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REMARKS ON HILBERT-SCHMIDT PERTURBATIONS OF ALMOST - NORMAL OPERATORS

by Dan Voiculescu

We have shown in a recent note [15], that quasi-triangularity relative to the Hilbert-Schmidt class can be used to give a new proof and an extension of the Berger-Shaw inequality [3]. This suggests that a study of almost normal operators modulo Hilbert-Schmidt perturbations may be interesting. When at least one of two almost normal operators, which differ by a Hilbert-Schmidt operator, has finite multicyclicity, we prove below that their Helton-Howe measures [8], or equivalently their Pincus-G-functions [5], are equal. On the other hand we have also other invariants: those of quasitriangularity and quasidiagonality relative to the Hilbert-Schmidt class. This leads to a question asked by R.G.Douglas in connection with [15], namely, whether the two kinds of invariants are related.

The present paper contains a few remarks in this direction. We show that the quasitriangularity of an almost-normal operator, relative to the Hilbert-Schmidt class is in a simple relation with that of its adjoint and with the quasidiagonality.

We prove that if an almost normal operator is quasi-triangular with respect to the Hilbert-Schmidt class then its Helton-Howe measure is ≤ 0.

For certain subnormal operators we obtain complete results: their adjoint is quasitriangular relative to the Hilbert-Schmidt class and we get exact formular for the quasitriangularity and

quasidiagonality invariants.

We conclude the paper with some questions which seem natural in connection with Hilbert-Schmidt perturbations of almost normal operators.

Throughout H will denote a compley separable infinite-dimensional Hilbert space. By $\mathcal{L}(H)$, $\mathcal{C}_p(H)$, $\mathcal{R}_1^+(H)$, $\mathcal{P}(H)$ (or simply $\mathcal{L}$, $\mathcal{C}_p$, $\mathcal{R}_1^+$, $\mathcal{P}$) we shall denote the bounded operators on H, the Schatten-von Neumann p-class, the finite rank positive contractions on H and respectively the finite rank orthogonal projections.

The norms on $\mathcal{L}(H)$ and $\mathcal{C}_p(H)$ will be denoted by $\| \ \|$ and respectively $| \ |_p$.

The class of almost-normal operators on H, i.e. the class of operators $T \in \mathcal{L}(H)$ such that $[T^*,T] \in \mathcal{C}_1$, will be denoted by $\mathcal{AN}(H)$ (or simply $\mathcal{AN}$). For $T \in \mathcal{AN}(H)$ we shall denote its Helton-Howe measure by P_T and its Pincus G-function by G_T ([8] [5] [6]). We recall, that it has been shown by J.D.Pincus that P_T is equal $\frac{1}{2\pi} G_T d\lambda$, where $d\lambda$ is Lebesgue measure on R^2.

Quasitriangular operators and quasidiagonal operators have been introduced by P.R.Halmos [9] and the corresponding notions relatve the other norm-ideals than the compacts have been considered in [12]

The analogs of Apostol's moduli of quasitriangularity and quasidiagonality, relative to a Schatten-von Neuman class ([1] [12]) are:

$$q_p(T) = \liminf_{P \in \mathcal{P}} |(I-P)TP|_p$$

$$qd_p(T) = \liminf_{P \in \mathcal{P}} |[P,T]|_p$$

where the lim infs are with respect to the natural order on $\mathcal{P}$.

We shall also consider the number

$$k_p(T)=\liminf_{A\in \mathcal{R}_1^+} |[A,T]|_p$$

from [13], where again the liminf is with respect to the natural order on $\mathcal{R}_1^+$. All these numbers are invariant with respect to $\mathcal{C}_p$-perturbations, i.e. $q_p(T+X)=q_p(T)$, $qd_p(T+X)=qd_p(T)$, $k_p(T+X)=k_p(T)$ for $X\in\mathcal{C}_p$.

Concerning q_p we recall the following inequality from [15]

$$q_p(T)\leq (m_T)^{1/p}\|T\|$$

where m_T denotes the multicyclicity of T.

For the class $\mathcal{AN}$ it seems that the class of perturbations which gives interesting results is $\mathcal{C}_2$. Many of the results will be derived in fact from the following easily checked relation, which was implicit in [15]

$$(*)\qquad |(I-P)TP|_2^2=\mathrm{Tr}(P[T^*,T]P)+|(I-P)T^*P|_2^2$$

where $T\in\mathcal{L}$, $P\in\mathcal{P}$

Proposition 1. Let $T\in\mathcal{AN}$. Then the following equalities hold:

$$(q_2(T))^2=(q_2(T^*))^2+\mathrm{Tr}\,[T^*,T]$$

$$(q_2(T))^2+(q_2(T^*))^2=(qd_2(T))^2$$

Proof. Let $P_n\uparrow I$, $P_n\in\mathcal{P}$ be such that

$$\lim_{n\to\infty}|(I-P_n)TP_n|_2=q_2(T)$$

By Proposition 1.1 of [12], we have

$\liminf_{n\to\infty}|(I-P_n)T^*P_n|\geq q_2(T)$ and hence using relation (*) and the fact that $[T^*,T]\in\mathcal{C}_1$ we infer:

$$(q_2(T))^2 \geq \mathrm{Tr}\,[T^*,T] + (q_2(T^*))^2$$

Comparing this inequality with the inequality which is obtained by replacing T with T we get

$$(q_2(T))^2 = \mathrm{Tr}\,[T^*,T] + (q_2(T^*))^2.$$

This implies also that $\liminf_{n \to \infty} |(I-P_n)T^*P_n|_2 = q_2(T^*)$.

But this gives then:

$$(qd_2(T))^2 \leq \liminf_{n \to \infty} |[P_n,T]|_2^2 =$$

$$= \liminf_{n \to \infty} (|(I-P_n)TP_n|_2^2 + |(I-P_n)T^* P_n|_2^2) =$$

$$= (q_2(T))^2 + (q_2(T^*))^2$$

where we used the easily checked relation $|[P,T]|_2^2 = |(I-P)TP|_2^2 + |(I-P)T^*P|_2^2$ for $P \in \mathcal{P}$, $T \in \mathcal{L}$. For the reverse inequality, let $Q_n \uparrow I$, $Q_n \in \mathcal{P}$ be such that $\lim_{n \to \infty} |[Q_n,T]|_2 = qd_2(T)$. Then we have:

$$(qd_2(T))^2 = \lim_{n \to \infty} (|(I-Q_n)TQ_n|_2^2 + |(I-Q_n)T^*Q_n|_2^2) \geq$$

$$\geq \liminf_{n \to \infty} |(I-Q_n)TQ_n|_2^2 + \liminf_{n \to \infty} |(I-Q_n)T^*Q_n|_2^2 \geq$$

$$\geq (q_2(T))^2 + (q_2(T^*))^2.$$

Q.E.D.

<u>Corollary</u> 2. <u>Let</u> $T \in \mathcal{AN}$ <u>be such that</u> $q_2(T) < \infty$. <u>Then</u> $k_2(T)=0$. <u>In particular if</u> $m_T < \infty$ <u>then</u> $k_2(T)=0$.

<u>Proof</u>. If $q_2(T) < \infty$, then

$(qd_2(T))^2 = 2(q_2(T))^2 + \mathrm{Tr}\,[T^*,T] < \infty$ and hence $k_2(T) \leq qd_2(T) < \infty$.

This implies $k_2(T)=0$, since we have shown in [14] that k_p for $1 < p < \infty$ can take only the values 0 or ∞.

If $m_T < \infty$, when $q_2(T) \leq (m_T)^{1/2} \| T \|$

Q.E.D.

Since in the next proposition we shall use the hypothesis $q_2(T) < \infty$ for $T \in \mathcal{AN}$, let us underline that, as in the proof of the preceding corollary, it is a consequence of Proposition 1, that $q_2(T) < \infty \Longleftrightarrow q_2(T^*) < \infty \Longleftrightarrow qd_2(T) < \infty$ and $m_T < \infty \Rightarrow q_2(T) < \infty$.

Proposition 3. Let $S \in \mathcal{AN}(H), T \in \mathcal{AN}(H)$ be such that $q_2(S) < \infty$ and $S-T \in \mathcal{C}_2$. Then we have:

$$P_S = P_T$$

Proof. In view of the definition of the Helton-Howe measure it is easily seen that all we have to prove is that for any $*$-polynomial $F(X,X^*)$ the traces of the self-commutators of $F(S,S^*)$ and $F(T,T^*)$ are equal. Since $F(S,S^*), F(T,T^*)$ $F(S,S^*)-F(T,T^*) \in \mathcal{C}_2$ and $q_2(S) < \infty \Rightarrow qd_2(S) < \infty \Rightarrow qd_2(F(S,S^*)) < \infty$ we see that the proof of this fact is the same as the proof of the equality $Tr[T^*,T] = Tr[S^*,S]$.

Since $qd_2(S) < \infty$ there is a sequence $(P_n)_{n=1}^{\infty} \subset \mathcal{P}$ $P_n \uparrow I$ such that $|(I-P_n)SP_n|_2 \leq C$, $|(I-P_n)S^*P_n|_2 \leq C$ for some constant C independent of n.

Using relation $(*)$ and the fact that $S,T \in \mathcal{AN}$, we have

$$\left| Tr[T^*, T] - Tr[S^*, S] \right| =$$

$$= \lim_{n \to \infty} \Big| |(I-P_n)TP_n|_2^2 - |(I-P_n)SP_n|_2^2 -$$

$$- |(I-P_n)T^* P_n|_2^2 + |(I-P_n)S^*P_n|_2^2 \Big| \leq$$

$$\leq \limsup_{n \to \infty} \Big| |(I-P_n)TP_n|_2^2 - |(I-P_n)SP_n|_2^2 \Big| +$$

$$+ \limsup_{n \to \infty} \Big| |(I-P_n)T^*P_n|_2^2 - |(I-P_n)S^*P_n|_2^2 \Big| \leq$$

$$\leq 2C\,(\limsup_{n\to\infty} |(I-P_n)(S-T)P_n|_2 +$$

$$+ \limsup_{n} |(I-P_n)(S-T)^* P_n|_2) = 0$$

where the last equality follows from $S-T \in \mathcal{C}_2$ and $P_n \uparrow I$.

Q.E.D.

Proposition 4. Let $T \in \mathcal{AN}(H)$ be such that $q_2(T)=0$. Then

$$P_T \leq 0$$

Proof. Since $q_2(T)=0$, there is a sequence $(P_n)_1^\infty \subset \mathcal{P}$ $P_n \uparrow I$, such that $\lim_{n\to\infty} |(I-P_n)TP_n|_2=0$.

Consider $T_n=(I-P_n)T(I-P_n)$, $X_n=(I-P_n)TP_n$, $Y_n=P_nT(I-P_n)$.

We have

$$[T_n^*,T_n] =(I-P_n)[T^*,T]\,(I-P_n)+X_nX_n^*-Y_n^*Y_n \,.$$

Thus for $A_n=(I-P_n)\,[T^*,T]\,(I-P_n)+X_nX_n^*$ and $B_n=Y_n^*Y_n$ we have

$$A_n \in \mathcal{C}_1 \,, B_n \in \mathcal{C}_1 \,, B_n \geq 0,$$

$$\lim_{n\to\infty} |A_n|_1 =0$$

$$[T_n^*,T_n] =A_n-B_n \,.$$

The remaining part of the proof will be similar to an argument used in the proof of a theorem of C.A.Berger (see [6] page 112).

Let r,s be real-valued polynomials on R.

Denoting by C_n and D_n the hermitian and respectively the antihermitian part of T_n ,consider:

$F_n=r(s(C_n)D_ns(C_n))C_nr(s(C_n)D_ns(C_n))+is(C_n)D_ns(C_n)$.

Then we have:

$$[F_n^*, F_n] = R_nS_n [T_n^*, T_n] S_nR_n$$

where

$$S_n=s(C_n), R_n=r(s(C_n)D_ns(C_n)).$$

Since $T-T_n \in \mathcal{C}_1$, we have $P_{T_n}=P_T$ and using the definition of the Helton-Howe-measure, we have:

$$\mathrm{Tr}[F_n^*, F_n] = 2\iint_{\mathbb{R}} r^2(s^2(x)y)s^2(x)dP_T$$

On the other hand

$$\lim_{n\to\infty} \mathrm{Tr}[F_n^*, F_n] = \lim_{n\to\infty} (\mathrm{Tr}(R_nS_nA_nS_nR_n)-\mathrm{Tr}(R_nS_nB_nS_nR_n)) \leq$$

$$\leq \lim_{n\to\infty} (\|R_nS_n\|^2 |A_n|_1)=0$$

Hence we have proved that

$$\iint_{\mathbb{R}^2} r^2(s^2(x)y)s^2(x)dP_T \leq 0.$$

Since r,s are arbitrary real polynomials, this gives $P_T \leq 0$.

Q.E.D.

<u>Proposition 5.</u> <u>Let</u> $T \in \mathcal{AN}$ <u>be subnormal and assume the spectrum of its minimal normal dilation is contained in the right essential spectrum of</u> T. <u>Then we have:</u>

$$q_2(T^*) = 0$$
$$(q_2(T))^2 = (qd_2(T))^2 = Tr[T^*, T]$$

<u>Proof</u>. In view of Proposition 1., it will be sufficient to prove that $q_2(T^*)=0$.

Let

$$N = \begin{pmatrix} T & X \\ 0 & S \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}')$$

be the minimal normal dilation of T. Since $T \in \mathcal{AN}$ we have $X \in \mathcal{C}_2$. Because of the assumption $\sigma(N^*) \subset \sigma_{le}(T^*)$ we can apply Proposition 3.3 of [12], which gives:

$$q_2(T^*) \leq q_2(T^* \oplus N^* \oplus N^* \oplus \ldots) \leq q_2(T^*) + q_2(N^* \oplus N^* \oplus \ldots).$$

In view of the fact that normal operators can be diagonalized modulo the Hilbert - Schmidt class [13], we have $q_2(T^*) = q_2(T^* \oplus N^* \oplus N^* \oplus \ldots)$.

For $T^* \oplus N^* \oplus \ldots$ acting on $\mathcal{H} \oplus (\mathcal{H} \oplus \mathcal{H}') \oplus (\mathcal{H} \oplus \mathcal{H}') \oplus \ldots$ the subspaces

$$K_n = \mathcal{H} \oplus \underbrace{(\mathcal{H} \oplus \mathcal{H}') \oplus \ldots \oplus (\mathcal{H} \oplus \mathcal{H}')}_{n\text{-times}} \oplus (0 \oplus \mathcal{H}') \oplus (0 \oplus 0) \oplus \ldots$$

are invariant subspaces. By Proposition 2.4 of [12] we have

$$q_2(T^* \oplus N^* \oplus N^* \oplus \dots) \leq \liminf_{n\to\infty} q_2((T^* \oplus N^* \oplus N^* \oplus \dots)|K_n) =$$

$$= \liminf_{n\to\infty} q_2(T^* \oplus S^* \oplus \underbrace{N^* \oplus \dots \oplus N^*}_{n\text{-times}}) = \liminf_{n\to\infty} q_2(\underbrace{N^* \oplus \dots \oplus N^*}_{(n+1)\text{-times}}) = 0$$

where we did use the fact that $T^* \oplus S^* - N^* \in \mathcal{C}_2$.

Q.E.D.

Proposition 6. Let

$$N = \begin{pmatrix} T & X \\ Y & S \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}')$$

be normal, and assume $X, Y \in \mathcal{C}_2$. Then $T \in AN$ and we have

$$q_2(T \oplus N \oplus N \oplus \dots) \leq |X|_2.$$

Proof. That $[T^*, T] \in \mathcal{C}_1$ follows immediately from

$$[T^*, T] = X X^* - Y^* Y.$$

Let $K_n = \mathcal{H} \oplus (\mathcal{H} \oplus \mathcal{H}') \oplus \dots \oplus (\mathcal{H} \oplus \mathcal{H}') \oplus (0 \oplus \mathcal{H}') \oplus (0 \oplus 0) \oplus \dots$

Then by an easy generalization of Proposition 2.4 of [12], we have $q_2(T \oplus N \oplus N \oplus \dots) \leq$

$$\leq \liminf_{n\to\infty} (q_2(P_{K_n}(T \oplus N \oplus \dots)|K_n) + |(I-P_{K_n})(T \oplus N \oplus \dots)P_{K_n}|_2)$$

$$= \liminf_{n\to\infty} (q_2(\underbrace{T \oplus S \oplus N \oplus \dots \oplus N}_{n\text{-times}}) + |X|_2) =$$

$$= \liminf_{n\to\infty} (q_2(\underbrace{N \oplus \dots \oplus N}_{(n+1)\text{-times}}) + |X|_2) = |X|_2 .$$

Q.E.D.

We think it is natural to conclude this paper by asking: to what extent do the Brown-Douglas-Fillmore theorem on essentially normal operators [4] and the Apostol-Foiaş-Voiculescu theorem on quasitriangular operators ([2], see also [7]) admit analogs for almost normal operators, relative to the Hilbert-Schmidt class ? The question concerning quasitriangularity is another way of stating R.G. Douglas' question mentioned in the introduction. Both theormes mentioned above have many particular cases, the analogs of which would be of interest, but which to state all would make a rather long list, so we think it may be useful to state the most far-fetched guesses about analogs of these theorems, with the hope that some small part may turn out to be true.

Guess A. *Let* $T_1 \in \mathcal{AN}(\mathcal{H})$, $T_2 \in \mathcal{AN}(\mathcal{H})$ *be such that* $P_{T_1} = P_{T_2}$ *Then there is a normal operator* $N \in \mathcal{L}(\mathcal{H})$ *and a unitary operator* $U \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ *such that*

$$U(T_1 \oplus N)U^* - T_2 \oplus N \in \mathcal{C}_2$$

Among the particular cases we mention:

A1) *If* $T \in \mathcal{AN}(\mathcal{H})$ *is such that* $P_T = 0$ *then there is a normal operator* N *such that:*

$$T \oplus N = \text{normal} + \text{Hilbert-Schmidt}.$$

A2) *If* $T \in \mathcal{AN}(\mathcal{H})$ *then there is* $S \in \mathcal{AN}(\mathcal{H})$ *such that*

$$T \oplus S = \text{normal} + \text{Hilbert-Schmidt}.$$

Guess B. *For* $T \in \mathcal{AN}$ *we have*

$$(q_2(T))^2 = 2 \iint_{R^2} dP_T^+$$

where P_T^+ is the postiive part of P_T . Among the particular cases we mention:

B1) T hyponormal, $T \in \mathcal{AN} \Rightarrow q_2(T^*)=0$.

B2) $P_T \leq 0 \Rightarrow q_2(T)=0$.

R E F E R E N C E S

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