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One of the classes of extensions which are more general than those of the ideal of compact operators K(H), for which we have the Brown-Douglas-Fillmore theory ([2] , [3]), are the extensions of $C_o(X) \otimes K(H)$ where X is locally compact. A class of such extensions, the homogeneous ones, for X compact have been studied in ([8] , [10]) (see [7] for a more general theory). The opposite case appears to be that of the singular extensions, i.e. those for which the extension is "localised" in a certain sense at infinity, in the Alexandrov compactification of X. Such extensions have been considered by Delaroche ([4]) and in connection with the C^*-algebra of the Heisenberg group, by several authors ([9] , [7], [11]). The structure of such extensions appears to be rather mysterious. This is due in part to the complicated structure of the "Calkin algebra" corresponding to a singular extension problem. This "Calkin algebra" is far from being simple and the aim of the present note is to classify its clossed two-sided ideals.

We begin with the notations.

Throughout, H will denote a complex separable infinite-dimensional Hilbert space and L(H), K(H) will denote the set of all bounded operators on H and respectively the ideal of compact operators on H.

Instead of a locally compact space X, it will be more convenient to consider a pointed compact space (Ω , ω), where X corresponds to $\Omega \setminus \{\omega\}$. We shall assume that Ω is metrizable

and finite-dimensional. By $\mathcal{B}(\Omega, \omega, H)$ (or simply $\mathcal{B}$) we shall denote the C^*-algebra of bounded norm-continuous functions $f: \Omega \setminus \{\omega\} \longrightarrow K(H)$ and by $\mathcal{J}(\Omega, \omega, H)$ (or simply $\mathcal{J}$) the C^*-algebra of norm-continuous functions $f: \Omega \longrightarrow K(H)$ such that $f(\omega)=0$. Clearly, the restriction to $\Omega \setminus \{\omega\}$ gives an isometric injection of $\mathcal{J}$ into $\mathcal{B}$, which we shall use to identify $\mathcal{J}$ with a sub-algebra of $\mathcal{B}$, which is in fact a closed two-sided ideal of $\mathcal{B}$. The singular extensions will correspond to $*$-monomorphisms into $\mathcal{B}/\mathcal{J}$, which is what might be called the "Calkin algebra" for the singular extensions of $C_o(\Omega \setminus \{\omega\}) \otimes \mathcal{K}(H) \simeq \simeq \mathcal{J}$. The problem we consider is the classification of the closed two-sided ideals of $\mathcal{B}/\mathcal{J}$ or equivalently the classification of the closed two-sided ideals of $\mathcal{B}$ containing $\mathcal{J}$.

For the sake of completeness we shall record as Lemma 1 a most likely well-known consequence of the finite-dimensionality of Ω.

Lemma 1. *Let Ω be a compact metrizable finite-dimensional space. Then there is a number N, depending only on the dimension of Ω such that for every open covering $\mathcal{U} = (U_j)_{j\in J}$ of $\Omega \setminus \{\omega\}$ there is a refinement $\mathcal{V} = (V_i)_{i\in I}$, which is a covering by open sets, with the following property:*

there is a partition $I = I_1 \cup \dots \cup I_N$ such that $V_p \cap V_q = \emptyset$ whenever $p \neq q$ belong to the same I_k.

For the next proposition we shall introduce some notations. By $E(\sigma; a)$ we shall denote for a positive operator $a \in L(H)$, the spectral projection corresponding to the Borel set $\sigma \in \mathbb{R}$. Another notation we shall use, is A_+ for the positive part of a C^*-algebra A.

Proposition 2. *Let $M \subset \mathcal{B}_+ = \mathcal{B}_+(\Omega, \omega, H)$ and $x \in \mathcal{B}_+$. Then, the following conditions are equivalent:*

(i) *x is in the closed two-sided ideal of $\mathcal{B}$ generated*

by $M \cup \mathcal{J}$.

(ii) for every $\varepsilon > 0$ there are $\delta > 0, n \in \mathbb{N}, y_1, \ldots, y_n \in M$ and $V \subset \Omega$ a neighborhood of ω such that

$$\operatorname{rank} E([\varepsilon, \infty); x(t)) \leq \sum_{j=1}^{n} \operatorname{rank} E([\delta, \infty); y_j(t))$$

for all $t \in V \setminus \{\omega\}$.

Proof. We shall use in the proof the following fact. Let $\tilde{\mathcal{B}} = \mathcal{B} + \mathbb{C} e$ denote the C^*-algebra obtained by adjoining a unit to $\mathcal{B}$, then (i) is equivalent with:

for every $\alpha > 0$ there are $n \in \mathbb{N}$, $y_1, \ldots, y_n \in M$, $b_1, \ldots, b_n \in \mathcal{B}$, $d \in \mathcal{J}_+$ such that

$$\alpha e + \sum_{j=1}^{n} b_j y_j b_j^* + d \geqslant x.$$

In view of the definition of $\mathcal{J}$ this gives that (i) is also equivalent to:

(i') for every $\alpha > 0$ there are $n \in \mathbb{N}, y_1, \ldots, y_n \in M, b_1, \ldots, b_n \in \mathcal{B}$, and $V \subset \Omega$ a neighborhood of ω such that

$$\alpha I_H + \sum_{j=1}^{n} b_j(t) y_j(t) b_j^*(t) \geqslant x(t)$$

for all $t \in V \setminus \{\omega\}$

With these preparations we can now pass to the proof of the proposition.

(i) $\Rightarrow$ (ii)

This will follow from (i) $\Leftrightarrow$ (i') and some remarks based on consequences of the mini-max principle.

Thus, using results in ch. II, §2 of [6] we have for $\gamma > 0$ the inequality:

$$\operatorname{rank} E([\gamma,\infty), b_j(t)y_j(t)b_j^*(t)) \leq \operatorname{rank} E([\tfrac{\gamma}{\|b_j\|^2},\infty), y_j(t))$$

Further, using Corollary 2.2 in § 2 of ch.II of [6] we have:

$$\operatorname{rank} E([\gamma,\infty), \sum_{j=1}^{n} b_j(t)y_j(t)b_j^*(t)) \leq$$

$$\leq \sum_{j=1}^{n} \operatorname{rank} E([\tfrac{\gamma}{n},\infty), b_j(t)y_j(t)b_j^*(t)) \leq$$

$$\leq \sum_{j=1}^{n} \operatorname{rank} E([\tfrac{\gamma}{n\|b_j\|^2},\infty), y_j(t)).$$

Assume now that

$$\alpha I_H + \sum_{j=1} b_j(t)y_j(t)b_j^*(t) \geqslant x(t),$$

then from the mini-max principle, it follows that for $\gamma > \alpha > 0$ we have

$$\operatorname{rank} E([\gamma-\alpha,\infty); \sum_{j=1}^{n} b_j(t)y_j(t)b_j^*(t)) \geqslant$$

$$\geqslant \operatorname{rank} E([\gamma,\infty); x(t)).$$

This, together with our previous remarks, gives:

$\operatorname{rank} E([\gamma,\infty); x(t)) \leq \sum_{j=1}^{n} \operatorname{rank} E([\frac{\gamma-\alpha}{\beta},\infty); y_j(t))$ where

$$\beta = n(\max_{1\leq j\leq n} \|b_j\|^2 + 1).$$

Thus, taking $\alpha = \varepsilon/2$, $\gamma = \varepsilon$ we see that (i') implies (ii) with $\delta = \varepsilon/2\beta$

(ii) => (i)

Let (i") denote condition (i') with M replaced by the closed two-sided ideal generated by M. It will be clearly sufficient

to prove that (ii) => (i"). Thus, assume (ii) holds. Then for every $t \in V \setminus \{\omega\}$ we can find $b_t^{(h)} \in \mathcal{B}$ $(1 \leq h \leq n)$ such that

$$\sum_{h=1}^{n} b_t^{(h)}(t) y_h(t) b_t^{(h)*}(t) + \varepsilon I_H \geq x(t),$$

$$\| b_t^{(h)} \| \leq \left(\frac{\|x\|}{\delta} \right)^{1/2}$$

But then, for every $t \in V \setminus \{\omega\}$ there is an open set $U_t \subset \Omega \setminus \{\omega\}$, $t \in U_t$ such that

$$\sum_{h=1}^{n} b_t^{(h)}(s) y_h(s) b_t^{(h)*}(s) + 2\varepsilon I_H \geq x(s)$$

for all $s \in U_t$

Assuming V is compact (which is no loss of generality) we can apply Lemma 1 and find a covering $(V_j)_{j \in J_1} \cup \ldots \cup (V_j)_{j \in J_N}$ by open subsets of $V \setminus \{\omega\}$. (in the relative topology of $V \setminus \{\omega\}$) such that $V_j \subset U_{t(j)}$ and $V_p \cap V_q = \emptyset$ whenever $p \neq q$ belong to the same set J_k. Let further $(g_j)_{j \in J_1 \cup \ldots \cup J_k}$ be a partition of unity subordined to this covering of $V \setminus \{\omega\}$. Then we may define bounded continuous K(H)-valued functions $c_k^{(h)}$ on $V \setminus \{\omega\}$ $(1 \leq k \leq N, 1 \leq h \leq n)$ by

$$c_k^{(h)}(s) = \sum_{j \in J_k} \sqrt{g_j(s)}\, b_{t(j)}^{(h)}(s),$$

We have:

$$\sum_{h=1}^{n} \left(\sum_{k=1}^{N} c_k^{(h)}(s) y_h(s) c_k^{(h)*}(s) \right) + 2\varepsilon I_H =$$

$$= \sum_{h=1}^{n} \sum_{k=1}^{N} \Big(\sum_{j\in J_k} g_j(s) b_{t(j)}^{(h)}(s) y_h(s) b_{t(j)}^{(h)*}(s)\Big) + 2\varepsilon I_H =$$

$$= \sum_{k=1}^{n} \sum_{j\in J_k} g_j(s)\Big(\sum_{h=1}^{n} b_{t(j)}^{(h)}(s) y_h(s) b_{t(j)}^{(h)*}(s) + 2\varepsilon I_H\Big) \geq$$

$$\geq \sum_{k=1}^{n} \sum_{j\in J_k} g_j(s) x(s) = x(s)$$

Remarking that the $K(H)$-valued functions $c_k^{(h)}$ can be prolonged from $V \setminus \{\omega\}$ to all of $\Omega\setminus\{\omega\}$ we see that we have proved that (ii) => (i").

Q.E.D.

We turn now to the classification of the closed two-sided ideals of $\mathcal{B}$ which contain $\mathcal{J}$. This will be achieved by exibiting a bijection between these ideals and the class of cones $\mathcal{C}$ of positive continuous functions on $\Omega\setminus\{\omega\}$, satisfying the following "completeness" property:

(*) If $f: \Omega\setminus\{\omega\} \longrightarrow [0,\infty)$ is a continuous function such that for every $\varepsilon > 0$ there exists a neighborhood V_ε of ω and a function $g_\varepsilon \in \mathcal{C}$ such that

$$f(t) \leq g_\varepsilon(t) + \varepsilon \quad \text{for all } t \in V_\varepsilon$$

then f belongs to $\mathcal{C}$.

We pass now to the construction of the correspondence between ideals and cones.

By $\mathcal{F}$ we shall denote the set of continuous functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$ such that $\operatorname{supp}\varphi \subset (0,\infty)$. Let further, for $\varepsilon > 0$, ψ_ε stand for the following particular function in $\mathcal{F}$
$\psi_\varepsilon(t) = \max(t-\varepsilon, 0)$.

For $x \in \mathcal{B}_+$ and $\varphi \in \mathcal{F}$ we get a continuous function $T_\varphi x$ on $\Omega\setminus\{\omega\}$ defined by $T_\varphi x(t) = \operatorname{Trace}\varphi(x(t))$.

Let us note the following properties of the functions $T_\varphi x$:

1) $T_\varphi x(t) \leq \|\varphi(x(t))\| \, .\mathrm{rank}\, E([\delta,\infty);x(t))$ where $\delta > 0$ is the greatest lower bound of supp φ.

2) $\varepsilon\, .\mathrm{rank}\, E([2\varepsilon,\infty);x(t)) \leq T_{\psi_\varepsilon} x(t)$

3) Assume $\varphi \in \mathcal{F}$ is an increasing function, then for $x,y \in \mathcal{B}_+$, $x \leq y$ we have $T_\varphi x(t) \leq T_\varphi y(t)$ for all $t \in \Omega \setminus \{\omega\}$.

The last property is a consequence of the mini-max principle, which shows that the n-th eigenvalue of $y(t)$ is greater than the n-th eigenvalue of $x(t)$ (eigenvalues being listed in decreasing order, multiple eigenvalues repeated), so that the same is true for the n-th eigenvalues of $\varphi(y(t))$ and $\varphi(x(t))$.

For a closed two-sided ideal $\mathcal{J}$ containing $\mathcal{I}$ we shall denote by $\mathcal{C}(\mathcal{J})$ the smallest cone of continuous positive functions satisfying property (*) containing all the functions $T_\varphi x$, where x runs over $\mathcal{J}_+$ and φ runs over $\mathcal{F}$.

Conversely, for a cone $\mathcal{C}$ satisfying (*), let $\mathcal{J}_+(\mathcal{C})$ be the set of all positive elements $x \in \mathcal{B}_+$ such that $T_\varphi(x) \in \mathcal{C}$ for all $\varphi \in \mathcal{F}$. $\mathcal{J}(\mathcal{C})$ will be the set of all elements $x \in \mathcal{B}$ such that $|x| = (x^* x)^{1/2} \in \mathcal{J}_+(\mathcal{C})$.

<u>Lemma</u> 3. <u>$\mathcal{J}(\mathcal{C})$ is a closed two-sided ideal of $\mathcal{B}$, which contains $\mathcal{I}$. Moreover $(\mathcal{J}(\mathcal{C}))_+ = \mathcal{J}_+(\mathcal{C})$</u>

<u>Proof</u>. Remark first that $x \in \mathcal{B}_+$ is in $\mathcal{J}_+(\mathcal{C})$ if $T_{\psi_\varepsilon} x \in \mathcal{C}$ for all $\varepsilon > 0$. This follows from the fact that every $\varphi \in \mathcal{F}$ is dominated by a function of the form $\alpha \psi_\varepsilon$ on the spectrum of x and from property (*).

Also, if $f: [0,\infty) \longrightarrow [0,\infty)$ is a continuous function such that $f(0)=0$, then $\varphi \circ f \in \mathcal{F}$ for every $\varphi \in \mathcal{F}$. Hence, if

$x \in \mathcal{J}_+(\mathcal{C})$ then also $f(x) \in \mathcal{J}_+(\mathcal{C})$

In particular, for $x \in \mathcal{B}_+$ we have that $x \in \mathcal{J}_+(\mathcal{C})$ if and only if $x^2 \in \mathcal{J}_+(\mathcal{C})$.

We will first show that $\mathcal{J}_+(\mathcal{C})$ is a closed convex hereditary cone in $\mathcal{B}$. To this end we apply the remarks preceding the Lemma and Corollary 2.2 in § 2 of ch.II of [6], to get:

$$T_{\psi_\varepsilon}(x+y)(t) \leq \|x+y\| .\text{rank } E([\varepsilon,\infty);x(t)+y(t)) \leq$$

$$\leq \|x+y\|(\text{rank } E([\tfrac{\varepsilon}{2},\infty);x(t))+\text{rank } E([\tfrac{\varepsilon}{2},\infty);y(t))) \leq$$

$$\leq \frac{4}{\varepsilon}\|x+y\|(T_{\psi_{\varepsilon/4}}x(t)+T_{\psi_{\varepsilon/4}}y(t)), \quad x,y \in \mathcal{B}_+.$$

$$T_{\psi_\varepsilon}\lambda x = \lambda T_{\psi_{\varepsilon/\lambda}}x, \quad x \in \mathcal{B}_+, \quad \lambda > 0.$$

Also, if $0 \leq x \leq y$ and $y \in \mathcal{J}_+(\mathcal{C})$ then $T_{\psi_\varepsilon} x \leq T_{\psi_\varepsilon} y$ since ψ_ε is increasing. This together with the preceding remarks yields that $\mathcal{J}_+(\mathcal{C})$ is a convex hereditary cone.

To see that $\mathcal{J}_+(\mathcal{C})$ is also closed, let x be in the closure of $\mathcal{J}_+(\mathcal{C})$. Then for any $\varepsilon > 0$ we can find $y \in \mathcal{J}_+(\mathcal{C})$ such that

$$x \leq y+\frac{\varepsilon}{2}e$$

where e is the unit of $\tilde{\mathcal{B}}$. It follows that

$$T_{\psi_\varepsilon}x \leq T_{\psi_\varepsilon}(y+\frac{\varepsilon}{2}e)=T_{\psi_{\varepsilon/2}}y$$

and the remark at the beginning of the proof yields the desired conclusion.

Also by one of the remarks at the beginning of the proof we have that

$$\mathcal{J}(\mathcal{C}) = \{x \in \mathcal{B} \mid x^*x \in \mathcal{J}_+(\mathcal{B})\}$$

A standard argument shows now that $\mathcal{J}(\mathcal{C})$ is a closed

left ideal of $\mathcal{B}$,such that $(\mathcal{J}(\mathcal{C}))_+ = \mathcal{J}_+(\mathcal{C})$ .Moreover since $T_\varphi x^*x = T_\varphi xx^*$ it follows that $\mathcal{J}(\mathcal{C})$ is self-adjoint and hence a two-sided ideal.

Since for $x \in J_+$ and $\varphi \in \mathcal{F}$, $\varphi(x)$ is zero on some neighborhood of ω , property ($*$) implies that $J_+ \subset \mathcal{J}_+(\mathcal{C})$ and hence $J \subset \mathcal{J}(\mathcal{C})$.

Q.E.D.

Theorem 4.The correspondence

$$\mathcal{C} \longmapsto \mathcal{J}(\mathcal{C})$$

is a bijection between cones satisfying property ($*$) and closed two-sided ideals of $\mathcal{B}$ containing J .The inverse of this bijection is

$$\mathcal{J} \longmapsto \mathcal{C}(\mathcal{J}).$$

Proof.It will be sufficient to prove that

$$\mathcal{J} \supset \mathcal{J}(\mathcal{C}(\mathcal{J}))$$
$$\mathcal{C} \subset \mathcal{C}(\mathcal{J}(\mathcal{C}))$$

the opposite inclusions being obvious.

To prove the first inclusion, let $x \in \mathcal{J}_+(\mathcal{C}(\mathcal{J}))$ and $\varepsilon > 0$ be fixed.Since $T_{\psi_\varepsilon} x$ is in $\mathcal{C}(\mathcal{J})$ we can find a neighborhood V_ε of ω ,functions $\varphi_1, \ldots, \varphi_n \in \mathcal{F}$ and $y_1, \ldots, y_n$ elements of $\mathcal{J}_+$ such that

$$T_{\psi_\varepsilon} x(t) \leq \sum_{i=1}^{n} T_{\varphi_i} y_i(t) + \varepsilon/4 \text{ for all } t \in V_\varepsilon$$

The remarks preceding Lemma 3 imply that

$$\varepsilon/2 \operatorname{rank} E([\varepsilon,\infty); x(t)) \leq c \sum_{i=1}^{n} \operatorname{rank} E([\delta,\infty); y_i(t)) + \varepsilon/4 .$$

for all $t \in V_\varepsilon$ and where $c = \max_{1 \le i \le n} (\sup_{t \in V_\varepsilon} \| \varphi(y_i(t)) \|)$ and $\delta = \inf(\bigcup_{1 \le i \le n} \operatorname{supp} \psi_i)$. Repeating the y_i's several times if necessary, we may assume that $2c < \varepsilon$ so that :

$$\operatorname{rank} E([\varepsilon, \infty); x(t)) \le \sum_{i=1}^{n} \operatorname{rank} E([\delta, \infty); y_i(t)) + 1/2$$

and since the rank of a projection is an integer, this gives:

$$\operatorname{rank} E([\varepsilon, \infty); x(t)) \le \sum_{i=1}^{n} \operatorname{rank} E([\delta, \infty); y_i(t)) \quad \text{for } t \in V_\varepsilon .$$

Using Proposition 2, we conclude that $x \in \mathcal{J}$. Using Lemma 3, we have

$$(\mathcal{J}(\mathcal{C}(\mathcal{J})))_+ = \mathcal{J}_+(\mathcal{C}(\mathcal{J})) \subset \mathcal{J}$$

and hence the desired conclusion.

To prove that $\mathcal{C} \subset \mathcal{C}(\mathcal{J}(\mathcal{C}))$ fix $f \in \mathcal{C}$. Consider further $e_1, e_2, \dots$, an orthogonormal basis of H and let E_i denote the orthogonal projection onto $\mathbb{C}e_i$. For $\varepsilon > 0$ let $f_{n,\varepsilon} : \Omega \setminus \{\omega\} \longrightarrow [0, \infty)$ be the functions defined recurrently by $f_{0,\varepsilon} = 0$, $f_{n+1,\varepsilon} = \min (f - \sum_{k=0}^{n} f_{n,\varepsilon}, \varepsilon)$. Define now $x_\varepsilon \in \mathcal{B}$ to be the element

$$x_\varepsilon(t) = \sum_{n \ge 1} f_{n,\varepsilon}(t) E_n \quad \text{for } t \in \Omega \setminus \{\omega\}$$

and note that Trace $(x_\varepsilon(t)) = f(t)$.

Our assertion will follow from property (*) once we have shown that $x_\varepsilon \in \mathcal{J}(\mathcal{C})$ and that there is $\varphi \in \mathcal{F}$ such that $f \le 2T_\varphi x_\varepsilon + \varepsilon$. Clearly $x_\varepsilon \in \mathcal{B}_+$ and $\| x_\varepsilon \| \le \varepsilon$

The inequality

$$T_{\psi_\delta} x_\varepsilon(t) \le \text{Trace } x_\varepsilon(t) = f(t) \text{ for every } \delta > 0,$$

together with one of the remarks at the beginning of the proof of Lemma 3 shows that $x_\varepsilon \in \mathcal{J}_+(\mathcal{C})$

For the remaining assertion, note that

$$\varepsilon + 2T_{\psi_{\varepsilon/2}} x_\varepsilon(t) \geq f(t)$$

for all $t \in \Omega$

Q.E.D.

Remark 5. Let $M \subset \mathcal{B}_+$,and let $\mathcal{J}$ be the closed two-sided ideal of $\mathcal{B}$ generated by $M \cup J$ .Then $\mathcal{C}(\mathcal{J})$ is the smallest cone with property (*) containing $\{T_{\psi_\varepsilon} y \mid \varepsilon > 0,\ y \in M\}$

Indeed, the smallest cone with property (*) containing the above set is clearly contained in $\mathcal{C}(\mathcal{J})$ and on the other hand the ideal corresponding to this cone contains M and hence $\mathcal{J}$, so that this cone must coincide with $\mathcal{C}(\mathcal{J})$.

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