

Men i belongs to E

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By \mathcal{F} we shall denote the set of continuous functions $\varphi:[0,\infty) \longrightarrow [0,\infty)$ such that supp $\P \subset (0,\infty)$. Let further, for $\xi > 0$, Ψ_{ε} stand for the following particular function in \mathcal{F} $\psi(t) = \max(t-\varepsilon,0)$.

For $x \in \mathcal{B}_{+}$ and $Y \in \mathcal{F}$ we get a continuous function $T \in X$ on $\Omega \setminus \{\omega\}$ defined by $T \notin X(t)$ =Trace Y(x(t)).

Let us note the following properties of the functions

T_{\phi} x:

- 1) $T_{\varphi} x(t) \leq \| \Psi(x(t)) \|$.rank $E(LS, \infty); x(t))$ where S > 0 is the greatest lower bound of supp Ψ .
 - 2) ε .rank $E([2\varepsilon, \infty); x(t)) \leq T_{\chi_{\varepsilon}} x(t)$
- 3) Assume $\gamma \in \mathcal{F}$ is an increasing function, then for $x,y \in \mathcal{B}_+$, $x \leq y$ we have $T_{\gamma} x(t) \leq T_{\gamma} y(t)$ for all $t \in \Omega \setminus \{\omega\}$.

The last property is a consequence of the mini-max principle, which shows that the n-th eigenvalue of y(t) is greater than the n-th eigenvalue of x(t) (eigenvalues being listed in decreasing order, multiple eigenvalues repeated), so that the same is true for the n-th eigenvalues of $\Upsilon(y(t))$ and $\Upsilon(x(t))$.

For a closed two-sided ideal $\mathcal J$ containing $\mathcal J$ we shall denote by $\mathcal C(\mathcal J)$ the smallest cone of continuous positive functions satisfying property (*) containing all the functions $T_{\mathcal F} x$, where x runs over $\mathcal J_+$ and $\mathcal F$ runs over $\mathcal F$.

Conversely, for a cone $\mathcal E$ satisfying (*), let $\mathcal F_+(\mathcal E)$ be the set of all positive elements $x \in \mathcal B_+$ such that $T_{\varphi}(x) \in \mathcal E$ for all $\P \in \mathcal F$. $\mathcal F_+(\mathcal E)$ will be the set of all elements $x \in \mathcal B$ such that $|x| \stackrel{\text{\tiny \pm}}{=} (x^* |x|)^{1/2} \in \mathcal F_+(\mathcal E)$.

Lemma 3. $J(\mathcal{E})$ is a closed two-sided ideal of \mathcal{B} , which contains J. Moreover $(J(\mathcal{E}))_+ = J_+(\mathcal{E})$

Proof. Remark first that $x \in \mathcal{B}_+$ is in $\mathcal{J}_+(\mathcal{C})$ if $T_{\gamma_{\varepsilon}} x \in \mathcal{C}$ for all $\varepsilon > 0$. This follows from the fact that every $\gamma \in \mathcal{F}$ is dominated by a function of the form $\alpha \uparrow_{\varepsilon}$ on the spectrum of x and from property (\star).

Also, if $f: [0,\infty) \longrightarrow [0,\infty)$ is a continuous function such that f(0)=0, then $\varphi \circ f \in \mathcal{F}$ for every $\varphi \in \mathcal{F}$. Hence, if

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- 1) $T_{\varphi} x(t) \leq \| \Psi(x(t)) \|$, rank $B(LS, \infty); x(t))$ where .S > 0 is the greatest lower bound of supp Ψ .
 - (t)x +T ≥ ((t)x;(∞, 35])E siner. 3 (5
- 3) Assume $P \in \mathcal{F}$ is an increasing function, then for $x,y \in \mathcal{S}_+$, $x \in y$ we have $T_{\varphi} x(t) \leq T_{\varphi} y(t)$ for all $t \in \Omega \setminus \{\omega\}$

The last property is a consequence of the mini-max principle which shows that the n-th eigenvalue of y(t) is greater than the n-th eigenvalue of x(t) (eigenvalues being listed in decreasing order, multiple eigenvalues repeated), so that the same is true for the n-th eigenvalues of Y(y(t)) and Y(x(t)).

For a closed two-sided ideal \mathcal{F} containing \mathcal{I} we shall denote by $\mathcal{E}(\mathcal{F})$ the smallest cone of continuous positive functions satisfying property (κ) containing all the functions $T_{\mathcal{F}}$ where x runs over $\mathcal{F}_{\mathcal{F}}$ and \mathcal{F} runs over $\mathcal{F}_{\mathcal{F}}$.

Conversely, for a cone \mathcal{E} satisfying (*), let $\mathcal{F}_*(\mathcal{E})$ be the set of all positive elements $x \in \mathcal{S}_+$ such that $T_*(x) \in \mathcal{E}$ for all $Y \in \mathcal{F}_+$, $\mathcal{F}_*(\mathcal{E})$ will be the set of all elements $x \in \mathcal{S}_+$ such that $(x) = (x^* x)^{1/2} \in \mathcal{F}_*(\mathcal{E})$.

Lemma 3. 2(8) is a closed two-sided ideal of 8 ,which contains 7 . Moreover (2(8)) = 2.(8)

Proof. Remark first that $x \in \mathcal{S}_n$ is in $\mathcal{J}_n(\mathcal{E})$ if $T_{\mathcal{E}_n} \times \mathcal{E}_n$ for all $\mathcal{E} > 0$. This follows from the fact that every $f \in \mathcal{F}$ is dominated by a function of the form $x \in \mathcal{E}_n$ on the spectrum of x and from property (*).

Also, if $f: [-0,\infty) \longrightarrow [0,\infty)$ is a continuous function such that f(0)=0, then $f\circ f\in \mathcal{F}$ for every $f\in \mathcal{F}$. Hence, if

 $x \in \mathcal{J}_{+}(\mathcal{C})$ then also $f(x) \in \mathcal{J}_{+}(\mathcal{C})$

In particular, for $x \in \mathcal{B}_+$ we have that $x \in \mathcal{J}_+(\mathcal{C})$ if and only if $x^2 \in \mathcal{J}_+(\mathcal{C})$ -

We will first show that $\mathcal{J}(\mathcal{C})$ is a closed convex hereditary cone in \mathcal{B} . To this end we apply the remarks preceding the Lemma and Corollary 2.2 in § 2 of ch.II of [6], to get:

 $T_{\psi_{\varepsilon}}(x+y)(t) \le ||x+y|| \cdot rank E([\varepsilon, \infty); x(t)+y(t)) \le ||x+y|| \cdot rank E([\varepsilon, \infty); x(t)+y([\varepsilon, \infty); x([\varepsilon, \infty); x([\varepsilon, \infty); x([\varepsilon, \infty); x([\varepsilon, \infty);$

 $\leq \|x+y\| (\operatorname{rank} E(\lfloor \frac{\varepsilon}{2}, \infty); x(t)) + \operatorname{rank} E(\lfloor \frac{\varepsilon}{2}, \infty); y(t))) \leq$

 $\leq \frac{4}{\epsilon} \| x + y \| (T_{Y_{\epsilon/4}} x(t) + T_{Y_{\epsilon/4}} y(t)), \quad x, y \in \mathcal{B}_{+}.$

 $T_{\psi_{\varepsilon}} \lambda = \lambda T_{\psi_{\varepsilon/\lambda}} x, x \in \mathcal{B}_{+}, \lambda > 0.$

Also, if $0 \le x \le y$ and $y \in \mathcal{F}_+(\mathcal{C})$ then $T_{\mathcal{F}_E} x \le T_{\mathcal{E}_E} y$ since \mathcal{F}_E is increasing. This together with the preceding remarks yields that $\mathcal{F}_+(\mathcal{C})$ is a convex hereditary cone.

To see that $\mathcal{J}_+(\mathcal{C})$ is also closed, let x be in the closure of $\mathcal{J}_+(\mathcal{C})$. Then for any E>0 we can find $y\in\mathcal{J}_+(\mathcal{C})$ such that

$$x \leq y + \frac{\varepsilon}{2} e$$

where e is the unit of $\widetilde{\mathcal{B}}$.It follows that

$$T_{\gamma_{\varepsilon}} \times \leq T_{\gamma_{\varepsilon}} (y + \frac{\varepsilon}{2} \theta) = T_{\gamma_{\varepsilon/2}} y$$

and the remark at the begining of the proof yields the desired conclusion.

Also by one of the remarks at the beginning of the proof we have that

$$J(6) = \{x \in B \mid x^*x \in J_+(B) \}$$

A standard argument shows now that J(C) is a closed

(3) % s (x) = outs new (5) % s =

in particular, for x a.S. we have that x & J_(E) if and only if x 2 E/(E):

where the property of the state of the state of the season of the season of the state of the state of the season of the state of the s

2 ((t)x+(t)x+(00,3)) a store, there is 2 (t)(x+x) to

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40. 3 E.X . ((4) 5 12+(4) 2 12) TEAR 1 2 2

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Also, if G s x s y and y s d (6) then T x s T y since

\$\frac{1}{2}\$ is increasing. This together with the preceding remarks yields

that \$\frac{1}{2}\$ (3) is a convex hereditary cone.

To see that d. (8). is also alosed, let x be in the olosure of d. (8). Then for ear E > 0 we can find y E J. (8) such that

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belies and the remark at the beginning of the proof yields the desired

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8 (8) = (xex) = (3) }

besolve at (3) best and another thomas becomes a

left ideal of \mathcal{B} , such that $(\mathcal{J}(\mathcal{C}))_+ = \mathcal{J}_+(\mathcal{C})$. Moreover since $T_{\varphi} x^* x = T_{\varphi} x x^*$ it follows that $\mathcal{J}(\mathcal{C})$ is self-adjoint and hence a two-sided ideal.

Since for $x \in J_+$ and $f \in \mathcal{F}$, f(x) is zero on some neighborhood of ω , property (*) implies that $J_+ \subset J_+$ (6) and hence $J \subset \mathcal{J}(\mathcal{C})$.

Q.E.D.

Theorem 4. The correspondence
$$\mathcal{E} \longrightarrow \mathcal{J}(\mathcal{E})$$

is a bijection between con es satisfying property (*) and closed two-sided ideals of $\mathcal B$ containing $\mathcal J$. The inverse of this bijection is

Proof. It will be sufficient to prove that $J \supset J(\mathcal{C}(J))$ $\mathcal{C}(\mathcal{C}(\mathcal{C}))$

the opposite inclusions being obvious.

To prove the first inclusion, let $x \in J_+(\mathcal{E}(J))$ and $\mathcal{E} > 0$ be fixed. Since $T_{\psi_{\mathcal{E}}}$ x is in $\mathcal{E}(J)$ we can find a neighborhood $V_{\mathcal{E}}$ of ω , functions $f_1, \ldots, f_n \in \mathcal{F}$ and y_1, \ldots, y_n elements of J_+ such that

$$T_{\gamma_{\epsilon}} x(t) \leq \sum_{i=1}^{n} T_{\gamma_{i}} y_{i}(t) + \frac{\varepsilon_{4}}{4} \text{ for all } t \in V_{\epsilon}$$

The remarks preceding Lemma 3 imply that

$$\epsilon_{2}$$
 rank $E([\epsilon,\infty); x(t)) \leq c \sum_{i=1}^{n} rank E([\epsilon,\infty); y_{i}(t)) + \epsilon_{4}$

left ideal of \mathcal{B} , such that $(\mathcal{J}(\mathcal{E}))_+ = \mathcal{J}_+(\mathcal{E})$. Moreover since $\mathbb{F}_p \times^n \times \mathbb{F}_p \times^n \mathbb{$

Since for $x \in \mathcal{I}_+$ and $\gamma \in \mathcal{F}_+$, $\gamma(x)$ is zero on some neighborhood of ω , property (*) implies that $\mathcal{I}_+ \subset \mathcal{F}_+$ (6) and hence $\mathcal{I} \subset \mathcal{F}(\mathcal{E})$.

Q.E.D.

Theorem 4. The correspondence

is a bidection between dones as satisfying property (*) and closed two-sided ideals of & containing J. The inverse of this bidection is

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Propi. It will be sufficient to prove that

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the opposite inclusions being obvious.

To prove the first inclusion, let $x \in \mathcal{F}(\mathcal{C}(\mathcal{F}))$ and E > 0 be fixed. Since $T_{\mathcal{F}_E} x$ is in $\mathcal{E}(\mathcal{F})$ we can find a neighborhood V_E of ω , functions $f_1, \dots, f_n \in \mathcal{F}$ and f_1, \dots, f_n elements of $\mathcal{F}_{\mathbb{F}}$ such that

$$\mathbb{T}_{f_{\xi}}\times(\mathfrak{t}) \ \leq \ \ \sum_{\mathfrak{t}=\mathfrak{t}} \ \mathbb{T}_{f_{\xi}} \ \mathbb{T}_{f_{\xi}} \ \mathbb{T}(\mathfrak{t}) + \ \mathcal{E}_{\mathcal{L}} \ \text{for all } \mathfrak{t} \in \ \mathbb{V}_{\xi}$$

The remarks preceding Lemma 3 imply that

$$\varepsilon_{ij}$$
 rank $E(1\varepsilon, \infty)$; $x(t)$) $\leq c\sum_{i=1}^{n} rank E(1\delta, \infty)$; $y_i(t)$) + ε_{ij}

for all $t \in V_{\epsilon}$ and where $c=\max$ (sup $\|\varphi(y_i(t))\|$) and $S = \inf(\bigcup_{1 \le i \le n} \sup_{1 \le i \le n} Y_i)$. Repeating the y_i 's several times if necessary, we may assume that $2c \in \epsilon$ so that

$$rank \ E([\epsilon, \infty); x(t)) \le \sum_{i=1}^{n} rank \ E([\delta, \infty); y_i(t)) + \frac{1}{2}$$

and since the rank of a projection is an integer, this gives:

 $rank\ E([\varepsilon,\infty);x(t))\leq \sum_{i=1}^n rank\ E([\delta,\infty);Y_i(t))\ \ for\ t\in V_\varepsilon.$ Using Proposition 2, we conclude that $x\in\mathcal{J}$. Using Lemma 3, we have

$$(J(\mathcal{E}(J)))_{+} = J_{+}(\mathcal{E}(J)) \subset J$$

and hence the desired conclusion.

To prove that $C \subset C(\mathcal{J}(\mathcal{E}))$ fix $f \in \mathcal{E}$ Consider further e_1, e_2, \ldots , an orthogonormal basis of H and let E_i denote the orthogonal projection onto Ce_i . For E > 0 let $f_n, e : \Omega \setminus \{\omega\} \longrightarrow [o, \infty)$ be the functions defined recurrently by $f_{o, E} = 0$, $f_{n+1}, e = \min (f - \sum_{k=0}^{n} f_{n, E}, e)$. Define now $x_e \in \mathcal{B}$

to be the element

$$x_{\varepsilon}(t) = \sum_{n > 1} f_{n, \varepsilon}(t) E_{n} \text{ for } t \in \Omega \setminus \{\omega\}$$

and note that Trace (x ; (t))=f(t).

Our assertion will follow from property (*) once we have shown that $x_{\xi} \in \mathcal{F}(\mathcal{E})$ and that there is $\Upsilon \in \mathcal{F}$ such that $f \leq 2T_{\varphi} x_{\xi} + \varepsilon$. Clearly $x_{\varepsilon} \in \mathcal{B}_{+}$ and $\|x_{\varepsilon}\| \leq \varepsilon$

The inequality

 $T_{\xi} x_{\xi}(t) \leq Trace x_{\xi}(t) = f(t) \text{ for every } \delta > 0,$ together with one of the remarks at the beginning of the proof of Lemma 3 shows that $x_{\xi} \in \mathcal{J}_{+}(\mathcal{C})$

For the remaining assertion, note that

for all $t \in V_c$ and where $o=\max$ (sup $\| \varphi(y_1(t)) \|$) and $S = \inf\{(\mathcal{Q}_{s_1} \otimes \varphi \varphi \mathcal{R}), \quad 1 \le t \le n \text{ to } V_c\}$ The y_1' is several times if necessary, we may assume that $2 \in C \in S$

$$rank \ E(L\varepsilon, \infty); x(t)) \le \sum_{i=1}^{n} rank \ E(L\delta, \infty); \chi_i(t)) + \frac{N_i}{N_i}$$

and since the rank of a projection is an integer, this gives:

rank $E([\varepsilon,\infty),x(t)) \le \sum_{i=1}^n \operatorname{rank} E([\delta,\infty),\chi(\mathcal{E})) \not= c t \in V_{\mathcal{E}}$. Using Proposition 2, we conclude that $x \in \mathcal{F}$. Using Lemma 3, we have

and hence the desired conclusion.

To prove that $\mathbb{C} \subset \mathbb{C} (f, e, f)$ that if f is a prove that f is an extraction onto f is a contraction of f is a

to be the element

$$I \omega / \Omega = I \text{ for } e^{\pm i\theta} = I \text{ or } e^{\pm i\theta} = X \times \{\omega\}$$

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Our assertion will follow from property (*) once we have shown that $x_{\mathcal{E}} \in \mathcal{J}(\mathcal{E})$ and that there is $Y \in \mathcal{F}$ such that $\mathcal{E} \in \mathcal{E}_{\varphi} \times_{\mathcal{E}} + \mathcal{E}_{\varphi}$. Clearly $x_{\mathcal{E}} \in \mathcal{S}_{\varphi}$ and $y_{\mathcal{E}} \times_{\mathcal{E}} \mathcal{E}_{\varphi}$.

The inequality

together with one of the remarks at the Deginning of the proof of Lemma 3 shows that $x_k \in \mathcal{F}_k(\mathcal{E})$

Nor the remaining essertion, note that

$$\varepsilon + 2T_{\psi_{\mathcal{E}/2}} \times_{\varepsilon} (t) \gg f(t)$$

for all $t \in \Omega$

Q.E.D.

Remark 5. Let M $\subset \mathcal{B}_+$, and let \mathcal{J} be the closed two-sided ideal of \mathcal{B} generated by M \cup J. Then $\mathcal{C}(\mathcal{J})$ is the smallest cone with property (*) containing $\{T_{\gamma_{\mathcal{E}}}y \mid \mathcal{E}>0, y \in M\}$

Indeed, the smallest cone with property (*) containing the above set is clearly contained in $\mathcal{C}(\mathcal{J})$ and on the other hand the ideal corresponding to this cone contains M and hence \mathcal{J} , so that this cone must coincide with $\mathcal{C}(\mathcal{J})$.

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Q.E.D.

namers 5. Dat M C St. and let & be the closed two-

Indeed, the smallest cone with property (\star) contains the above set is clearly contained in $\mathcal{E}(\zeta)$ and on the other hand the ideal corresponding to this cone contains M and hence ζ so that this cone must coincide with $\mathcal{E}(\zeta)$.

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