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On quasi-analytic vectors for some classes of operators.

by

Ioana Cioranescu.

Abstract. We present in this work an extension of Nussbaum's quasi-analytic vector theorem to the setting of distribution semigroups in Banach spaces; we give also an application to the construction of some quasi-analytic classes of functions.

§ 1. Introduction. We begin by recalling some classical facts concearning the theory of quasi-analytic functions.

It is well known that two analytic functions, all of whose derivatives agree at a point, must coincide; further, a C^∞ function is analytic if and only if its successive derivatives satisfy growth conditions given by Cauchy's estimates. It was so quaite natural (Hadamard rised this question) to seek less restrictive growth conditions on the derivatives of a C^∞ function that imply the above mentioned property of „quasi-analyticity."

More precisely, a subclass C of C^∞ is called a quasi-analytic class if $f, g \in C$ and and $x_o \in R$ satisfy: $f^{(n)}(x_o) = g^{(n)}(x_o)$ for all $n=0,1,2,\ldots$, then $f=g$.

Let $\{M_n\}_{n\geq 0}$ be a sequence of nonnegative numbers and let us define the class

$$C\{M_n\} = \{\ f \in C^\infty;\quad \|f^{(n)}\|_\infty \leq L^n M_n \quad \text{for some } L>0\}.$$

The above problem recived the following answer:

Denjoy-Carleman Theorem. $C\{M_n\}$ is a quasi-analytic class if and only if the least nonincreasing majorant of the series $\sum_{n=0}^{\infty} M_n^{-1/n}$ diverges.

Let us remark that the condition $\sum_{n=0}^{\infty} M_n^{-1/n} = +\infty$ is equivalent to $\sum_{n=0}^{\infty} \frac{M_{n-1}}{M_n} = +\infty$.

Moreover, if the sequence $\{M_n\}$ is logarithmic convex, that is $M_n^2 \leq M_{n-1}M_{n+1}$, then the condition in the above theorem means that the series $\sum_{n=0}^{\infty} M_n^{-1/n}$ itself diverges.

Further it was natural to look for other quasi-analytic classes, associated to other norms as the $\|\cdot\|_\infty$ and to other differential operators besides the derivation. Along these lines many results were obtained by analytical means, on which we don't insist. We shall present here shortly an operator theoretic point of view which leads to interesting generalizations of the quasi-analytic classes, on one hand, and which provides some new facts concearning the spectral theory, on the other hand.

Let A be an unbounded operator with domain D(A) in the Banach space X.

A vector $x \in X$ is <u>a C^∞ vector for A</u> if x belongs to $D^\infty(A) = \bigcap_{n=0}^{\infty} D(A^n)$.

<u>An analytic</u>, respectively <u>semi-analytic vector for A</u> is a C^∞ vector $x \in X$ such that the series

$$\sum_{n=0}^{\infty} (t^n/n!)\,\|A^n x\|, \text{ respectively } \sum_{n=0}^{\infty} (t^n/(2n)!)\,\|A^n x\|$$

converges for some $t > 0$.

<u>A quasi-analytic</u>, respectively <u>Stieltjes vector for A</u> is a C^∞ vector $x \in X$ such that the least nonincreasing majorant of the series $\sum_{n=1}^{\infty} \|A^n x\|^{-1/n}$, respectively $\sum_{n=1}^{\infty} \|A^n x\|^{-1/2n}$ diverges.

Let us remark that if A is a symmetric operator on a Hilbert space, then the sequence $\{\|A^n x\|\}_{n \geq 0}$ is logarithmic convex. This is not true in general.

Analytic vectors were introduced by E.Nelson in 1959, [10], semi-analytic vectors by B.Simon in 1971, [13], quasi-analytic and Stieltjes vectors by A.E.Nussbaum in 1965, [11]. Stieltjes vectors were independently defined by D.Masson and W.Mc.Clary in [9], to whom the terminology is due, suggested by the Stieltjes moment problem.

Let us use the following notations :

$D^{a}(A)$ = the set of all analytic vectors of A ;

$D^{sa}(A)$ = the set of all semi-analytic vectors of A ;

$D^{qa}(A)$ = the set of all quasi-analytic vectors of A ;

$D^{s}(A)$ = the set of all Stieltjes vectors of A .

It is clear that

$$\begin{array}{ccc} D^{a}(A) & \subset & D^{qa}(A) \\ \cap & & \cap \\ D^{sa}(A) & \subset & D^{s}(A) \end{array}$$

We still remark that $D^{a}(A)$ and $D^{sa}(A)$ are linear subspaces of $D^{\infty}(A)$ but it is not true for $D^{qa}(A)$ and $D^{s}(A)$.

In [11] E.Nussbaum proved the following result:

<u>Theorem QA</u>. <u>Let</u> A <u>be a symmetric operator on a Hilbert space</u> H <u>If</u> A <u>has a total set of quasi-analytic vectors, then its closure is selfadjoint.</u>

This is a generalization of Nelson's similar result which requires that A has a dense set of analytic vectors [10].

It is interesting to remark that in [7] M.Hasegawa has obtained a very simple proof of Theorem QA, avoiding the classical Hamburger moment-problem method of Nussbaum by a direct use of Denjoy-Carleman's theorem on quasi-analytic functions. On the other hand we remark that the sufficiency of Denjoy-Carleman criterion for quasi-analyticity can be derived from Theorem QA as a special case; all these facts are completely disscussed by P.R.Chernoff in [3] and [4]. Moreover Hasegawa has generalized the Theorem QA to the context of contraction semigroups on a Hilbert space, giving the following

<u>Theorem D</u>. <u>Let</u> A <u>be a closed, densely defined dissipative operator on a Hilbert space</u> H. <u>If</u> A <u>has a total set of quasi-analytic vec-</u>

tors, then A is the generator of a contraction semi-group.

In [3] P.R.Chernoff extende this result to general strongly continuous semigroups on Banach spaces, giving the

Theorem (C_0). Let A be a closed operator on a Banach space X such that:

(i) A has a total set of quasi-analytic vectors;

(ii) A has an extension $\tilde{A}$ which generates a (C_0) semigroup.

Then $A = \tilde{A}$.

Using this theorem, the L^p version of Denjoy-Carleman theorem follows as a Corollary.

Concearning Stieltjes vectors we have the following correspondent to theorem QA :

Theorem S. Let A be a symmetric and semibounded operator operator on a Hilbert space H. If A has a total set of Stieltjes vectors, then its closure is selfadjoint.

This result was proved by Nussbaum [11] and rediscovered by Masson and Mc Clary [9] and applied to quantum field theory.

A simplified proof of Theorem S has been given by B.Simon [13]; he also proved the analogue of Nelson's theorem replacing in Theorem S the condition on A by the density of the set $D^{sa}(A)$.

In [4] P.R.Chernoff showed how via some operator-theoretic techniques, Theorem S can be derived from Theorem QA. He also generalized Theorem S from semibounded symmetric operators to the larger context of generators of bounded holomorphic semigroups :

Theorem HS. Let A be a closed operator on a Banach space X such that :

(i) A has a total set of Stieltjes vectors;

(ii) A has an extension $\tilde{A}$ which generates an holomorphic semigroup for $\operatorname{Re}\lambda > 0$ which is uniformly bounded for $\operatorname{Re}\lambda \geq 0$.

Then $A = \tilde{A}$.

The aim of this work is to extend Theorem (C_0) and HS to distribution,respectively holomorphic distribution semigroups on Banach spaces and to give,related to these facts,some examples of new quasi-analytic classes.

§ 2. Quasi-analytic vectors and generators of distribution semigroups.

We recall first some facts concearning distribution semigroups. Rather then Lions's definition for a distribution semigroup from [8] ,more usefull for our considerations is an equivalent definition due to T.Ushijima [15] .

Let A be a closed linear operator in a Banach space X such that $D^{\infty}(A)$ is dense in X.We put $\|x\|_n = \|A^n x\|$ for any $x \in D^{\infty}(A)$ and n=0,1,2,...;then the set $D^{\infty}(A)$ can be considered as a Fréchet space Y with the topology determined by the semi-norm system $\{\|x\|_n\}_{n\geq 0}$ .We denote by A^{∞} the restriction of A to Y.Clearly $A^{\infty} \in \mathcal{L}(Y)$.

We say that A is A^{∞}-well posed,or shortly $A \in (A^{\infty})$,if the operator A^{∞} is the generator of a semigroup $\{U_t\}_{t\geq 0}$ of class (C_0) in Y .

We remark two important facts:

i) for any $x \in Y$, $t \longrightarrow U_t x$ is in $C^{\infty}((0,+\infty);Y))$;

ii) $\{U_t\}_{t\geq 0}$ is locally equi-continuous in $\mathcal{L}(Y)$,that is for any $s>0$,there exist an integer n_s and a constant $c_s>0$ such that :

$$\|U_t x\| \leq c_s \|x\|_{n_s} \text{ , for any } t \in [0,s] \text{ and any } x \in Y.$$

By results of T.Ushijima [15] and J.Chazarain [2] ,we have :

Theorem. For the operator A the following two conditions are equivalent :

1) $A \in (A^{\infty})$ and the resolvent set $\rho(A) \neq \emptyset$;

2) the resolvent $R(\lambda;A)$ exists for λ belonging to the logarithmic region

$$\Lambda_{\alpha,\beta,\gamma} = \{ \lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda \geq \alpha \ln |\mathrm{Im}\,\lambda| + \beta,\ \mathrm{Re}\,\lambda \geq \gamma \}$$

defined by some constants $\alpha > 0$, $\beta, \gamma \in R$ and satisfies

$$\| R(\lambda;A) \| \leq C(1+|\lambda|)^k$$

with some constant $C > 0$ and an integer k.

An operator satisfying the equivalent conditions from the above theorem is called the generator of the distribution semigroup defined as follows

let $\mathcal{D} = \{ \varphi \in C^\infty;\ \mathrm{supp}\,\varphi \subset R \text{ compact} \}$ and let

$$\mathcal{E}(\varphi) = \frac{1}{2\pi i} \int_{\Gamma_{\alpha,\beta,\gamma}} \tilde{\varphi}(\lambda) R(\lambda;A)\, d\lambda ,$$

where $\Gamma_{\alpha,\beta,\gamma}$ is the boundary of $\Lambda_{\alpha,\beta,\gamma}$ and $\tilde{\varphi}(\lambda) = \int_{-\infty}^{+\infty} e^{\lambda t} \varphi(t)\, dt$. Then we have

(I) $\mathcal{E} \in \mathcal{L}(\mathcal{D};\mathcal{L}(X))$, $\mathrm{supp}\,\mathcal{E} \subset (0,+\infty)$, $\mathcal{E}(\varphi)x \in D(A)$ for any $x \in X$

and

(II) $\mathcal{E}' - A\mathcal{E} = \delta \otimes I_X$, $\mathcal{E}' - \mathcal{E}A = \delta \otimes I_{D(A)}$.

Reciprocally, if for the closed and densely defined operator A there is $\mathcal{E} \in \mathcal{L}(\mathcal{D};\mathcal{L}(X))$ with the above properties, then $A \in (A^\infty)$ and $\rho(A) \neq \emptyset$. Conditions (I) and (II) represents the initial definition of L.J.Lions [8] of a distribution semigroup.

Let us finally remark that we have [14], [15]:

$$\mathcal{E}(\varphi) = \int_{-\infty}^{+\infty} \varphi(t) U_t x\, dt \quad \text{for any } x \in Y \text{ and} \quad .$$

We can now give our main result

Theorem 1. Let A be a closed operator on a Banach space X such that

(i) A has a total set of quasi-analytic vectors ;

(ii) A has an extension $\tilde{A}$ which generates a distribution semigroup.

Then $A = \tilde{A}$.

Proof. We shall use similar arguments as in the proof of Theorem

(C_0).

Let $\lambda \in \rho(\tilde{A})$ which by hypothesis is non void. The operator $\lambda - \tilde{A}$ is surjective and is an extension of $\lambda - A$; A beeing closed, the range of $\lambda - A$ is closed, so that if we can prove that the range of $\lambda - A$ is dense in X, then it is clear that $\lambda - A = \lambda - \tilde{A}$, that is $A = \tilde{A}$.

Suppose now that $x^* \in X^*$ annihilates the range of $\lambda - A$; then

$$\langle x^*, Ax \rangle = \lambda \langle x^*, x \rangle \quad \text{for every } x \in D(A) .$$

Let $x \in D^{\infty}(A)$ a quasi-analytic vector for A and let $f(t) = \langle x, U_t x \rangle$, $t \geq 0$, where $\{U_t\}_{t \geq 0}$ is the locally equi-continuous semigroup generated by $\tilde{A}$ in $Y = D^{\infty}(\tilde{A})$.

It is clear that $f \in C^{\infty}$ for $t > 0$ and that

$$f^{(n)}(t) = \langle x^*, U_t A^n x \rangle , \quad t > 0$$

$$f^{(n)}(0_+) = \langle x^*, A^n x \rangle = \lambda^n \langle x^*, x \rangle$$

for every $n = 0, 1, 2, \ldots$.

We define now

$$g(t) = \begin{cases} f(t) - e^{\lambda t} \langle x^*, x \rangle & \text{for } t \geq 0 \\ 0 & \text{for } t \leq 0 \end{cases} .$$

It is easy to verify that $g \in C^{\infty}$ and that

$$g^{(n)}(t) = \langle x^*, U_t A^n x \rangle - \lambda^n e^{\lambda t} \langle x^*, x \rangle , \quad \text{for } t \geq 0 .$$

Using now the equi-continuity of the semigroup $\{U_t\}_{t \geq 0}$ we can find for every $s > 0$, a constant $c_s > 0$ and an integer n_s such that

$$\sup_{0 \leq t \leq s} | g^{(n)}(t) | \leq c_s (\| A^n x \|_{n_s} + |\lambda|^n) = c_s (\| A^{n+n_s} x \| + |\lambda|^n) .$$

Further, it is known that if $x \in X$ ia a quasi-analytic vector for A, then so is Ax (see [3]); hence if we denote $K_n = \sup_{0 \leq t \leq s} | g^{(n)}(t) |$ we get as in [3], Theorem 3.1., that $\sum_{n=0}^{\infty} K_n^{-1/n} = +\infty$.

Thus g belongs to a quasi-analytic class on every finite interval and as g vanishes for $t \leq 0$, from Denjoy-Carleman's theorem it follows that g is identically zero. Therefore

$$\langle x^*, U_t x \rangle = e^{\lambda t} \langle x^*, x \rangle , \quad \text{for } t \geq 0 \text{ and } x \in D^{qa}(\tilde{A}) .$$

But this implies by derivation

$$\langle x^*, Ax \rangle = \lambda \langle x^*, x \rangle .$$

As the above relation holds for x belonging to a total subset in X, it results that $x^* \in D(\tilde{A}^*)$ and that

$$\tilde{A}^* x^* = \lambda x^* .$$

But as $\lambda \in \mathcal{S}(\tilde{A}) = \mathcal{S}(\tilde{A}^*)$, we necessarilly get x = 0. Thus $\lambda - A$ has a dense range.

q.e.d.

We shall next generalize Theorem H S to holomorphic distribution semigroups.

Using the terminology of the beginning of this paragraph, we say that A is A_e^∞-well posed or that $A \in (A_e^\infty)$ if there is $\omega > 0$ such that the operator $A^\infty - \omega$ is the generator of an equi-continuous semigroup $\{U_t\}_{t \geq 0}$ in $\mathcal{L}(Y)$. If $A \in (A_e^\infty)$ and $\mathcal{S}(A) \neq \emptyset$, then we say that A is the generator of an exponential distribution semi-gruop, in the sense that the associated distribution semigroup has the property that $e^{-\xi t}\mathcal{E}$ is an $\mathcal{L}(X)$-valued tempered distribution, for $\xi > \omega$.

Exponential distribution semigroups were introduced by J.L.Lions in [8] and further studied by Fujiwara [5] (see also [6]) who considered also the notion of an holomorphic distribution semi-group, giving the spectral characterisation of its generator.

We say that $A \in (A_e^\infty)$ with $\mathcal{S}(A) \neq 0$ is the generator of an holomorphic distribution semigroup in the sector $S_\alpha = \{\lambda;\ |\arg| \leq \alpha;\ 0 < \alpha < \frac{\pi}{2}\}$ if A^∞ is the generator of an holomorphic semigroup $\{U_t\}_{t \geq 0} \subset \mathcal{L}(Y)$ in this sector .

In this case holds :

for every $\varepsilon > 0$, there is $\xi_\varepsilon > 0$, $c_\varepsilon > 0$ and an integer n_ε with

$$\| e^{-\xi \lambda} U_\lambda x \| \leq c_\varepsilon \| x \|_{n_\varepsilon}$$

for every $x \in D^{\infty}(A)$, $\xi > \xi_{\varepsilon}$, $\lambda \in \overline{S_{\alpha-\varepsilon}}$.

Thus it is natural to introduce the following notion :

we say that $A \in (A_e^{\infty})$ is a bounded holomorphic distribution semigroup for $\mathrm{Re}\,\lambda > 0$ if A generates an holomorphic distribution semigroup for $\mathrm{Re}\,\lambda > 0$ with the property :-

there is $c_0 > 0$ and $n_0 \in N$ such that

$$\| U_\lambda x \| \leq c_0 \| x \|_{n_0} \quad \text{for every } x \in D^{\infty}(A) \text{ and } \mathrm{Re}\,\lambda \geq 0 .$$

Then Theorem HS can be generalized in the following way :

Theorem 2. Let A be a closed operator on a Banach space X such that:

(i) A has a total set of Stieltjes vectors ;

(ii) A has an extension $\tilde{A}$ which generates an bounded holomorphic distribution semigroup for $\mathrm{Re}\,\lambda > 0$.

Then $A = \tilde{A}$.

The proof can be carried out adapting as in the proof of Theorem 1 the arguments used to prove Theorem HS;we don't insist on the details.

§ 3. On a quasi-analytic class of functions.

We shall show how using Theorem 1 from § 2,new classes of quasi-analytic functions can be derived.

In [1] two example of distribution semigroups,generalizing the translation and the Gauss-Weierstrass semigroups,are studied;we remark that these examples were suggested us by U.Mosco and we present here shortly the principal facts.

Let us define the following function spaces :

$$\Sigma = \left\{ \sigma : R \to R \, ; \, \sigma = \sigma_0 + \sigma_1 , \ \sigma_0 , \sigma^{-1} \in L^{\infty} , \ \sigma_1 \in L^2 , h(t) = \int_{t-\alpha}^{t} \sigma_1^2(s)\,ds \in L^1 \right.$$
for any $\alpha > 0$

$$\Sigma_0 = \left\{ \sigma \in \Sigma \, ; \, \exists t_\sigma > 0 \ \text{ with } \sigma^{-1} \tau_{t_\sigma} \sigma \notin L^{\infty} \right\}$$

where $\sigma^{-1} = 1/\sigma$, $L^{\infty} \equiv L^{\infty}(R)$, $L^1 \equiv L^1(R)$ and τ_t is the translation operator.

Here is an example of a function belonging to Σ_0:

$$\sigma(t) = \begin{cases} 1+\sqrt{n} & \text{for } t\in[n-n^{-3},n] \ , \ n\geq 2 \\ 1 & \text{for } t\in \mathbb{R}\setminus \bigcup_{n=2}^{\infty}[n-n^{-3},n] \end{cases} .$$

For a fixed $\sigma\in\Sigma$ let us define the operator A on L^2 by

$$D(A) = \{ f\in L^2 ; \sigma f \text{ is absolutely continuous and } \sigma^{-1} d/_{dt}(\sigma f)\in L^2\}$$

$$Af = -\sigma^{-1} d/_{dt}(\sigma f) \quad \text{for } f\in D(A)$$

and the operator $B = A^2$ by

$$D(B) = \left\{ f\in L^2; \ f, d/_{dt}(\sigma f) \text{ are absolutely continuous and } \sigma^{-1} d^2/_{dt^2}(\sigma f)\in L^2 \right\}$$

$$Bf = \sigma^{-1} d^2/_{dt^2}(\sigma f) \quad \text{for } f\in D(B) \ .$$

Let $\mathcal{Y}$ be the Heaviside function and for $\varphi\in\mathcal{D}$ denote by $\varphi_+ = \mathcal{Y}\varphi$. We shall also use the notation $\langle g,\varphi\rangle = \int_{-\infty}^{+\infty} g(t)\varphi(t)dt$ for $\varphi\in\mathcal{D}$ and $g\in L^1_{loc}(\mathbb{R})$. Then we have (see [1]) :

<u>For a fixed</u> $\sigma\in\Sigma$ <u>the maps</u>

$$\mathcal{E}_1(\varphi)f = \sigma^{-1}(\varphi_+ * \sigma f) \quad , \ \varphi\in\mathcal{D} , \ f\in L^2 ,$$

<u>and</u>

$$\mathcal{E}_2(\varphi)f = \frac{1}{2\sqrt{\pi}}\sigma^{-1}(\sigma f_u * \langle \frac{e^{-u^2/4\xi}}{\sqrt{\xi}} , \varphi_+ \rangle), \varphi\in\mathcal{D}, f\in L^2$$

<u>are exponential distributions semigroups on L^2 of generators A, respectively</u> B.

Moreover, if $\sigma\in\Sigma_0$, then $\mathcal{E}_1$ and $\mathcal{E}_2$ aren't usual semigroups of bounded operators on L^2 .

We still remark that for $\sigma \equiv 1$, the distribution semigroup $\mathcal{E}_1$ coincides with the semigroup of translations on L^2 and $\mathcal{E}_2$ coincides with the semigroup of Gauss-Weierstrass.

Denoting by $C[T_1;T_2]$ the commutator of two operators on L^2, it is easy to get that

$$A = -d/_{dt} + \sigma^{-1}C[\sigma ; -d/_{dt}] \ , \ B = d^2/_{d^2t} + \sigma^{-1}C[\sigma ; d^2/_{d^2t}]$$

In particular, if σ is a.e. differentiable, then

$$Af = -{}^{df}/_{dt} - {}^{d\log|\sigma|}/_{dt} \quad , \quad f \in L^2 \quad ,$$

that is A is the perturbation of the derivation operator by the multiplication with a certain function.

If σ is twice differentiable, then

$$Bf = d^2f/_{dt^2} + 2\, {}^{d\log|\sigma|}/_{dt} \cdot {}^{df}/_{dt} +$$

$$\left[{}^{d^2\log|\sigma|}/_{dt^2} + \left({}^{d\log|\sigma|}/_{dt} \right)^2 \right] f \quad ,$$

that is B is the perturbation of the operator $d^2/_{dt^2}$ by a differential operator of order one with variable coefficients.

We can now give the following

<u>Proposition</u> 1. <u>Let $\sigma \in \Sigma$ and f be a C^∞ function. Assume that</u>

$$\sigma^{-1}\, d^n/_{dt^n}(\sigma f) \in L^2 \quad , \quad \left[\sigma^{-1} d^n/_{dt^n}(\sigma f)\right](0) = 0, \; n=0,1,..$$

<u>and that for every integer k, the least nonincreasing majorant of the series $\sum_{n=0}^{\infty} \| \sigma^{-1} d^n/dt^n (\sigma \tau_k f)\|_{L^2}^{-1/n}$ diverges</u> .

<u>Then f is identically zero.</u>

<u>Proof</u>. Let A_0 be the operator defined by

$$D(A_0) = \left\{ f \in L^2; \; f \text{ is absolutely continuous}, \sigma^{-1} d/_{dt}(\sigma f) \in L^2 \right.$$
$$\left. \text{and } f(0) = 0 \right\}$$

$$A_0 f = \sigma^{-1} d/_{dt}(\sigma f) \quad , \; f \in D(A_0) \quad .$$

It is clear that A_0 is closed and can be extended to an operator A which is the generator of a distribution semigroup , such that $A_0 \neq A$.

The hypothesis says that f is a quasi-analytic vector for A_0. It is easy to prove that for every $\alpha \in \mathbb{R}$, $e^{i\alpha t}f$ is also a quasi-analytic vector for A_o, because of the estimate :

$$\| A_o^n e^{i\alpha t} f \|_{L^2} = \| \sigma^{-1} d^n/_{dt^n} (\sigma e^{i\alpha t} f) \|_{L^2} \leq (1+|\alpha|)^n \cdot$$

$$\cdot \max_{1 \leq i \leq n} \| \sigma^{-1} d^n/_{dt^n} (\sigma f) \|_{L^2} \cdot$$

Thus $L^2(\mathrm{supp} f)$ is contained in the closed span of $D^{qa}(A)$.

As for every integer k, $\tau_k f$ is also a quasi-analytic vector for A_o, by the above reasoning we get that if $f \neq 0$, then $D^{qa}(A_o)$ has a dense span in L^2. By Theorem 1 it follows that $A_o = A$, which is impossible.

q.e.d.

By the above method many other classes of quasi analytic functions can be constructed.

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