INSTITUTUL DE MATEMATICA

INSTITUTUL NATIONAL PENTRU CREATIE STIINTIFICA SI TEHNICA

ISSN 0250-3638

A REGULAR STRATIFICATION FOR THE JET SPACE $J^3(2,2)$

by

Alexandru DIMCA

PREPRINT SERIES IN MATHEMATICS
No.31/1980

BUCURESTI

A REGULAR STRATIFICATION FOR THE JET SPACE $J^3(2,2)$

by

Alexandru DIMCA*)

June 1980

*) Department of Mathematics, National Institute for Scientific and Technical Creation, Bd.Pacii 220, 77538 Bucharest, ROMANIA

A REGULAR STRATIFICATION FOR THE JET SPACE $J^3(2,2)$

by

Alexandru Dimca

There is an increasing interest in the problem of stratifying (complex) jet spaces by strata invariant under the K-equivalence (see [5], [1], [3], [2]).

However there are still very few examples of such stratifications - perhaps only three.

The purpose of this paper is to add a new example of complex jet space having a natural stratification and to show its basic similarity with the known examples. As a by-product we gain some new insight into the geometry of pencils of binary cubic forms.

The author wishes to express his whole gratitude to Professor Christopher Gibson for suggesting this type of problem.

1. Listing the orbits

We shall describe the orbits of the action of the contact group K^3 on our jet space $J=J^3(2,2)$. Let M^k denote the group of invertible 2x2 matrices A having as entries polynomials in x,y of degree $\leq k$ over $\mathbb{C}$ (and truncated multiplication as product) and D^k the group of k-germs of diffeomorphisms $h:(\mathbb{C}^2,0) \longrightarrow (\mathbb{C}^2,0)$.

Since $K^3=M^2 x D^3$, we see that K^3 is a connected affine group. The action $\mu:K^3 x J \longrightarrow J$, $(A,h)\cdot f=A\cdot(f\circ h)$ is obviously algebraic and it follows that its orbits are constructible connected smooth submanifolds in J [4].

We shall list these orbits by the first nonvanishing jet of their representatives.

Proposition 1

If $f \in J$ and $j^1 f \neq 0$, then f is equivalent to one of the following 4 normal forms:

Table 1

Normal form	Codimension	Boardman symbol
(x,y)	0	Σ^{o}
(x,y^2)	1	$\Sigma^{1,o}$
(x,y^3)	2	$\Sigma^{1,1,o}$
$(x,0)$	4	$\Sigma^{1,1,1}$

Proof. Since $j^1 f \neq 0$, we have that $\mathrm{rk}\,df(0) \geqslant 1$ and in this case it is known that f is equivalent to some constant unfolding $g(x,y)=(x,g^2(y))$ and the normal forms are given by the order of g^2. The computations for the codimension of the orbits and for their Boardman symbols are standard and we shall omit them.

Proposition 2

If $f \in J$, $j^1 f=0$ and $j^2 f \neq 0$, then f is equivalent to one of the following 11 normal forms:

Table 2

Type of $j^2 f$	Normal form	Codimension	Boardman symbol
(x^2,y^2)	(x^2,y^2)	4	$\Sigma^{2,o}$
(xy,y^2)	(xy,y^2+x^3)	5	
	(xy,y^2)	6	
$(x^2+y^2,0)$	(x^2+y^2,y^3)	6	
	$(x^2+y^2,0)$	8	
$(x^2,0)$	(x^2,y^3)	7	$\Sigma^{2,1,0}$

Table 2 (continuation)

Type of j^2f	Normal form	Codimension	Boardman symbol
$(x^2,0)$	(x^2+y^3,xy^2)	8	
	(x^2+y^3,xy^2+y^3)	8	
	(x^2,xy^2)	9	
	$(x^2+y^3,0)$	9	
	$(x^2,0)$	11	$\Sigma^{2,1,1}$

Proof. By the classification of pencils of binary quadrics [4] we know that f is equivalent to some $g=(Q^1+C^1,\ Q^2+C^2)$, where (Q^1,Q^2) is a pair of quadrics in the first column of Table 2 and (C^1,C^2) is a pair of binary cubic forms. We note also that in this case the action of K^3 is the same as the action of the simpler group $M^1 x D^2$ and by a courageous straightforward computation we find that the pair (C^1,C^2) can be reduced to the special types displayed by Table 2.

As an illustration we shall give some details in the case $(Q^1,Q^2)=(x^2,0)$. We shall denote in this paper by L, Q and C general linear, quadratic and cubic forms in x, y.

If $h\in D^2$ and $A\in M^1$ are given by

$$h(x,y)=(x+Q,y) \qquad A=\begin{pmatrix}1 & 0\\ L & 1\end{pmatrix}$$

then $(A,h)\cdot g=(x^2+2xQ+C^1,\ x^2L+C^2)$ and we see that we can take in g $C^1=ay^3$, $C^2=bxy^2+cy^3$.

i) a=0. If b=c=0 then $f\sim(x^2,0)$ and if c=0 and $b\neq0$ then $f\sim(x^2,xy^2)$. Next if $c\neq0$ using $h(x,y)=(x,y-\frac{b}{3c}x)$ we find that $f\sim(x^2,y^3)$

ii) $a\neq0$. If b=0 then for c=0 $f\sim(x^2+y^3,0)$ and for $c\neq0$ $f\sim(x^2,y^3)$. If $b\neq0$ it is clear that $f\sim g_t$, where $g_t=(x^2+y^3,\ xy^2+ty^3)$. And we find that $g_t\sim g_s$ iff s=t=0 or $s\neq0$ and $t\neq0$ and so the only new cases are $f \sim g_t$ for $t=0,1$.

Proposition 3. If $f \in J$ and $j^2 f=0$ then f is equivalent to one of the following 11 normal forms, when $\lambda \in \mathbb{C} \setminus \{0,1,9\}$:

Table 3

Normal form	Codimension	Boardman symbol
$(x^3+3x^2y,\ 3xy^2+\lambda y^3)$	11	$\Sigma^{2,2,0}$
$(x^3+x^2y,\ xy^2)$	11	
$(x^2y,\ xy^2)$	12	
$(x^3,\ xy^2+y^3)$	12	
$(x^3,\ xy^2)$	12	
$(x^3,\ y^3)$	12	
$(x^3,\ x^2y)$	13	
$(x^3+y^3,\ 0)$	13	
$(x^2y,\ 0)$	14	
$(x^3,\ 0)$	15	$\Sigma^{2,2,1}$
$(0,\ 0)$	18	$\Sigma^{2,2,2}$

Proof. In fact we have to classify the pencils of binary cubic forms. To a cubic form $C=ax^3+3bx^2y+3cxy^2+dy^3$ we can associate its hessian

$$H=\frac{1}{36}\begin{vmatrix} \frac{\partial^2 C}{\partial x^2} & \frac{\partial^2 C}{\partial x\partial y} \\ \frac{\partial^2 C}{\partial x\partial y} & \frac{\partial^2 C}{\partial y^2} \end{vmatrix} = \alpha x^2+\beta xy+\gamma y^2$$

where $\alpha=ac-b^2$, $\beta=ad-bc$, $\gamma=bd-c^2$ and also its discriminant $\Delta=\Delta(C)=\beta^2-4\alpha\gamma$.

In the projective space P^3 of all nonzero cubic forms we shall consider the quartic surface $V=V(\Delta)$ and the twisted cubic curve $W=V(\alpha,\beta,\gamma)$, which is precisely the singular part of V.

The induced action $G\times P^3 \longrightarrow P^3$, where $G=Gl(2)$ has 3 orbits: $G\cdot x^3=W$, $G\cdot x^2y=V\setminus W$ and $G\cdot(x^3+y^3)=P^3\setminus V$. In particular, V and W are G-invariant. Now let $f=(f^1,f^2)$ be a pencil of binary cubic forms. If f^1 and f^2 are linearly dependent, then f is clearly equivalent to one of the last 4 normal forms.

From now on we shall suppose that f^1 and f^2 are distinct points in P^3 and denote by L_f the line determined by them. By the above remark it follows that the position of L_f with respect to V and W is a geometric invariant of the equivalence class of f and in fact we shall prove that this invariant determines this class.

In order to control the intersection multiplicities of L_f and V we introduce the following binary quartic form $\Delta_f(\mu^1,\mu^2)=$ $=(\mu^1 f^1+\mu^2 f^2)$.

Essential to our study is

<u>Lemma 4</u>. $L_f \cap V$ has at least 2 distinct points.

<u>Proof</u>: Let us suppose that $L_f \cap V$ consists of only one point f^1 (which exists by dimensional reasons). By the homogeneity property of P^3 under G we can suppose that i) $f^1=x^2y$ or ii) $f^1=x^3$. We shall make the proof only in the case i), the other case being completely similar.

We shall work in the open affine set

$$D(b)=\{C\in P^3;\ \ b\neq 0\}\simeq \mathbb{C}^3$$

Here f^1 corresponds to the origin 0 and we can suppose that $f^2\in D(b)$, $f^2=(a,c,d)$.

Then $L_f \cap D(b)=\{t(a,c,d);\ t\in\mathbb{C}\}$ and $V\cap D(b)$ has equation $\Delta(a,1,c,d)=0$.

It follows that the intersection $L_f\cap V\cap D(b)$ is given by the equation in t:

$$t^4(a^2d^2+4ac^3)-6t^3acd-3t^2c^2+4td=0$$

Since this equation must have only the solution t=0, all of its coefficients but one have to be zero. If $d\neq 0$, then a=c=0 and $L_f=(x^2y,\ y^3)$. If $c\neq 0$ then a=d=0 and $L_f=(x^2y,\ xy^2)$. And finally if c=d=0 then $L_f\subset V$ and we have thus proved that the case i) cannot take place.

By Lemma 4, for any pencil $f=(f^1,f^2)$ we can choose $f^1, f^2\in V$ and this is the key to finding normal forms. The intersection $L_f\cap V$ is most precisely described in terms of the divisor D of degree 4 induced by Δ on L_f. To determine this divisor we have computed for each normal form f its associated quartic form $\Delta_f(\mu^1,\mu^2)$ and here is the complete description of the orbits (the points $P_i\in L_f\cap V$ are distinct and in $V\setminus W$ if not contrary stated):

a) D is not defined (i.e. $L_f\subset V$) iff $f\sim(x^3,x^2y)$.

b) $D=P_1+P_2+P_3+P_4$ iff $f\sim f_\lambda=(x^3+3x^2y,\ 3xy^2+\lambda y^3)$ for some $\lambda\in\mathbb{C}\setminus\{0,1,9\}$.

c) $D=2P_1+P_2+P_3$

i. $P_1\in W$ iff $f\sim(x^3,xy^2+y^3)$.

ii. $P_1\in V\setminus W$ iff $f\sim(x^3+x^2y,xy^2)$.

d) $D=2P_1+2P_2$

i. P_1, $P_2\in W$ iff $f\sim(x^3,y^3)$

ii. P_1, $P_2\in V\setminus W$ iff $f\sim(x^2y,xy^2)$

e) $D=3P_1+P_2$ and $P_1 \in W$ iff $f \sim (x^3, xy^2)$.

And we have finished the proof of Prop.3.

We define $\Sigma = \{f \in J;\ f \sim f_\lambda$ for some $\lambda \in \mathbb{C} \setminus \{0,1,9\}\}$. It is clear by our description b. that Σ is a Zariski open subset of the vector space $\{ f \in J;\ j^2 f = 0 \}$ and therefore Σ is a constructible connected smooth submanifold in J.

We have obtained thus a partition $\mathcal{S}$ of J consisting of Σ and the remaining 25 orbits.

2. The regularity of the stratification

We shall prove now the main result.

Theorem 5. The partition $\mathcal{S}$ is a constructible Whitney stratification of the complex jet space $J^3(2,2)$.

Proof. Because the single stratum which is not an orbit is Σ, we have only to verify regularity of the orbits listed in Table 1 and 2 over Σ.

We shall first deal with the strata from Table 1. The stratum (x,y) is open in J and hence its regularity over Σ is obvious.

Let Y be the stratum $(x,0)$ and let $f_n \in Y$ such that $\lim f_n$ exists ($=f_o$ say) and $j^2 f_o = 0$. It is easy to see that f_n must have the form

$$f_n = (a_n^1 L_n + L_n^1 L_n + a_n^1 Q_n + a_n^1 C_n + L_n^1 Q_n + Q_n^1 L_n\ , \quad a_n^2 L_n + a_n^2 Q_n + L_n^2 L_n + a_n^2 C_n + L_n^2 Q_n + Q_n^2 L_n)$$

where $a_n^i \in \mathbb{C}$ and L, Q, C are 1-,2-,3-forms in x,y such that $L_n \neq 0$ for any n. A moment thought shows that $f_o \notin \Sigma$, since f_o^1 and f_o^2 have at least a linear common factor.

Therefore $\bar{Y} \cap \Sigma = \emptyset$ and thus the regularity of Y over Σ is proved.

To prove regularity for the strata $Y_1=(x,y^2)$ and $Y_2=(x,y^3)$ we shall use the method of [1]. A slice S at f_λ has the form

$$* \qquad f=(f_\lambda^1+\alpha x^3+Q^1+L^1, \quad f_\lambda^2+Q^2+L^2) \qquad \alpha \in \mathbb{C}.$$

It is sufficient to prove that $N_i=Y_i \cap S$ for i=1,2 are A- and B-regular over $N=\Sigma \cap S$, which is a line in S. A-regularity follows by the simple remark that at any point $f \in N_i$ the direction given by the line N is a tangent direction at that point. Indeed if $f \in N_1$ then $g(t)=f+t(x^3,0) \in N_1$ for all $t \in \mathbb{C}$, since $N_1=\Sigma^{1,0}$ which is a condition on the 2-jet only. And if $f \in N_2$ then $g(t) \in N_2$ for all sufficiently small $t \in \mathbb{C}$, since $N_2=\Sigma^{1,1,0}$ which is an open condition on the 3-jet.

To prove B-regularity we shall show that N_1 and N_2 are nicely positioned over N i.e. if π is orthogonal projection on N, then for any $f \in N_i$, $f-\pi(f) \in T_f N_i$. To show this we note that if f is given by (*) then

$$f-\pi(f)=(Q^1+L^1, \quad Q^2+L^2)$$

and for t sufficiently close to 1 we have

$$g(t)=(f_\lambda^1+\alpha x^3+t(Q^1+L^1), \quad f_\lambda^2+t(Q^2+L^2)) \in N_i \quad .$$

It follows that $f-\pi(f)=\dot{g}(1) \in T_f N_i$.

In order to prove the regularity of a stratum M from table 2 over Σ we can restrict our considerations from J to the vector subspace J_o formed by all jets f with $j^1f=0$. In J_o a slice S for f_λ is given by (*) in which we take $L^1=L^2=0$. As above, it is sufficient to prove that $P=M \cap S$ is A- and B-regular over the line $N=\Sigma \cap S$ and for doing that we shall use the method of [3].

A-regularity. Let $\emptyset_t$ be an analytic path in the slice starting at f_λ and lying in P, T_t denote the tangent space to M at $\emptyset_t$ and T the tangent space to the orbit of f_λ. We know [3] that $\lim_{t\to 0} T_t$ exists ($=T_o$ say) and $T_o \supset T$. Since T has condimension 1 in T_1=tangent space at f_λ to Σ, to prove A-regularity (i.e. $T_o \supset T_1$) it is enough to find a 3-homogeneous element $f \in T_o \setminus T$.

Let $\emptyset_t=(f^1_\lambda+\alpha(t)x^3+Q^1_t \ , \ f^2_\lambda+Q^2_t)$, where

$$Q^i_t=a^i(t)x^2+b^i(t)xy+c^i(t)y^2 \qquad \text{for } i=1,2.$$

Since $\emptyset_t$ lies in P, at least one of the Q^i_t is not identically zero. Let us suppose $Q^1_t \neq 0$ and let

$$r=\text{min.ord}\ \{a^1(t),\ b^1(t),\ c^1(t)\}$$

Define $Q=\lim_{t\to 0} t^{-r}Q^1_t$ and notice that Q is a nonzero quadratic form. Since $(xQ^1_t\ ,0)$, $(yQ^1_t\ ,0)$, $(0,xQ^1_t)$ and $(0,\ yQ^1_t)$ are all vectors in T_t, it follows that the corresponding four vectors obtained by replacing Q^1_t by Q are all in T_o. And the proof of A-regularity is ended by using the following:

<u>Lemma 6</u>. If for some quadratic form Q, the four vectors associated to it as above are all in T, then Q=0.

<u>Proof</u>. Let $C=ax^3+bx^2y+cxy^2+dy^3$ be a general cubic form. By a straightforward computation, using a base for T, we find that

$$(C,0)\in T \quad \text{iff} \quad 2\lambda^2(3a-b)+(\lambda+3)(\lambda c-3d)=0$$

$$(0,C)\in T \quad \text{iff} \quad \lambda(\lambda-3)(3a-b)+2(\lambda c-3d)=0$$

Writing that the four vectors associated to Q are in T we get a linear homogeneous system of four equations in the coefficients of Ω. Since the rank of the system is 3 for any $\lambda \neq 0$, the only solution of it is the trivial one.

B-regularity. It is sufficient to show that P is nicely positioned over N and this is proved exactly as above. In this case we find that $g(t) \in P$ for $t \neq 0$ by using the contact transformation given by change of coordinates $(x,y) \longmapsto (t^{-1}x, t^{-1}y)$ and multiplication by t^3.

R e f e r e n c e s

1. Bedford, J.O., Gibson, C.G., On stratifying 3 jets of functions. Proc.Cambridge Philos.Soc. 82, (1977), 389-400.

2. Bedford, J.O., Gibson, C.G., Regularity conditions over the first simple elliptic family. J.London Math.Soc. (2), 20 (1979), 343-346.

3. Bruce, J.W., Stratifying jet spaces-some special cases. Proc.Cambridge Philos.Soc. 85 (1979), 439-444.

4. Gibson, C.G., Singular points of smooth mappings, Research Notes in Mathematics, 25, Pitman, 1975.

5. Lander, L.C., Stratification by orbits, with an application to $J^2(2,2)$, J.London Math.Soc. 9 (1974), 443-450.