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Th.FRIEDRICH and H.KURKE

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Th. FRIEDRICH*) and H. KURKE*)

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*) Sektion Mathematik der Humboldt Universität zu Berlin, Berlin, DDR

# Compact four-dimensional self-dual Einstein manifolds with positive scalar curvature.

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Th. Friedrich and H. Kurke

Sektion Mathematik der Humboldt-Universität zu Berlin

## 1. Introduction

N. Hitchin [7] described in 1974 the possible topological type of all 4-dimensional compact self-dual Einstein spaces $X^4$ with scalar curvature $\tau = 0$. In fact, he proved that such a space is either flat or a K3-surface, an Enriques-surface or the orbit space of an Enriques-surface by an anti-holomorphic involution. On the other hand, it follows from the solution of the Calabi-conjecture by S. T. Yau that every K3-surface admits a self-dual Einstein metric with vanishing scalar curvature (see [7] and [24]). In the present paper we study the 4-dimensional compact self-dual Einstein manifolds of positive scalar curvature and we particularly prove the following

<u>Theorem.</u> A compact four-dimensional self-dual Einstein manifold with positive scalar curvature is either isometric to the sphere $S^4$ or diffeomorphic to the complex projective plane $P^2(\mathbb{C})$.

The canonical metric of $P^2(\mathbb{C})$ is a self-dual Einstein metric with positive scalar curvature. Therefore, the topological classification given in this theorem is complete. It seems to be an open question whether $P^2(\mathbb{C})$ admits further self-dual Einstein metrics of positive scalar curvature. Using for example non-trivial conformal changes of the canonical metric of $P^2(\mathbb{C})$ one only gets non-Einstein metrics on $P^2(\mathbb{C})$ (see [23]).

The present investigations were specially inspired by the paper [1] containing an interpretation of Penrose's twistor programme. In that work the authors proved that the almost complex structure on the projective spinor bundle $P^-$ of a Riemannian manifold $X^4$ is integrable if and only if $X^4$ is self-dual. Starting from this result we decide the question under which conditions the metric on $P^-$ naturally defined by the metrics of the basis $X^4$ and of the fibres $P^1(\mathbb{C})$ is a Kähler metric. This situation occurs if and only if $X^4$ is a self-dual Einstein space with positive scalar curvature. Furthermore, a calculation of the Ricci-tensor in this case shows that $P^-$ then is a Kähler-Einstein-manifold of positive scalar curvature. This relation between self-dual Einstein manifolds of dimension four and Kähler-Einstein manifolds of complex dimension three is the basic idea of our argumentation. In fact, this observation yields that every compact self-dual Einstein space $X^4$ with positive scalar curvature is simply connected and - after some calculations in the cohomology ring $H^*(P^-;Z)$ - the quadratic form $H^2(X^4;Z)$ is positive definite with dimension $\sigma \leq 3$. Therefore, $X^4$ must have the same homotopy type as $S^4$, $P^2(\mathbb{C})$, $P^2(\mathbb{C}) \# P^2(\mathbb{C})$ or $P^2(\mathbb{C}) \# P^2(\mathbb{C}) \# P^2(\mathbb{C})$ (see [12]). The first case is simple and gives the result that $X^4$ is isometric to $S^4$. On the other hand, we exclude the third and fourth case by a thorough study of the 1-dimensional complex vector bundle $T(P^-/X^4)$ of all vertical vectors in $P^-$. Since the Ricci curvature is positive, it follows by a result of Kodaira that the higher cohomology groups of $P^-$ with coefficients in the sheaf of holomorphic functions must vanish. Therefore, it follows that $H^2(P^-;Z)$ classifies linear equivalence classes of divisors, and since the cohomology ring is generated by $H^2(P^-;Z)$, it follows that $H^*(P^-;Z)$ is the Chow-ring of $P^-$ modulo numerical equivalence, the cup-product corresponds to the intersection product of algebraic cycles. Using Kodaira's vanishing theorem, Bertini's theorems, the classification of algebraic surfaces, and the enumeration of algebraic varieties of small degree we can deduce from the structure of the Chow-ring that $P^-$ must be one of the following varieties: A double covering of $\mathbb{P}^3$ ramified along a K3-surface if $\sigma = 3$ and a complete intersection of two quadrics in $\mathbb{P}^5$ if $\sigma = 2$. In both cases the calculation of the Euler characteristic entails that such a variety cannot be the projective spinor bundle $P^-$ of a self-dual Einstein space with positive scalar curvature.

In case $\sigma = 1$, such a variety is analytic isomorphic to the flag manifold $\mathbb{F}(1,2)$ and we can describe the spinor fibration explicitely, the base must be diffeomorphic to the complex projective plane.

Finally, let us remark that the same method yields the following result: If $X^4$ is a compact 4-dimensional self-dual Riemannian manifold such that $c_1(P^-)$, which is always to $2c_1(T(P^-/X^4))$, is positive, then $X^4$ is isometric to $S^4$ or diffeomorphic to $\mathbb{P}^2(\mathbb{C})$ and the spinor bundle is analytic isomorphic to $\mathbb{P}^3(\mathbb{C})$ or to $\mathbb{F}(1,2)$.

## 2. The SO(4)-action on $P(\Delta^-)$

Let $\Delta$ be the spinor representation of the group Spin(4) and denote its decomposition into irreducible components by $\Delta = \Delta^+ + \Delta^-$. Spin(4) and SO(4) transitively act on the 1-dimensional complex projective space $P(\Delta^-)$. Therefore, we can express $P(\Delta^-)$ as a symmetric space $P(\Delta^-) = SO(4)|H^-$ and we can describe the metric and the complex structure of $P(\Delta^-)$ in a complement $n^-$ of the Lie algebra $\underline{H}^-$ in $\underline{so}(4)$. If $E_{ij}$ $(i < j \leq 4)$ is the standard basis of the Lie algebra $\underline{so}(4)$, then we have

<u>Theorem 1</u> (see for example [4]). $H^-$ is a connected subgroup of SO(4) with Lie algebra

$$\underline{H}^- = \left\{ \sum_{i<j} a_{ij} E_{ij} :\ a_{13} + a_{24} = 0,\ a_{14} - a_{23} = 0 \right\}.$$

If we choose

$$n^- = \mathcal{L}in(E_{13} + E_{24},\ E_{14} - E_{23}),$$

then $\underline{so}(4) = \underline{H}^- + n^-$, $[\underline{H}^-, n^-] \subset n^-$, $[n^-, n^-] \subset \underline{H}^-$. The metric of the symmetric space $P(\Delta^-) = SO(4)|H^-$ is given by the condition that $\{E_{13}+E_{24}, E_{14}-E_{23}\}$ is an orthonormal basis. Furthermore, the complex structure $J : n^- \longrightarrow n^-$ of the complex line $P(\Delta^-)$ is described by

$$J(E_{13}+E_{24}) = -(E_{14}-E_{23}), \quad J(E_{14}-E_{23}) = E_{13}+E_{24}.$$

In the Lie algebra $\underline{so}(4)$ we introduce the following basis:

$$Y_1 = E_{12}, \quad Y_2 = E_{34}, \quad Y_3 = E_{13} - E_{24},$$

$$Y_4 = E_{14} + E_{23}, \quad Y_5 = E_{13} + E_{24}, \quad Y_6 = E_{14} - E_{23}.$$

The elements $Y_1, Y_2, Y_3, Y_4$ span the Lie algebra $H^-$ and $Y_5, Y_6$ belong to $n^-$. Finally, we have the following commutator relations:

$$
\begin{array}{lll}
[Y_1,Y_2] = 0, & [Y_1,Y_3] = Y_4, & [Y_1,Y_4] = -Y_3, \\
[Y_1,Y_5] = -Y_6, & [Y_1,Y_6] = Y_5, & [Y_2,Y_3] = Y_4, \\
[Y_2,Y_4] = -Y_3, & [Y_2,Y_5] = Y_6, & [Y_2,Y_6] = -Y_5, \\
[Y_3,Y_4] = 2Y_1+2Y_2, & [Y_3,Y_5] = 0, & [Y_3,Y_6] = 0, \\
[Y_4,Y_5] = 0, & [Y_4,Y_5] = 0, & [Y_5,Y_6] = -2Y_1+2Y_2.
\end{array}
$$

## 3. The Kähler condition for the projective spinor bundle of a Riemannian manifold

Let $X^4$ be a 4-dimensional oriented Riemannian manifold and let $(Q,\pi,X^4)$ denote the principal SO(4)-bundle of all orthonormal frames. We consider the projective spinor bundle

$$P = Q \times_{SO(4)} P(\Delta^-)$$

which is a $P^1(\mathbb{C})$-fibration over $X^4$. The Levi-Civita-connection introduces a decomposition of the tangent bundle $TP = T_vP + T_hP$ into vertical and horinzontal vectors. There exists an almost-complex structure $J : TP \longrightarrow TP$ preserving this decomposition such that J coincides with the complex structure of the fibres $P^1(\mathbb{C})$ on vertical vectors. Furthermore, on a horizontal vector $\xi \in (T_hP)|_\psi$ at the point $\psi \in P$, J is defined (using the Clifford-multiplication between vectors and spinors) by the formula:

$$[\pi_* J(\xi)]\cdot\psi = i(\pi_*(\xi)\cdot\psi), \quad i = \sqrt{-1}.$$

It is well known (see [1] or [4]) that J is a complex structure on P if and only if $X^4$ is a self-dual Riemannian manifold (the negative part $W_-$ of the conformally invariant Weyl tensor W vanishes). Now we introduce a hermitian metric on P by pulling back the metric of $X^4$ to the horizontal subspaces and by adding the $\lambda$-fold of the metric of the fibres.

Theorem 2. $(P,J,g^\lambda)$ is a Kähler manifold if and only if $X^4$ is a self-dual Einstein space with positive scalar curvature $\tau = 48/\lambda$.

Before we shall prove this theorem let us look at some examples.

Example 1. $X^4 = S^4$ is a self-dual Einstein manifold with positive scalar curvature and P is analytic-isometric to $P^3(\mathbb{C})$.

Example 2. $X^4 = P^2(\mathbb{C})$ is a self-dual Einstein manifold with positive scalar curvature and P is isomorphic to the flag manifold F(1,2).

Example 3. $X^4 = S^3 \times S^1$ is a conformally flat space with positive scalar curvature, but it is not an Einstein space. We calculate the first Betti-number of P using the $S^2$-fibration $\pi : P \to X^4$:

$$b_1(P) = b_1(X^4) = 1.$$

Therefore, P does not admit Kähler structures at all.

Example 4. Let $\Gamma \subset \mathcal{E}(3)$ be the group generated by the Euclidian transformations $\alpha, \beta, \gamma, t_1, t_2, t_3$, which are given in an orthonormal basis $e_i$ by the formulas:

$$t_1(x) = x + e_1,$$

$$\alpha(x) = A(x+\tfrac{1}{2}e_1), \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\beta(x) = B(x+\tfrac{1}{2}e_2+\tfrac{1}{2}e_3), \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\gamma(x) = C(x+\tfrac{1}{2}e_1+\tfrac{1}{2}e_2+\tfrac{1}{2}e_3), \quad C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$X^4 = R^3|\Gamma \times S^1$ is a flat Riemannian manifold with the homology group $H_1(X^4;Z) = Z + Z_4 + Z_4$ (see [22]). With respect to $b_1(P) = b_1(X^4) = 1$ P does not admit Kähler structures at all.

Proof of Theorem 2. $(P,J,g^\lambda)$ is a Kähler manifold if and only if the 2-form $\Omega^\lambda(\xi_1,\xi_2) = g^\lambda(J\xi_1,\xi_2)$ is closed. Let $f : Q \to P = Q \times_{SO(4)} (SO(4)|H^-)$ be the submersion defined by $f(s) = [s,I]$, where $s = (s_1,\dots,s_4)$ is an orthonormal basis tangent to $X^4$. Then $d\Omega^\lambda = 0$ is equivalent to $d(f^*\Omega^\lambda) = 0$. On the manifold Q we consider the vertical fundamental vector fields $Y_\alpha$ induced by the elements $Y_\alpha$ of the Lie algebra $\underline{so}(4)$. Furthermore, we define four horizontal vector fields $X_1$, $X_2$, $X_3$, $X_4$ on Q by the formula

$$\pi_*(X_i(s)) = s_i.$$

Let $\{\varkappa^i, \eta^\alpha\}$ $(1 \le i \le 4,\ 1 \le \alpha \le 6)$ be the dual reper of 1-forms. Then the formula

$$f^*\Omega^\lambda = \lambda\eta^5\wedge\eta^6 - \varkappa^1\wedge\varkappa^2 + \varkappa^3\wedge\varkappa^4$$

immediately follows from the construction of the almost complex structure $J$ and of the metric $g^\lambda$. Now we fix a point $s^0 = (s_1^0, s_2^0, s_3^0, s_4^0)$ in $Q$ over $x^0 = \pi(s^0)$. Using parallel displacement along geodesic lines we get an orthonormal reper $s$ of vector fields in a neighbourhood $U$ of $x^0$ with $s(x^0) = s^0$. Let $w_{kl} = g(\nabla s_k, s_l)$ denote the local forms of the Levi-Civita connection. The reper $s$ gives locally a trivialization $Q|_U \approx U \times SO(4)$. If now

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & \dots & a_{14} \\ \vdots & & \vdots \\ a_{41} & \dots & a_{44} \end{pmatrix}.$$

is an element of $SO(4)$, then

$$X_i(x,A) = a_{ij}(s_j(x) - w_{kl}(s_j)(x)E_{kl})$$

defines horizontal vector fields on $Q|_U = U \times SO(4)$ with the property $\pi_*(X_i(s)) = s_i$ for all $s \in Q|_U$. Since $w_{kl}(s_j)(x^0) = 0$, it follows that

$$\langle 1\rangle \qquad [X_i, X_j](s^0) = R_{jikl}(x^0)E_{kl}$$

and

$$\langle 2\rangle \qquad [X_i, Y_\alpha](s^0) = -a_{ij}(Y_\alpha)s_j^0,$$

where $a_{ij}(Y_\alpha)$ denotes the element of the matrix at the place $(i,j)$. Since the commutator of vertical vector fields is vertical, we get

$$\langle 3\rangle \qquad \varkappa^i[Y_\alpha, Y_\beta] = 0.$$

Finally, the commuator between a fundamental vector field and a horizontal vector field is a horizontal vector field. This yields

$\langle 4\rangle \qquad \eta^{\alpha}[X_i, Y_\beta] = 0.$

From $\langle 1\rangle$ – $\langle 4\rangle$ it now follows by a direct calculation – using the equalities $a_{ij}(Y_\alpha) = -a_{ji}(Y_\alpha)$, $a_{13}(Y_\alpha) + a_{24}(Y_\alpha) = 0$, $a_{14}(Y_\alpha) = a_{23}(Y_\alpha)$ for $1 \leq \alpha \leq 4$ and using the commutator relations between the $Y_\alpha$'s written down above – that $d(f^*\Omega^\lambda) = 0$ is equivalent to the following system of equations for the components of the curvature tensor at $x^0 \in X^4$:

$$R_{1214} = R_{1223}, \quad R_{1213} = -R_{1224}, \quad R_{1414} - R_{1423} = -4/\lambda,$$

$$R_{1314} = R_{1323}, \quad R_{1413} = -R_{1424}, \quad R_{2314} - R_{2323} = 4/\lambda,$$

$$R_{2414} = R_{2423}, \quad R_{2313} = -R_{2324}, \quad R_{1313} + R_{1324} = -4/\lambda,$$

$$R_{3414} = R_{3423}, \quad R_{3413} = -R_{3424}, \quad R_{2413} + R_{2424} = -4/\lambda.$$

Clearly, every relation occuring, if we apply an even permutation to one of the twelve equations, holds too. Now it is easy to see that $d(f^*\Omega^\lambda) = 0$ if and only if $X^4$ is a self-dual Einstein space with scalar curvature $\tau = 48/\lambda$.

q.e.d.

## 4. Some curvature properties of $(P, g^\lambda)$

In this section we study the curvature of $(P, g^\lambda)$ in case of it being a Kähler manifold.

Theorem 3. If $X^4$ is a self-dual Einstein space with positive scalar curvature $\tau = 48/\lambda$, then $(P, J, g^\lambda)$ is a Kähler-Einstein manifold with scalar curvature $\tau(P) = \tau(X^4)$.

To prove this theorem we need some well-known formulas connected with the change of the curvature tensor in Riemannian submersions. Let $f : M^n \longrightarrow \bar{M}^m$ be a Riemannian submersion and let $N$ denote its normal bundle. On $M^n$ we locally choose an orthnormal reper $E_A$ $(A, B, \ldots = 1, \ldots, n)$ such that $f_*(E_\alpha) = 0$ $(\alpha, \beta, \ldots = m+1, \ldots, n)$ and such that $\bar{E}_i = f_*(E_i)$ $(i, j, \ldots = 1, \ldots, m)$ is an orthonormal reper on $\bar{M}^m$. Let $\sigma^A$, $\bar{\sigma}^i$, respectively, be their dual repers. Then we have the structure equations

$$d\sigma^A = w_{AB} \wedge \sigma^B,$$

$$dw_{AB} = w_{AC} \wedge w_{CB} + \frac{1}{2} R_{ABCD} \sigma^C \wedge \sigma^D,$$

$$d\bar{\sigma}^i = \bar{w}_{ij} \wedge \sigma^j,$$

$$d\bar{w}_{ij} = \bar{w}_{il} \wedge \bar{w}_{lj} + \frac{1}{2} \bar{R}_{ijkl} \bar{\sigma}^k \wedge \bar{\sigma}^l.$$

<u>Lemma 0.</u> $\bar{R}_{ijpq} = R_{ijpq} + w_{i\alpha} \wedge w_{\alpha j}(E_p, E_q) + \sigma^\alpha [E_p, E_q] w_{ij}(E_\alpha)$.

<u>Proof.</u> In the equation

$$f^* d\bar{w}_{ij} = f^* \bar{w}_{il} \wedge f^* \bar{w}_{lj} + \frac{1}{2} \bar{R}_{ijkl} \sigma^k \wedge \sigma^l$$

we put the vector fields $E_p$, $E_q$. With respect to $w_{ij} = f^* \bar{w}_{ij}$ on $N$ we get

$$w_{il}\wedge w_{lj}(E_p,E_q) + \bar{R}_{ijpq} =$$

$$= f^{*}d\bar{w}_{ij}(E_p,E_q)$$

$$= E_p(f^{*}\bar{w}_{ij}(E_q)) - E_q(f^{*}\bar{w}_{ij}(E_p)) - f^{*}\bar{w}_{ij}[E_p,E_q]$$

$$= E_p(w_{ij}(E_q)) - E_q(w_{ij}(E_p)) - f^{*}\bar{w}_{ij}[E_p,E_q]$$

$$= dw_{ij}(E_p,E_q) + w_{ij}[E_p,E_q] - f^{*}\bar{w}_{ij}[E_p,E_q]$$

$$= R_{ijpq} + w_{i\alpha}\wedge w_{\alpha j}(E_p,E_q) + \sigma^{\alpha}[E_p,E_q]w_{ij}(E_\alpha).$$

q.e.d.

<u>Proof of Theorem 3.</u> Let $X^4$ be a self-dual Einstein space with scalar curvature $\tau = 48/\lambda$. We introduce a metric on $Q$ by pulling back the metric of $X^4$ to the horizontal subspaces and by adding the metric of the fibres defined by the condition that $1/\sqrt{\lambda}\, Y_\alpha$ is orthonormal. Then $\pi : Q \longrightarrow X^4$ and $f : Q \longrightarrow P$ are Riemannian submersions. In an arbitrary point $x^0 \in X^4$ we fix an orthonormal basis $\overset{0}{s}_1, \overset{0}{s}_2, \overset{0}{s}_3, \overset{0}{s}_4$, which defines a local reper $s = (s_1,s_2,s_3,s_4)$ by parallel displacement along geodisic lines. Then $X_i = s_i - w_{kl}(s_i)E_{kl}$ are orthonormal horizontal vector fields on $Q$. For the calculation we now use the following convention: we designate a quantity connected with $Q$ by $^{*}$ and a quantity of $P$ by $^{-}$. Let us consider the orthonormal reper on $Q$,

$$E^{*}_1 = X_1, \qquad E^{*}_2 = X_2, \qquad E^{*}_3 = X_3, \qquad E^{*}_4 = X_4,$$

$$E^{*}_5 = \frac{1}{\sqrt{\lambda}}Y_5, \quad E^{*}_6 = \frac{1}{\sqrt{\lambda}}Y_6, \quad E^{*}_7 = \frac{1}{\sqrt{\lambda}}Y_1, \quad E^{*}_8 = \frac{1}{\sqrt{\lambda}}Y_2,$$

$$E^{*}_9 = \frac{1}{\sqrt{\lambda}}Y_3, \quad E^{*}_{10} = \frac{1}{\sqrt{\lambda}}Y_4.$$

If we apply Lemma [illegible] to the Riemannian submersions $\pi : Q \longrightarrow X^4$ and

$f : Q \longrightarrow P$, then we get $(i,j,p,q \leq 4)$:

$$R_{ijpq} = R^{*}_{ijpq} + \sum_{\alpha=5}^{10} w^{*}_{i\alpha}\wedge w^{*}_{\alpha j}(E^{*}_p,E^{*}_q) + \sum_{\alpha=5}^{10} \sigma^{*\alpha}[E^{*}_p,E^{*}_q]\, w^{*}_{ij}(E^{*}_{\alpha}),$$

$$\bar{R}_{ijpq} = R^{*}_{ijpq} + \sum_{\alpha=7}^{10} w^{*}_{i\alpha}\wedge w^{*}_{\alpha j}(E^{*}_p,E^{*}_q) + \sum_{\alpha=7}^{10} \sigma^{*\alpha}[E^{*}_p,E^{*}_q]\, w^{*}_{ij}(E^{*}_{\alpha}).$$

This yields

$$R_{ijpq} - \bar{R}_{ijpq} = \sum_{\alpha=5}^{6} w^{*}_{i\alpha}\wedge w^{*}_{\alpha j}(E^{*}_p,E^{*}_q) + \sum_{\alpha=5}^{6} \sigma^{*\alpha}[E^{*}_p,E^{*}_q]\, w^{*}_{ij}(E^{*}_{\alpha}).$$

With respect to

$$w^{*}_{AB}(E^{*}_C) = \frac{1}{2}\left\{\langle E^{*}_B,[E^{*}_C,E^{*}_A]\rangle + \langle E^{*}_A,[E^{*}_B,E^{*}_C]\rangle - \langle E^{*}_C,[E^{*}_A,E^{*}_B]\rangle\right\}$$

and since $[E^{*}_i,E^{*}_{\alpha}](s^0) = 0 \quad (i \leq 4, \alpha \geq 5)$, we get

$$\begin{aligned} w^{*}_{ij}(E^{*}_5) &= -\frac{1}{2\sqrt{\lambda}}\langle Y_5,[X_i,X_j]\rangle \\ &= \frac{1}{2\sqrt{\lambda}} R_{ijkl}\langle Y_5,E_{kl}\rangle \\ &= \frac{\sqrt{\lambda}}{4}(R_{ij13}+R_{ij24}), \end{aligned}$$

$$w^{*}_{ij}(E^{*}_6) = \frac{\sqrt{\lambda}}{4}(R_{ij14}-R_{ij23}),$$

$$w^{*}_{ij}(E^{*}_{\alpha}) = w^{*}_{i\alpha}(E^{*}_j), \qquad i,j \leq 4, \alpha \geq 5,$$

at the point $s^0$.

Now $X^4$ is a self-dual Einstein space with scalar curvature $\tau = 48/\lambda$ and so we have the equations from the end of Section 3. Using this relations we get $(i < j \leq 4)$

$$w^{*}_{ij}(E^{*}_{5}) = w^{*}_{i5}(E^{*}_{j}) = \begin{cases} -\frac{1}{\sqrt{\lambda}}, & i = 1,\ j = 3 \\ -\frac{1}{\sqrt{\lambda}}, & i = 2,\ j = 4 \\ 0, & \text{otherwise} \end{cases},$$

$$w^{*}_{ij}(E^{*}_{6}) = w^{*}_{i6}(E^{*}_{j}) = \begin{cases} -\frac{1}{\sqrt{\lambda}}, & i = 1,\ j = 4 \\ -\frac{1}{\sqrt{\lambda}}, & i = 2,\ j = 3 \\ 0, & \text{otherwise} \end{cases}.$$

Since $\sigma^{*\alpha}[E^{*}_{p}, E^{*}_{q}] = w^{*}_{q\alpha}(E^{*}_{p}) - w^{*}_{p\alpha}(E^{*}_{q})$, it follows that

$$\langle 1\rangle \qquad \sum_{i=1}^{4} R_{ijpi} - \sum_{i=1}^{4} \bar{R}_{ijpi} = \frac{6}{\lambda}\delta_{jp}.$$

$w^{*}_{AB}(E^{*}_{C})(s^{0})$ vanishes if exactly two indices are greater than 4. Therefore, we get the equations $(i, k \leq 4)$

$$\langle 2\rangle \qquad \begin{aligned} \bar{R}_{ki\alpha k} &= R^{*}_{ki\alpha k}, \quad \alpha = 5, 6, \\ \bar{R}_{\alpha i k\alpha} &= R^{*}_{\alpha i k \alpha}, \quad \alpha = 5, 6, \\ \bar{R}_{5ik6} &= R^{*}_{5ik6}, \\ \bar{R}_{5i56} &= R^{*}_{5i56}, \\ \bar{R}_{6i56} &= R^{*}_{6i56}. \end{aligned}$$

Now we calculate on Q the components of the curvature tensor $R^{*}$ at $s^{0}$:

$$\langle 3\rangle \qquad \begin{aligned} R^{*}_{5ik5} &= R^{*}_{6ik6} = \frac{1}{\lambda}\delta_{ik}, \\ R^{*}_{ki5k} &= R^{*}_{ki6k} = R^{*}_{5ik6} = R^{*}_{5i56} = R^{*}_{6i56} = 0. \end{aligned}$$

For example, we prove the first equation and remark that one gets the others in the same way.

$$R_{5ik5} = \langle \nabla^{*}_{E^{*}_5} \nabla^{*}_{E^{*}_i} E^{*}_k - \nabla^{*}_{E^{*}_i} \nabla^{*}_{E^{*}_5} E^{*}_k - \nabla^{*}_{[E^{*}_5, E^{*}_i]} E^{*}_k, E^{*}_5 \rangle$$

$$= \langle \nabla^{*}_{E^{*}_5} w^{*}_{kC}(E^{*}_i) E^{*}_C - \nabla^{*}_{E^{*}_i} w^{*}_{kC}(E^{*}_5), E^{*}_5 \rangle .$$

With respect to $E^{*}_5(w^{*}_{kC}(E^{*}_i)) = 0$, it follows that

$$R^{*}_{5ik5} = w^{*}_{kC}(E^{*}_i) w^{*}_{C5}(E^{*}_5) - E^{*}_i(w^{*}_{k5}(E^{*}_5)) - w^{*}_{kC}(E^{*}_5) w^{*}_{C5}(E^{*}_i)$$

$$= \sum_{C=5}^{10} w^{*}_{kC}(E^{*}_i) w^{*}_{C5}(E^{*}_5) - E^{*}_i(w^{*}_{k5}(E^{*}_5)) - \sum_{C=1}^{4} w^{*}_{kC}(E^{*}_5) w^{*}_{C5}(E^{*}_i)$$

$$= \sum_{C=5}^{10} w^{*}_{kC}(E^{*}_i) \langle E^{*}_5, [E^{*}_5, E^{*}_C] \rangle - E^{*}_i(w^{*}_{k5}(E^{*}_5)) + \frac{1}{\lambda}\delta_{ik}$$

$$= -E^{*}_i(w^{*}_{k5}(E^{*}_5)) + \frac{1}{\lambda}\delta_{ik}$$

$$= -E^{*}_i(\langle E^{*}_5, [E^{*}_5, E^{*}_k] \rangle) + \frac{1}{\lambda}\delta_{ik}$$

$$= -E^{*}_i(\langle E^{*}_5, [E^{*}_5, s_k - w_{pq}(s_k) E_{pq}] \rangle) + \frac{1}{\lambda}\delta_{ik}$$

$$= E^{*}_i(w_{pq}(s_k)) \langle E^{*}_5, [E^{*}_5, E_{pq}] \rangle + \frac{1}{\lambda}\delta_{ik}$$

$$= \frac{1}{\lambda}\delta_{ik}.$$

The fibres of the Riemannian submersions $\pi : Q \longrightarrow X^4$ and $\pi : P \longrightarrow X^4$ are totally geodesic submanifolds. Therefore, we can calculate $\bar{R}_{5665}$ only in one fibre and then we get

$$\langle 4 \rangle \qquad \bar{R}_{5665} = 4/\lambda.$$

Now from ⟨1⟩ - ⟨4⟩ it follows that

$$\bar{R}_{ij} = \frac{8}{\lambda}\delta_{ij} \qquad (i,j \leq 6)$$

and this means that $(P,J,g^{\lambda})$ is a Kähler-Einstein manifold with scalar curvature $\tau(P) = 6 \cdot \frac{8}{\lambda} = \frac{48}{\lambda} = \tau(X^4)$.

q.e.d.

A compact Kähler manifold $(M,J,g)$ is called a Hodge manifold if its fundamental form $\Omega(\xi_1,\xi_2) = g(J\xi_1,\xi_2)$ represents an integer cohomology class. By Kodaira's theorem a Hodge manifold is algebraic-projective (see [21]). Let Ric : TM ⟶ TM denote the Ricci-tensor. Then the 2-form

$$\Omega^{Ric}(\xi_1,\xi_2) = \frac{1}{2\pi} g(J \circ Ric(\xi_1),\xi_2)$$

represents the first Chern-class of M:

$$\{\Omega^{Ric}\} = c_1(M).$$

If we apply this formula to $(P,J,g^{48/\tau})$, then we get

$$c_1(P) = \Omega^{Ric} = \frac{\tau}{12\pi} g^{48/\tau}(J\cdot, \cdot).$$

Corollary 1. If $X^4$ is a compact self-dual Einstein space with positive scalar curvature, then $(P,J,\frac{\tau}{12\pi} g^{48/\tau})$ is a Kähler-Einstein manifold with the integer class $\{\Omega\} = c_1(P)$. In particular P is algebraic-projective.

Corollary 2. A compact self-dual Einstein space with positive scalar curvature is simply connected.

Proof. Since $\pi_1(P) = \pi_1(X^4)$, it is sufficient to prove that P is simply connected. In our situation P is a Kähler manifold with positive definite Ricci tensor and $\pi_1(P) = 0$ follows - by an idea of S. Kobayashi (see [9]) - from Myer's theorem ($\pi_1(P)$ is finite), Bochner's theorem ($h^{p,0}(P) = 0,\ p > 0$), and from the Hirzebruch-Riemann-Roch formula.

## 5. The cohomology structure of the projective spinor bundle P

Let $X^4$ be a compact self-dual Einstein space with positive scalar curvature and let $\pi : P \longrightarrow X^4$ denote the $S^2$-fibration of the projective spinor bundle. Since $X^4$ is simply connected, there exists an orientation class $w \in H^2(P;Z)$ such that its restriction to the fibres is the generator of $H^2(\text{fibre};Z) = H^2(S^2;Z)$. We remark that w is defined up to elements of $\pi^* H^2(X^4;Z)$. From the Thom-Gysin sequence it immediately follows (see [19]) that:

$$H^{2i}(P;Z) = \pi^* H^{2i}(X^4;Z) + w\cdot\pi^* H^{2i-2}(X^4;Z), \quad H^{2i+1}(P;Z) = 0.$$

Let us consider the fundamental form

$$\Omega(\xi_1,\xi_2) = \frac{\tau}{12\pi}\, g^{48/\tau}(J\xi_1,\xi_2)$$

of the Kähler-Einstein manifold P, which represents the first Chern-class $c_1(P)$. Restricting $\Omega$ to the fibres $P^1(\mathbb{C})$ one gets

$$\Omega|\text{fibre} = \frac{\tau}{12\pi}\frac{48}{\tau}\Omega^{P^1(\mathbb{C})} = 4w|\text{fibre}$$

and hence

$$\Omega \equiv 4w \text{ modulo } \pi^{\varkappa} H^2(X^4;Z).$$

We calculate the characteristic classes of P and denote by $\sigma$ and $\chi$ the signature and the Euler characteristic of $X^4$, respectively.

Theorem 4. Let $T_v$ and $T_h$ be the complex vector bundles of all vertical and horizontal vectors tangent to P. Then:

1. $c_1(T_v) = c_1(T_h) = \Omega/2$. We denote this element by $\gamma$.
2. $c_1(P) = 2\gamma$, $c_2(P) = 3(\sigma-\chi)\pi^{\varkappa}[X^4]$, $c_3(P) = -\chi\gamma\cdot\pi^{\varkappa}[X^4]$.
3. $p_1(P) = (6\sigma-2\chi)\pi^{\varkappa}[X^4]$.
4. $\gamma^2 = (3\sigma-2\chi)\pi^{\varkappa}[X^4]$.

Proof. Since the Euler characteristic of the fibres $P^1(\mathbb{C})$ equals two, we get

$$c_1(T_v) \equiv 2w \text{ modulo } \pi^{\varkappa}H^2(X^4;Z)$$

and hence

$$2c_1(T_v) \equiv \Omega \text{ modulo } \pi^{\varkappa}H^2(X^4;Z).$$

Consider now the anti-holomorphic involution $\mu : P \longrightarrow P$ mapping each fibre into itself and corresponding to the antipodal map of $S^2$ on each fibre. Since $\mu$ preserves the decomposition $TP = T_v + T_h$ and since the complex structure $J^{\mu(\psi)}$ at the point $\mu(\psi)$ equals $-J^{\psi}$,

$$\mu^{*}\Omega = -\Omega$$

results. If we apply $\mu^{*}$ to $2c_1(T_v) \equiv \Omega$, then we get $-2c_1(T_v) \equiv -\Omega$ and $4c_1(T_v) = 2\Omega$. Therefore,

$$c_1(T_v) = \Omega/2 \quad \text{and} \quad c_1(T_h) = c_1(TP) - c_1(T_v) = \Omega - \Omega/2 = \Omega/2$$

hold. Next we calculate the second Chern-class of $T_h$ using the functoriality of the Euler-class e and using the fact that $\pi_{*} : T_h \longrightarrow TX^4$ changes the orientation (P is the bundle of all "negative" projective spinors):

$$c_2(T_h) = e(T_h) = e(\pi^{*}TX^4) = -\pi^{*}(e(TX^4)) = -\chi\pi^{*}[X^4].$$

With respect to $p_1 = -2c_2 + c_1^2$ it follows that

$$3\sigma\pi^{*}[X^4] = p_1(\pi^{*}TX^4) = p_1(T_h) = -2c_2(T_h) + c_1^2(T_h)$$

and

$$\gamma^2 = (3\sigma - 2\chi)\pi^{*}[X^4].$$

Now we get the Chern classes of P:

$$c(P) = c(T_v)c(T_h) = (1+\gamma)(1+\gamma-\chi\pi^{*}[X^4]),$$

$$c_2(P) = \gamma^2 - \chi\pi^{*}[X^4] = 3(\sigma-\chi)\pi^{*}[X^4],$$

$$c_3(P) = -\chi\gamma\pi^{*}[X^4].$$

Finally, we calculate the first Pentrjagin-class using $p_1 = -2c_2 + c_1^2$ once more:

$$\begin{aligned} p_1(P) &= p_1(T_v) + p_1(T_h) \\ &= \gamma^2 + \gamma^2 + 2\chi\pi^*[X^4] \\ &= (6\sigma - 2\chi)\pi^*[X^4] \end{aligned}$$

q.e.d.

<u>Corollary 3.</u> If $X^4$ is a compact self-dual Einstein space with positive scalar curvature, then the quadratic form $H^2(X^4;Z) \times H^2(X^4;Z) \longrightarrow Z$ is positively definite with the discriminant 1. Furthermore, the second Betti-number $b_2(X^4)$ is bounded, $0 \leq b_2(X^4) \leq 3$.

<u>Proof.</u> Since $X^4$ is simply connected, the quadratic form $H^2(X^4;Z)$ is non-singular. Let $b^{\pm}$ denote the dimension of the subspaces of $H^2$ on which the quadratic form is positively or negatively definite. Then

$$\sigma = b^+ - b^-, \quad \chi = 2 + b^+ + b^-.$$

On the other hand, P is a Kähler-Einstein manifold with positive scalar curvature. By Bochner's theorem ($h^{p,0}(P) = 0$ if $p > 0$) and from the Hirzebruch-Riemann-Roch formula we get

$$1 = \sum_p (-s)^p h^{p,0}(P) = \frac{1}{24}\int_P c_1 c_2 = \frac{\sigma - \chi}{4}\int_P \gamma\pi^*[X^4].$$

With respect to $\gamma|\text{fibre} = 2w|\text{fibre}$ and using the Fubini integration we conclude $1 = (\chi - \sigma)/2$ and have $b^- = 0$. This means that the quadratic form $H^2(X^4;Z)$ is positively definite. Now for every compact oriented Einstein space $X^4$ the formula (see [4])

$$\chi \pm \frac{3}{2}\sigma = \frac{1}{4\pi^2}\int_{X^4} |W_\pm|^2 + \frac{\tau^2 \mathrm{vol}(X^4)}{192\pi^2}$$

holds. The proof of this formula follows after some calculations from the Gauß-Bonnet formula

$$\chi = \frac{1}{32\pi^2}\int_{X^4} |R|^2 - 4|Ric|^2 + \tau^2$$

and the Hirzebruch signature formula

$$\sigma = \frac{1}{3}\int_{X^4} p_1 = \frac{1}{12\pi^2}\int_{X^4} |W_+|^2 - |W_-|^2.$$

If $X^4$ is a self-dual Einstein space with positive scalar curvature, then we get $\chi - \frac{3}{2}\sigma > 0$, and since $b^- = 0$, i.e. $\sigma = b_2$, it follows that $b_2 < 4$.

q.e.d.

<u>Remark.</u> For a self-dual Einstein space with positive scalar curvature we proved that $H^2$ is positively definite using the projective spinor bundle P, Bochner's theorem, and the Hirzebruch-Riemann-Roch formula. However, one can also prove this property directly in the geometry of $X^4$. In fact, let $\Lambda^2 = \Lambda^2_+ + \Lambda^2_-$ be the decomposition of the bundle $\Lambda^2$ under the Hodge operator $\ast$. The Laplace operator $\Delta = d\delta + \delta d$ preserves this decomposition and induces two operators $\Delta_\pm$ : $\Gamma(\Lambda^2_\pm) \to \Gamma(\Lambda^2_\pm)$. With respect to $b_1(X^4) = 0$ we immediately get by the Hodge theory that

$$\sigma = a - \mathrm{ind}(d+\delta) = \dim \ker \Delta_+ - \dim \ker \Delta_-,$$

$$\chi = 2 + \dim \ker \Delta_+ + \dim \ker \Delta_-$$

and hence

$$\dim \ker \Delta_- = \frac{1}{2}(\chi - \sigma - 2) = b^-.$$

On the other hand, for every 2-form $u = \frac{1}{2} u_{i_1 i_2} dx^{i_1} \wedge dx^{i_2}$, the Lichnerowicz-formula

$$\int_{X^4} \langle u, \Delta u \rangle = \int_{X^4} |\nabla u|^2 + \int_{X^4} F_2(u),$$

where

$$F_2(u) = R_{rs} u^{ri_2} u^s_{i_2} + \frac{1}{2} R_{rsjl} u^{rs} u^{jl},$$

holds. If $X^4$ is an Einstein space and if $u$ is a section in $\Lambda^2_-$, we can simplify this formula to:

$$F_2(u) = ((W_- + \frac{\tau}{6})u, u), \quad u \in \Gamma(\Lambda^2_-).$$

Furthermore, if this Einstein space is self-dual with positive scalar curvature, then $b^- = \dim \ker \Delta_- = 0$ immediately follows.

<u>Corollary 4.</u> A compact self-dual Einstein space $X^4$ with positive scalar curvature and vanishing Betti number $b_2(X^4) = 0$ is isometric to the sphere $S^4$.

Proof. If $b_2 = 0$, then $\sigma = 0$. Furthermore, we have

$$0 = \sigma = \frac{1}{12\pi^2}\int_{X^4} |W_+|^2 - |W_-|^2 = \frac{1}{12\pi^2}\int_{X^4} |W_+|^2$$

and hence $W_+ = W_- = 0$. This means that $X^4$ is a conformally flat Einstein space. But then $X^4$ is a space of constant positive sectional curvature (see [3]), hence isometric to the sphere $S^4$ (see [22]).

q.e.d.

With respect to Corollaries 3 and 4 we now have to study such self-dual Einstein spaces with positive scalar curvature that the second Betti number $b := b_2(X^4)$ satisfies $1 \leq b \leq 3$. In the next sections of this paper we will prove that the cases $b = 2,3$ are impossible. Furthermore, $b = 1$ occurs if and only if $X^4$ is diffeomorphic to the complex projective plane $P^2(\mathbb{C})$. First of all we describe the cohomology ring $H^*(P;Z)$ of the projective spinor bundle.

Theorem 5. Let $e_1, \ldots, e_b$ be an orthonormal basis of $H^2(X^4;Z)$. One can choose the orientation class $w \in H^2(P;Z)$ of the $S^2$-fibration in such a way that $H^*(P;Z) = H^*(X^4;Z)[w]$ and

$$w^2 + (x_1 + \ldots + x_b)w + x_1^2 = 0, \quad \gamma = 2w + x_1 + \ldots + x_b,$$

where $x_i = \pi^* e_i$.

Proof. The class $w$ is defined up to an element of $\pi^* H^2(X^4;Z)$ and $\gamma \equiv 2w$ modulo $\pi^* H^2(X^4;Z)$. Therefore, we can assume without loss of generality that $\gamma = 2w + x_1 + \ldots + x_r$ and $0 \leq r \leq b$. Since $x_i^2 = \pi^*[X^4]$, we get

$$\gamma^2 = 4w^2 + 4w(x_1+\ldots+x_r) + rx_1^2 = 4[w^2 + w(x_1+\ldots+x_r)] + rx_1^2.$$

On the other hand, by Theorem 4 and Corollary 3, we have

$$\gamma^2 = (3\sigma-2\chi)\pi^*[X^4] = (b-4)x_1^2.$$

This yields

$$4[w^2 + w(x_1+\ldots+x_r)] = (b-4-r)x_1^2.$$

Hence $b - 4 - r$ is divisible by 4. But $1 \leq b \leq 3$ and $0 \leq r \leq b$, so that is possible in the case $b = r$ only.

q.e.d.

## 6. Analysis of the linear system $|\gamma|$

We keep the notations and assumptions of the previous section. By $\omega_P$ we denote the canonical sheaf on P, i.e. the sheaf of holomorphic 3-forms, by $\underline{O}_P$ we denote the sheaf of holomorphic functions on P.

Since $H^{oq} = H^q(P, \underline{O}_P) = 0$ for $q > 0$, the exponential map: $O_P \to O_P^*$ (= sheaf of nowhere vanishing holomorphic functions on P) $f \mapsto \exp(2\pi i f)$ yields (via the exact cohomology sequence) an isomorphism

$$\mathrm{Pic}(P) = H^1(P, O_P^*) \to H^2(P, \mathbb{Z}).$$

Moreover, since P is a Hodge-manifold, hence a projective algebraic variety, the elements of Pic (P) correspond to the divisor classes (by [17]) on P, and since the cohomology ring $H^*(P, \mathbb{Z})$ is generated by $H^2(P, \mathbb{Z})$, the canonical homomorphism of the Chow-ring of P (the ring of algebraic cycles modulo rational equivalence with the intersection product) into the cohomology ring is surjective. For a cycle z or a class of a cycle of codimension 3 we denote its degree by $(z) \in \mathbb{Z}$. If $\alpha$ is an element of $H^2(X, \mathbb{Z})$, $|\alpha|$ denotes the corresponding linear system on P, i. e. (set-theoretically) the set of all non-negative divisors D on P representing the class $\alpha$. If we calculate intersection products, we often do not distinguish between algebraic cycles and their cohomology classes. We note the following formulas which are consequences of theorem 4 and theorem 5

<u>Lemma 1</u> $(\gamma^3) = 2(4-\sigma)$

$$\gamma^2 = (\sigma - 4)\, x_1^{\ 2} = \dots = (\sigma - 4)\, x^{\ 2}$$

$$(w\, x_1^{\ 2}) = \dots = (w\, x_\sigma^{\ 2}) = -1$$

$$(w^2 x_1) = \dots = (w^2 x_\sigma) = 1$$

If D is an effective divisor on P, then $(D \cdot \gamma^2) = m\,(4-\sigma) > 0 \quad (m \in \mathbb{Z})$ and the divisor class of D has the form

$$m\,w + \sum_{i=1}^{\sigma} a_i\, x_i \quad , \quad a_i \in \mathbb{Z}$$

(The assertion about D follows because $\gamma$ is ample, and because of

$$\left(\left(m\,w + \sum_{i=1}^{\sigma} a_i\, x_i\right) \cdot \gamma^2\right) = m\,(4-\sigma)\,)$$

From theorem 4 we infere that the divisor class $\gamma$ is ample and $c_1\,(\omega_P) = -2\,\gamma$ .(If X is a Kähler-Einstein manifold, then either $\omega_X$ or $\omega_X^{-1}$ is ample, since $c_1(X)$ can be expressed by the Ricci tensor. To decide which of the sheaves is ample, compute $(C \cdot \omega_X)$, where C is any algebraic curve on X.)

We will show that in the case $\sigma = 1$ the associated rational map is the canonical embedding of the flag manifold $\mathbb{F}\,(1,\,2) \subset \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, and that the cases $\sigma = 2$ or 3 are not possible.

<u>Lemma 2</u> Let L be an invertible $O_P$-module representing the class $\gamma$. Then

$$H^q(P, L^{\otimes\gamma}) = 0 \quad \text{for} \quad q > 0, \quad \gamma \geq -1,$$

$$\dim H^0(P, L^{\otimes\gamma}) = \frac{2(4-\sigma)}{6}(\gamma+1)^3 + \frac{2(\sigma-1)}{6}(\gamma+1)$$

$$\text{for } \gamma \geq -1.$$

Prove: The first assertion is Kodairas vanishing theorem [21] since $L^{\otimes\gamma} \otimes \omega_P^{-1} (\simeq L^{\otimes\gamma+2})$ is ample. The second assertion follows from the Hirzebruch-Riemann-Roch formula [6],

$$\dim H^0(P, L^{\otimes\gamma}) = \frac{1}{6}(\gamma+1)^3(\gamma^3) - \frac{1}{24}(\gamma+1)\ (\gamma \cdot p_1)$$

and from theorem 4.

<u>Corollary</u> The linear system $/\gamma/$ has dimension $9 - 2\sigma$.

By B we will denote the set of base points of $/\gamma/$, more precisely the subschema of P defined by the sheaf of ideals

$$\mathcal{J} = \text{image}\,(H^0(P, L) \otimes L^{-1} \longrightarrow \mathcal{O}_P).$$

Then $/\gamma/$ defines a rational morphism

$$\bar{\Phi} \quad P \setminus B \longrightarrow \mathbb{P}^{9-2},$$

by Y we shall denote the Zariski-closure of the image of $\bar{\Phi}$ in $\mathbb{P}^{9-2}$.

Since Y is not contained in a proper linear subspace, we infere by a well known formula

$$\deg(Y) \geq \operatorname{codim}(Y) + 1 = 10 - 2\sigma - \dim(Y)$$

In the next sections we shall prove that $/\gamma/$, B, $\bar{\Phi}$, and $Y \subset \mathbb{P}^{9-2}$ must have the following properties:

(A) Each divisor of $/\gamma/$ splits into at most 2 components. If $/\gamma/$ contains a linear subsystem of the form $/V_1/ + V_2$, $V_2 > 0$, $\dim /V_1/ > 0$, then $\sigma = 1$ (and $\dim /V_1/ = 2$).

(B) If the linear system $/\gamma/$ has no base points, then either $\sigma = 3$ and $\Phi$ restricted to any smooth connected surface $V \in /\gamma/$ is finite of degree 2 or $\sigma < 3$ and $\Phi$ restricted to any smooth connected surface $V \in /\gamma/$ is a closed embedding.

(C) If $/\gamma/$ has base points, then B is an irreducible curve and $\Phi$ is birational onto its image Y, moreover degree $(Y) = 7 - 2\sigma$.

In this section we will deduce the following consequence of these properties:

<u>Theorem 6:</u> If $X^4$ is an oriented 4 dimensioned self-dual compact Einstein manifold with positive scalar curvature, then $b_2 = \sigma \leq 1$.

Proof: I) The case $\sigma = 3$: Consider P and $\Phi$ as above. If $\Phi : P \longrightarrow \mathbb{P}^3$ is a 2-fold covering, the branch locus of $\Phi$ well be a smooth quartic in $\mathbb{P}^3$ (since we know $\omega_P \simeq \Phi^* \mathcal{O}_{\mathbb{P}^3}(-2)$ and $\Phi^* \omega_{\mathbb{P}^3} \simeq \Phi^* \mathcal{O}_{\mathbb{P}^3}(-4)$ ), we will denote it by Z.

We recall the following facts about double coverings $\Phi : V \longrightarrow U$ of smooth complete varieties: There exists an algebraic line bundle $L \xrightarrow{\pi} U$ on U such that V is isomorphic to a closed subvariety of L and $\Phi$ is induced by the projection .

Moreover there exists a fibre product diagram

$$\begin{array}{ccc} V & \xrightarrow{\Phi} & U \\ \downarrow & & \downarrow j \\ L & \xrightarrow{q} & L^{\otimes 2} \end{array} ,$$

where $j$ is an algebraic section and q the morphism
$q(e) = e^2 = e \otimes e$.

The ramification locus W of $\Phi$ is the divisor of zeros of $j$, which is always smooth. The inclusion $\Phi^*\omega_{\mathcal{U}} \subset \omega_V$ yields a section of $\omega_V \otimes \Phi^*\omega_{\mathcal{U}}^{-1} \simeq \Phi^* \underline{O}_{\mathcal{U}}(L)$, which corresponds to a divisor Z and $\Phi/Z$ is an isomorphism onto W.

There is a canonical exact sequence

$$0 \longrightarrow \Omega^1_{\mathcal{U}} \longrightarrow \Omega^1_V \longrightarrow \underline{O}_Z \otimes \underline{O}_{\mathcal{U}}(L^{-1}) \longrightarrow 0$$

which gives for the Chern classes

$$c(\Omega^1_V) = \Phi^*(c(\Omega^1_{\mathcal{U}})\ \ c(L^{-1})\ \ c(L^{-2})^{-1})$$
$$= \Phi^*(c(\Omega^1_{\mathcal{U}})\ \ (1 + c_1(L) + 2c_1(L)^2 + 4c_1(L)^3 + \\ + \ldots))$$

In our case: $V = P$, $W = \mathbb{P}^3$, $L = \underline{O}_{\mathbb{P}^3}(2)$ we get

$\chi(P) = - c_3(\Omega^1_V) = -8(\gamma^{3}) = -16$ which contradicts
$\chi(P) = 2\chi(X^4) = 10.$

The case where $/\gamma/$ has a base curve B, $\Phi$ is a birational morphism $P \smallsetminus B \longrightarrow \mathbb{P}^3$ is left.

We also consider the inverse birational transformation $\Psi = \Phi^{-1}$, which is a morphism $\mathbb{P}^3 \smallsetminus B' \longrightarrow P$, where B' is a Zariski closed subset of codimension $\geq 2$. The canonical inclusion induced by $\Phi$

$$\Phi^*\omega_{\mathbb{P}^3} \simeq \Phi(\underline{O}_{\mathbb{P}^3}(-4)) \longrightarrow \omega_P \,|\, P \smallsetminus B \simeq \Phi^*(\underline{O}_{\mathbb{P}^3}(-2)) \,|\, P \smallsetminus B$$

tensorized with $\Phi^*\underline{O}_{\mathbb{P}^3}(4)$ gives a holomorphic section of

$\Phi^*\underline{O}_{\mathbb{P}^3}(2)$ on $P \smallsetminus B$ which extends uniquely to a holomorphic section of $\Phi^*\underline{O}_{\mathbb{P}^3}(2)$, since codim $B > 1$. The divisor of zeros E of this section is therefore an element of $/2\gamma/$

and supp (E) $\smallsetminus$ B is precisely the locus where $\bar{\Phi}$ is not an open embedding.

In the same way we define a divisor F on $\mathbb{P}^3$ such that supp (F) $\smallsetminus$ B' is the locus where $\Psi$ is not an open embedding. We claim that the morphism $\bar{\Phi}$ induces an isomorphism P$\smallsetminus$supp(E) $\xrightarrow{\sim}$ $\mathbb{P}^3 \smallsetminus$ supp(F). To see this we firstly prove that $\Psi$ induces an open embedding $\mathbb{P}^3 \smallsetminus \mathrm{supp}(F) \to P \smallsetminus$ supp(E). Since E is ample, the variety P $\smallsetminus$ supp(E) is affine, hence it is sufficient to show that for any rational function f on P being holomorphic on P$\smallsetminus$supp(E) the function $f \circ \Psi$ is holomorphic on $\mathbb{P}^3 \smallsetminus$supp(F). Hence we have to show that for any prime divisor W' on $\mathbb{P}^3$ such that W' $\not\subset$ supp(F) the function $f \circ \Psi$ is contained in the local ring $\underline{O}_{\mathbb{P}^3, W'}$. Since W' $\not\subset$ supp(F), exactly one prime divisor of P corresponds to W' such that $\Psi$ induces an isomorphism of the local rings $\underline{O}_{P,W} \simeq \underline{O}_{\mathbb{P}^3, W'}$. Because of this isomorphism we see that W $\not\subset$ supp(E), hence $f \in \underline{O}_{P,W}$ and $f \circ \Psi \in \underline{O}_{\mathbb{P}^3, W'}$.

This proves that $\Psi$ is an open embedding $\mathbb{P}^3 \smallsetminus \mathrm{supp}(F) \to$ P $\smallsetminus$ supp(E), especially $F \neq 0$ and therefore we can apply the same argument to $\bar{\Phi}$ and F instead of $\Psi$ and E (any divisor $F > 0$ on $\mathbb{P}^3$ is ample).

Because of this isomorphism we get an isomorphism of the (algebraic) Picard groups Pic(P $\smallsetminus$supp(E)) $\simeq$ Pic($\mathbb{P}^3 \smallsetminus$ supp (F)). If $E_1 \dots E_m$ are the irreducible components of E, we can define an exact sequence

$$\mathbb{Z}^m \longrightarrow \mathrm{Pic}(P) \xrightarrow[\text{restriction}]{} \mathrm{Pic}\,(P \smallsetminus \mathrm{supp}(E)) \longrightarrow 0$$

$$(a_1, \dots, a_m) \longmapsto \text{class of } (\textstyle\sum a_i E_i).$$

The restriction map is surjective since any divisor on $P \setminus \mathrm{supp}(E)$ extends to a divisor on P. The analogous sequence for $\mathbb{P}^3$ and F entails that Pic $(P \setminus \mathrm{supp}(E)) \cong$ Pic $(\mathbb{P}^3 \setminus \mathrm{supp}(F))$ is a finite cyclic group.

Because of Pic (P) $= \mathbb{Z}^4$ the number of components of E is therefore at least 4, but since $E \in |2\gamma| = |4w + x_1 + x_2 + x_3|$, it must be exactly 4 and $E = E_1 + E_2 + E_3 + E_4$, $(E \cdot \gamma^2) = 2(\gamma^3) = 4 = (E_1 \cdot \gamma^2) + \dots + (E_4 \cdot \gamma^2)$, hence $(E_i \cdot \gamma^2) = 1$.

Since any surface $V \in |\gamma|$ is irreducible, the intersection product $E_i \cdot V$ is defined and must be an irreducible curve (because of $(E_i \cdot \gamma^2) = (E_i \cdot V \cdot \gamma) = 1$).

The image $L_i = \Phi(E_i \setminus B)$ is consequently a point or a line in $\mathbb{P}^3$, because it is always a point or a curve, and if it was a curve of higher degree, $E_i \cdot V$ would split into at least 2 components.

Moreover, if H is a hyperplane of $\mathbb{P}^3$ containing $L_i$, the inverse image $\Phi^* H = V$ would contain $E_i$ as a component. But we know that $|\gamma|$ does not contain a linear subsystem of the form $|V'| + E_i$, $\dim |V'| > 0$, hence the case of a base curve is not possible.

This proves that the case $\sigma = 3$ is impossible.

II) The case $\sigma = 2$:

If $|\gamma|$ has no base curve, the map $\Phi$ will be a finite birational morphism onto a threefold $Y \subset \mathbb{P}^5$ of degree 4, moreover, for generic hyperplanes $H \subset \mathbb{P}^5$ the map $\Phi$ will induce an isomorphism $V = \Phi^{-1}H \longrightarrow H \cap Y$ (since $\Phi^{-1}H$ is smooth and irreducible by Bertini's theorem).

Therefore H does not meet the singular locus of Y, hence Y has at most isolated singularities.

Because of $\dim /\underline{O}_{\mathbb{P}^5}(2)/ = 20$ and $\dim /2\gamma/ = 18$ there are at least 2 distinct quadrics $Q_1$, $Q_2$ of $\mathbb{P}^5$ containing Y. As Y is not contained in a hyperplane, any quadric containing Y will be irreducible.

Because of $Y \leq Q_1 \cdot Q_2$ and $\deg Y = \deg Q_1 \cdot Q_2 = 4$ we infere $Y = Q_1 \cdot Q_2$.

Thus Y is a 3-dimensional complete intersection with at most isolated singularities, this implies that Y is a normal variety [18]. But since $\bar{\Phi}$ is finite and birational, it must be an isomorphism, i.e. $P$ is isomorphic to an intersection of 2 quadrics in $\mathbb{P}^5$.

For a complete intersection Y of two hyper surfaces in $\mathbb{P}^5$ of degree $d_1$, $d_2$ we can compute the Euler characteristic by using the exact sequence

$$0 \to \Theta_Y \longrightarrow \underline{O}_Y \otimes \Theta_{\mathbb{P}^5} \to \underline{O}_Y(d_1) \oplus \underline{O}_Y(d_2) \longrightarrow 0$$

(where $\Theta$ denotes the sheaf of holomorphic vector fields), which yields for the Chern classes

$$c(Y) = c(\underline{O}_Y \otimes \Theta_{\mathbb{P}^5})\, c(O_Y(d_1))^{-1}\, c(O_Y(d_2))^{-1}$$

$$= (1+\gamma)^6 (1+d_1\gamma)^{-1} (1+d_2\gamma)^{-1}$$

$$= (1+6\gamma+15\gamma^2+20\gamma^3)\,[\,1-(d_1+d_2)\gamma+(d_1^2+d_2^2+d_1d_2)\gamma^2-(d_1^3+d_2^3+d_1^2d_2+d_1d_2^2)\gamma^3\,]$$

In the case $d_1 = d_2 = 2$ we get therefore

$\chi(Y) = c_3(Y) = 0,$

which contradicts $\chi(P) = 2\chi(X^4) = 16$.

If on the other hand $/\gamma/$ has a base curve B, the image Y will be a variety of degree 3 = codim Y + 1 in $\mathbb{P}^5$. However, such varieties (degree Y = codim Y + 1) are classified (see [20]), and checking the list of these varieties, Y would be one of the following ones:

(a) $Y = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ (Segre embedding)

(b) Y cone over the rational scroll $F_1 \subset \mathbb{P}^4$ (blowing up of one point of $\mathbb{P}^2$, embedded into $\mathbb{P}^4$ by the linear system of quadrics through this point)

(c) Y cone over $\mathbb{P}^1 \subset \mathbb{P}^3$ (3-fold Veronese embedding).

But in each of these cases the linear system $/\gamma/$ would contain a linear subsystem of the form $/V_1/ + V_2$ , $V_2 > 0$, $\dim /V_1/ > 0$.

In case (a) we could take

$V_1 = \Phi^*(P \times \mathbb{P}^2)$ ($P \in \mathbb{P}^1$ a point)

$V_2 = \Phi^*(\mathbb{P}^1 \times H)$ ($H \subset \mathbb{P}^2$ a line)

in case (b) we could take

$V_1 = \Phi^*(\hat{F})$ $\hat{F}$ cone over the strict transform of a line passing through the centre in $\mathbb{P}^2$

$V_2 = \Phi(\hat{H})$ $\hat{H}$ cone over the strict transform of a line in $\mathbb{P}^2$ not passing through the centre and in case (c)

$V_1 = \Phi^*(C_o)$  $C_o$ cone over a point $P_o \in \mathbb{P}^1$

$V_2 = \Phi^*(C)$  $C$ cone over 2 points $P_1 + P_2$, $P_i \in \mathbb{P}^1$

This proves that the case $\sigma = 2$ is not possible.

7. Proof of property (A)

Lemma 3 Each divisor $D \in /\gamma/$ splits into at most 2 components and $\dim Y \geq 2$. If the linear system $/\gamma/$ contains a linear subsystem of the form $/V_1/ + V_2$, $V_2 > 0$, $\dim /V_1/ > 0$, then it follows: $\sigma = 1$, $\dim /V_1/ = 2$, $/V_1/$ has no base points, $(V_1^2 \cdot \gamma) = 1$, $(V_1^3) = 0$ and each curve $C \in /V_1/ . V_1$ resp. $C' \in /V_1/ . V_2$ is isomorphic to $\mathbb{P}^1$.

Proof: Assume $D \in /\gamma/$ and $D = V_1 + V_2$, $V_1, V_2 > 0$. Since $\gamma$ is ample, we must have $(D . \gamma) > 0$ for any effective divisor D on P and $(C . \gamma^2) > 0$ for any curve C on P.

In our case

$$\begin{aligned}(\gamma^3) &= (D . \gamma^2) = 2\,(4 - \sigma) \\ &= (V_1 . \gamma^2) + (V_2 . \gamma^2) \\ &= (\sigma - 4)\,(\,(V_1 . x_1^2) + (V_2 . x_1^2)\,)\end{aligned}$$

and $(V_i . x_1^2) = \frac{1}{\sigma - 4}\,(V_i . \gamma^2) < 0$, hence $(V_1 . x_1^2) = (V_2 . x_1^2) = -1$ and $V_1$, $V_2$ cannot split into further summands and $(V_i . \gamma^2) = 4 - \sigma$.

Now we can prove $\dim Y \geq 2$. If Y were a curve, we would have $\deg(Y) \geq 9 - 2\sigma \geq 3$, i.e. a generic hyperplane

$H \subset \mathbb{P}^{9-2}$ would meet the curve in at least 3 points and a generic divisor $D = \Phi^{-1} H$ would therefore split into at least 3 components, which is a contradiction.

By Bertini's theorems [8] the following two possibilities remain:

a) generic divisors of $/\gamma/$ are irreducible

b) $/\gamma/ = /V_1/ + V_2$ , $V_2$ a fixed component, and $\dim /V_1/ = 9 - 2\sigma > 0$

But if $/V_1/ + V_2$ is any linear subsystem of $/\gamma/$ such that $V_2 > 0$ and $\dim /V_1/ > 0$, we have $(V_1^2 \cdot \gamma) \geq 0$ since $\gamma$ is ample, and $(V_1 \cdot V_2 \cdot \gamma) = (V_1 \cdot (\gamma - V_1) \cdot \gamma) = (V_1 \cdot \gamma^2) - (V_1^2 \cdot \gamma) \geq 0$,

i. e.

$0 \leq (V_1^2 \cdot \gamma) \leq (V_1 \cdot \gamma^2) = 4 - \sigma$

Because of $(V_1 \cdot \gamma^2) = 4 - \sigma$ the cohomology class of $V_1$ has the form

$$v_1 = w + \sum_{i=1}^{\sigma} a_i x_i \qquad , a_i \in \mathbb{Z}$$

(by lemma 1), hence

$$v_1^2 = w^2 + 2w \sum_{i=1} a_i x_i + \left(\sum_{i=1} a_i^2\right) x_1^2$$

$$= w \sum_{i=1}^{\sigma} (2a_i - 1) x_i + \left[\left(\sum_{i=1}^{\sigma} a_i^2\right) - 1\right] x_1^2$$

and

$$(v_1^2 \cdot \gamma) = 2 \sum_{i=1}^{\sigma} (a_i - a_i^2) - \sigma + 2.$$

If $\sigma = 3$, these inequalities have no solution, for $\sigma < 3$ the only possible solutions are $v_1 = w$ , $v_1 = w + x_1$ or $v_1 = w + x_1 + x_2$.

If $\sigma = 2$, we would have $(V_1^2 . \gamma) = 0$ , i.e. $v_1^2 = 0$ , which contradicts the structure of the ring $H^*(P, \mathbb{Z})$.

In the case $\sigma = 1$ we would have $(V_1^3) = 0$ and $(V_1^2 . \gamma) = 1$, consequently, for any two distinct surfaces $V_1, V_1' \in /V_1/$, the intersection $V_1 \circ V_1'$ would be an irreducible curve.

Because of $H^1(P, \underline{O}_P) = 0$ we infere that $/V_1/ . V_1$ is a full linear system on the surface $V_1$ , and any curve $C \in /V_1/ . V_1$ is irreducible (because of $(V_1^2 . \gamma) = 1$). By the adjunction formula we find for the canonical sheaf (dualizing sheaf) on C:

$$\omega_C = \underline{O}_C \otimes \omega_P \otimes \underline{O}_P(2 V_1) \simeq \underline{O}_C \otimes \underline{O}_P(-2 V_2),$$

$$\deg(\omega_C) = - 2(C . V_2) = - 2(V_1^2 . V_2) = - 2$$

hence $C \simeq \mathbb{P}^1$.

In the same way we see that any curve $C' \in /V_1/ . V_2$ is isomorphic to $\mathbb{P}^1$.

Let $L_1$ be the line bundle $\underline{O}_P(V_1)$, then the sequence $0 \to L_1^{-1} \to \underline{O}_P \to \underline{O}_{V_1} \to 0$ corresponding to the section defining $V_1$ is exact and thus $H^0(P, L_1^{-1}) = H^1(P, L_1^{-1}) = 0$ and $H^1(V_1, \underline{O}_{V_1}) \cong H^2(P, L_1^{-1})$ , $H^2(V_1, \underline{O}_{V_1}) \simeq H^3(P, L_1^{-1})$ by the exact cohomology sequence.

By Serre duality $H^3(P, L_1^{-1}) \simeq H^0(P, \omega_P \otimes L_1)^* = 0$ and by the Hirzebruch-Riemann-Roch formula $\dim H^2(P, L_1^{-1}) =$

$= \frac{1}{6}(-V_1 + \gamma)^3 = 0$ (observe $p_1(P) = 0$).

Now let $\varphi$ be the section of $L_1$ corresponding to $V_1$, $C = V_1' \cdot V_1 \in /V_1/ \cdot V_1$ and $\varphi'$ the section of $L_1$ corresponding to $V_1'$. Then we have the exact sequences

$$0 \longrightarrow \underline{O}_P \xrightarrow{\varphi} L_1 \longrightarrow \underline{O}_{V_1} \otimes L_1 \longrightarrow 0$$

$$0 \longrightarrow \underline{O}_{V_1} \xrightarrow{1\otimes\varphi'} \underline{O}_{V_1} \otimes L_1 \longrightarrow \underline{O}_C \otimes L_1 \longrightarrow 0$$

and $\deg(\underline{O}_C \otimes L_1) = (C \cdot L_1) = (V_1^3) = 0$, hence $\underline{O}_C \otimes L_1 \simeq \underline{O}_C$ (since $C \simeq \mathbb{P}^1$). By the exact cohomology sequences we get therefore $\dim H^0(V_1\, \underline{O}_{V_1} \otimes L_1) = 2$ and $\dim H^0(P, L_1) = 3$.

This proves $\dim /V_1/ = 2$ and $\dim /V_1/ \cdot V_1 = 1$.

Then $/V_1/ \cdot V_1$ has no fixed component and because of $(V_1^3) = 0$ the linear system $/V_1/$ has no base points.

Because of $\dim /V_1/ = 2$ the surface $V_2$ cannot be a fixed component of $/\gamma/$, since this would imply $/\gamma/ = /V_1/ + V_2$, $\dim /\gamma/ = 2$ q.e.d.

<u>Lemma 4</u> If $\dim Y = 3$, we have

a) for the case $\sigma < 3$: The morphism $\Phi$ is birational and $7 - 2\sigma \leq \deg Y \leq 8 - 2\sigma$

b) for the case $\sigma = 3$: The morphism $\Phi$ is birational or of degree 2 (and $Y = \mathbb{P}^3$).

Proof: Because of $\operatorname{codim}(B) > 1$ we conclude (see [5]) that

$$\begin{aligned} H^0(P, L^{\otimes\nu}) &= H^0(X - B, L^{\otimes\nu}) \\ &\simeq H^0(X - B, \Phi^*\underline{O}_Y(\nu)) \\ &\simeq H^0(Y, (\Phi_*\underline{O}_X) \otimes_{\underline{O}_Y} \underline{O}_Y(\nu)), \end{aligned}$$

where $\underline{O}_Y(\gamma)$ is the restriction of the sheaf of hyperplanes $\underline{O}_{\mathbb{P}}9-2$ $(\gamma)$.

There holds $\underline{O}_Y \subset \Phi_* \underline{O}_X$ , and if d = deg $(\Phi)$, there exists an integer $n_o \gg 0$ and an embedding $\underline{O}_Y \oplus \underline{O}_Y(-n_o)^{d-1}$ $\subset \Phi_* \underline{O}_X$ (corresponding to a choice of a base of the field of rational functions of X over the subfield of rational functions of Y). Consequently

$$\dim H^o(Y, \underline{O}_Y(\gamma)) + (d-1) \dim H^o(Y, \underline{O}_Y(\gamma - n_o))$$

$$\dim H^o(X, L^{\otimes \gamma}) = \frac{2(4-\sigma)}{6}(v+1)^3 + \frac{2(\sigma-1)}{6}(\gamma+1)$$

Since for $\gamma \gg 0$ the function $\gamma \mapsto \dim H^o(Y, O_Y(\gamma))$ is polynomial with the leading coefficient $\frac{\deg(Y)}{3!}$ , this inequality implies

$$d \deg(Y) \leq 2(4-\sigma)$$

As on the other hand $\deg(Y) \geq 7 - 2\sigma$ , the lemma follows immediately.

## 8. Proof of the properties (B), (C)

In this section we will prove that $|\gamma|$ has no base points and that the morphism $\Phi$ is a finite morphism.

Lemma 5 If $|\gamma|$ contains a smooth irreducible surface V, the base locus B is empty and dim Y = 3. Furthermore $\Phi/V$ is finite and a closed embedding if $\sigma \leq 2$.

Proof: Step I: V is a rational surface. By the adjunction formula

$$\omega_V = \omega_P \otimes \underline{O}_P(V) \otimes \underline{O}_V \simeq \underline{O}_P(-V) \otimes \underline{O}_V$$

(since $\omega_P = \underline{O}_P(-2V)$).

Therefore the anticanonical class $\omega_V^{-1}$ is ample and because of $H^1(P, \underline{O}_P) = 0$ the anticanonical linear system is $/\omega_V^{-1}/ = /\gamma/ . V$.

By the Lefschetz theorem on hyperplane sections (see [13]) it follows that $H^1(V, \mathbb{Z}) = 0$.

By Castelnuovo's criterion we can consequently conclude that V is rational (since $P_2 = \dim H^0(V, \omega_V^{\otimes 2}) = 0$ , $q = \frac{1}{2} \dim H^1(V, \mathbb{R}) = 0$) ([2]).

Step II V is the blowing up of $\mathbb{P}^2$ in $2\sigma + 1$ points $P_0, \dots, P_2$ such that no 3 of these points are colinear and no 6 of these points lie on a quadric.

The minimal rational surfaces are the surfaces $\mathbb{P}^2$, $F_0$, $F_2$, $F_3, \dots$, where $F_n = P(\underline{O}_{\mathbb{P}^1} \oplus \underline{O}_{P^1}(n))$ (rational scrolls).

Since $(\omega_{\mathbb{P}^2}{}^2) = 9$ , $(\omega_{F_n}{}^2) = 8$, and $(\omega_V{}^2) = (\gamma^3) = 8 - 2\sigma$ , the surface V is obtained from $\mathbb{P}^2$ by blowing up $2\sigma + 1$ points or from a surface $F_n$ by blowing up $2\sigma$ points (since the blowing up of one point diminishes $(\omega_V{}^2)$ by 1).

On the ruled surfaces $F_n \longrightarrow \mathbb{P}^1$ there exists a distinguished section $s : \mathbb{P}^1 \longrightarrow F_n$ such that the curve $B = s(P^1) \subseteq F_n$ has the property $(B^2) < 0$. The Picard group of $F_n$ is $\mathbb{Z}^2$ with generators b = class of B and f = class of a fibre F, and

$(F^2) = 0$ , $(F . B) = 1$ , $(B^2) = - n$.

The divisors $- 2B - (n + 2)F$ are canonical (i.e. represent $c_1(\omega_{F_n})$ ), and if $\psi : V \longrightarrow V'$ is the blowing up of one point and $E \subset V$ the exceptional curve on $V$, an isomorphism

$$\omega_V \cong \psi^* \omega_{V'} \otimes \mathcal{O}_V(E)$$

holds.

If $\psi : V \longrightarrow F_n$ is a sequence of blowing up of points and if $E$ denotes the exceptional divisor of $\psi$, then the divisor $-2\,\psi^* B - (n+2)\,\psi^* F + E$ is canonical on $V$.

If $\omega_V^{-1}$ is ample, we have

$$\begin{aligned} 0 < (\omega_V^{-1} \cdot \psi^* B) &= ((2\,\psi^* B + (n+2)\,\psi^* F - E) \cdot \psi^* B) \\ &= 2(B^2) + (n+2)\,(F \cdot B) - (E \cdot \psi^* B) \\ &= -2n + n + 2 \\ &= -n + 2 \end{aligned}$$

Consequently only $n = 0$ is possible in our case, $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

If we blow up one point on the surface $F_0$, we get a surface which is isomorphic to the blowing up of two points of $\mathbb{P}^2$. The surface $V$ is therefore always obtained by blowing up $\mathbb{P}^2$ in $2\sigma + 1$ points $P_0, \ldots, P_{2\sigma+1}$ (perhaps infinitely near points).

Let $L$ be a line in $\mathbb{P}^2$ and $\psi : V \longrightarrow \mathbb{P}^2$ the sequence of blowing up, $E$ the exceptional curve. The divisors $-3L$ on $V$ and $-3\,\psi^* L + E$ on $V$ are canonical, if $\widetilde{L}$ is the strict transform of $L$ on $V$, then

$$\begin{aligned} 0 < ((3\,\psi^* L - E) \cdot \widetilde{L}) &= 3(\psi^* L \cdot \widetilde{L}) - (E \cdot \widetilde{L}) \\ &= 3 - (E \cdot \widetilde{L}), \end{aligned}$$

hence $(E \cdot \widetilde{L}) < 3$.

Therefore $L$ cannot contain more than two points of

$\{P_0, \dots P_2\}$ .

Similarly, if Q is a quadric in $\mathbb{P}^2$, the divisor $-\psi^* Q - \psi^* L + E$ on V is canonical, if $\tilde{Q}$ is the strict transform of Q on V, then

$$0 < ((\psi^* Q + \psi^* L - E) . \tilde{Q}) = 4 + 2 - (E . \tilde{Q})$$

$$(E . \tilde{Q}) < 6,$$

hence Q cannot contain more than five points of $\{P_0, \dots, P_2\}$.

<u>Step III</u> Description of the surface V and the linear system $/\omega_V^{-1}/$

The linear system $/\omega_V^{-1}/$ corresponds to the linear system $\Lambda$ of all cubics in $\mathbb{P}^2$ passing through the points $P_0, \dots P_{2\sigma}$ ($\sigma$ = 1, 2 or 3) (the cubic C corresponds to the divisor $\psi^* C - E$ on V). If $\sigma \leq 2$, this linear system defines an embedding of V.

Now consider the case $\sigma = 3$.

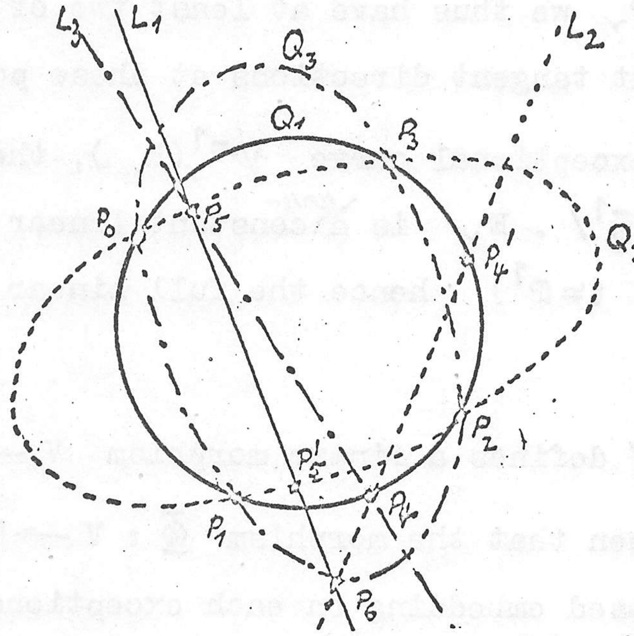
It is easy to see that for any point $P \neq P_\nu$ there exists a cubic through $P_0, \dots, P_6$ which does not contain the point P.

Hence $/\omega_V^{-1}/$ has no base point outside the exceptional locus.

For any point $P_\nu$ there are cubics C, C' through $P_0, \dots, P_6$ which are non-singular in $P_\nu$ and have different tangent directions in $P_\nu$ . Therefore $/\omega_V^{-1}/$ has no base points on V.

Let $Q_\nu$ be the quadric through the points $P_0$, $P_1$, $P_2$, $P_3$ and $P_{3+\nu}$ ($\nu$ = 1, 2, 3) and $L_\nu$ the line through the remaining 2 points $\{P_4, P_5, P_6\} \setminus \{P_{3+\nu}\}$.

The linear system $\Lambda$ is spanned by the cubics $C_\nu = Q_\nu + L_\nu$ ($\nu = 1, 2, 3$).




Consider for example $C_1 \cdot C_2 = Q_1 \cdot Q_2 + Q_1 \cdot L_2 + Q_2 \cdot L_1 + L_1 \cdot L_2$,

then

$$C_1 \cdot C_2 = \sum_{\nu=0}^{6} P_\nu + P_4' + P_5'$$

if $Q_1 \cdot L_2 = P_4 + P_4'$ , $Q_2 \cdot L_1 = P_5 + P_5'$.

If for example $P_4' \in C_3 = Q_3 + L_3$, then $P_4' \in Q_1 \cap Q_3 \cap L_2$ or $P_4' \in L_2 \cap L_3 \cap Q_1$. In the first case $P_4'$ would be a point $P_\alpha$ , $\alpha = 0, 1, 2$ or $3$, hence $P_\alpha$, $P_4$, $P_6$ would be colinear, which is impossible, in the second case $P_4'$ would be the point $P_4$.

The tangent directions of the cubics $C_\nu$ in the points $P_0$, $P_1$, $P_2$, $P_3$ are different, since the cubics transversally meets in these points (observe that no line $L_\nu$ passes through these points).

In $P_4$ (resp. $P_5$, resp. $P_6$) the cubics $C_2$, $C_3$ (resp. $C_1$, $C_3$, resp. $C_1$, $C_2$) intersect transversally.

In all points $P_\nu$ we thus have at least two of the cubics having different tangent directions at these points.

If $E_\nu$ is the exceptional curve $\psi^{-1}(P_\nu)$, then $(\omega_V^{-1} \cdot E_\nu) = 1$, hence $|\omega_V^{-1}| \cdot E_\nu$ is a non-constant linear system of degree 1 on $E_\nu$ $(\simeq \mathbb{P}^1)$ hence the full linear system $|\omega_V^{-1} \otimes O_{E_\nu}|$.

<u>Step IV</u> $|\omega_V^{-1}|$ defines a finite morphism $V \longrightarrow \mathbb{P}^2$ $(\sigma = 3)$

We have just seen that the morphism $\Phi : V \longrightarrow \mathbb{P}^2$ defined by $|\omega_V^{-1}|$ is a closed embedding on each exceptional curve $E$ .

Let $\tilde{C}$ be an irreducible curve on $V$, $\tilde{C} \neq E_\nu$ for $\nu = 0,\ldots,6$ , then it is the strict transform of a plane curve $C$. If $C$ has degree $d$ and multiplicity $m_\nu$ at the point $P_\nu$, we can compute $\dim |\omega_V^{-1}| \cdot \tilde{C}$ as follows:

$$\dim |\omega_V^{-1}| \cdot \tilde{C} = \dim H^0(V,\omega_V^{-1}) - \dim H^0(V,\omega_V^{-1}(-\tilde{C})) - 1$$

$$= 2 - \dim H^0(V,\omega_V^{-1}(-\tilde{C}))$$

and $\tilde{C} = \psi^* C - \sum_{\nu=0}^{6} m_\nu E_\nu$ , hence

$$\omega_V^{-1}(-\tilde{C}) \simeq \psi^*(\omega_{\mathbb{P}^2}^{-1}(C)) \otimes \underline{O}_V \Big(\sum_{\nu=0}^{6} (m_\nu - 1) E_\nu\Big)$$

$$\simeq \psi^* \underline{O}_{P^2}(3-d) \otimes \underline{O}_V \Big(\sum_{\nu=0}^{6} (m_\nu - 1) E_\nu\Big)$$

$$H^0(V,\omega_V^{-1}(-\tilde{C})) = H^0\Big(\mathbb{P}^2, \underline{O}_{\mathbb{P}^2}(3-d) \otimes \psi_* \underline{O}_V \Big(\sum_{\nu=0}^{6} (m_\nu - 1) E_\nu\Big)\Big)$$

If $I_\nu$ denotes the sheaf of ideals of the point $P_\nu$, then

$$\psi_* \underline{O}_V \Big( \sum_{\nu=0}^{6} (m_\nu - 1) E_\nu \Big) = \bigcap_{P_\nu \notin C} I_\nu , \text{ hence } H^0(V, \omega_V^{-1}$$

$(-\tilde{C})) = 0$ if $d > 3$ or if $d = 3$ and $C \in \Lambda$.

If $d = 3$ and $C \in \Lambda$, we get $\dim H^0(V, \omega_V^{-1}(-\tilde{C})) = 1$, and if $d < 3$ the sections of $H^0(V, \omega_V^{-1}(-\tilde{C}))$ correspond to the curves of degree $3 - d$ passing through all points $P_\nu \notin C$.

Hence $\dim H^0(V, \omega_V^{-1}(-\tilde{C})) \leq 1$ for any curve $\tilde{C}$ on $V$ and

$$\dim |\omega_V^{-1}| . \tilde{C} \geq 1.$$

Therefore no curve on $C$ is contracted to a point under the morphism $\Phi$, hence $\Phi$ is finite q.e.d.

<u>Corollary</u> If the linear system $|\gamma|$ contains a smooth irreducible surface $V$, the morphism $\Phi : P \to \mathbb{P}^{9-2\sigma}$ is finite.

Proof: The linear system $|\gamma|$ has no base point, consequently $\Phi$ is defined everywhere on $P$, and $\Phi / V$ is finite for any smooth irreducible surface $V \in |\gamma|$.

Let $E \subset P$ be the locus of points $x \in P$ which are not isolated in their fibre $\Phi^{-1}(\Phi(x))$, by Zariski's main theorem this is a Zariski closed subset in $P$ (cf. for example [10], [11]). If $E \neq \phi$, the image $\Phi(E)$ is of dimension $\leq 1$, if $H \subset \mathbb{P}^{9-2\sigma}$ is a generic hyperplane, the set $H \cap \Phi(E)$ is finite, and $\Phi^{-1}(H) = V \in |\gamma|$ is smooth and irreducible (by Bertini's theorem). But we have seen that $\Phi / V$ is finite, hence $V \cap E$ must be finite and thus dim E = 1, dim $\Phi(E) = 0$. In this case $H \cap \Phi(E) = \phi$ for any sufficiently general hyperplane, hence $V \cap E = \phi$, which is a contradiction since V is ample.

<u>Lemma 6</u> dim Y = 3 and if B is not empty, then B is an irreducible curve. In this case, the variety $Y \subset \mathbb{P}^{9-2\sigma}$ is of degree $7 - 2\sigma$ and the morphism $\Phi : \mathbb{P} \setminus B \longrightarrow Y$ is birational.

Proof: The assumption dim Y = 2 would imply $\deg(Y) \geq 8 - 2\sigma$, hence any two generic surfaces of $/\gamma/$ would have an intersection consisting of at least $8 - 2\sigma$ components, and at least $9 - 2\sigma$ components if $/\gamma/$ has a base curve. But since $\gamma$ is ample and $(\gamma^3) = 8 - 2\sigma$, the second case is impossible, i.e. $/\gamma/$ has at most finitely many base points and 2 generic surfaces of $/\gamma/$ have an intersection $C_1 + C_2 + \dots + C_{8-2\sigma}$. If V is a different surface of $/\gamma/$, then

$$(\gamma^3) = 8 - 2\sigma = (C_1 \cdot V) + \dots + (C_{8-2\sigma} \cdot V),$$

hence $(C_i \cdot V) = 1$ for $i = 1, \dots, 8 - 2\sigma$. Then each curve $C_i$ has exactly one point in common with the surface V and they intersect transversally in this point. Especially these intersection points are non singular on $C_i$ and on V. Therefore any base point of $/\gamma/$ must be a nonsingular point on V for $V \in /\gamma/$. By Bertini's theorem it follows that there exist smooth irreducible surfaces $V \in /\gamma/$, but then, by lemma 5, we would have dim Y = 3.

Hence we have proved dim Y = 3, $7 - 2\sigma \leq \deg Y \leq 8 - 2\sigma$.

Assume that $C \subset B$ is an irreducible curve, then for any 2 surfaces $V_1, V_2 \in /\gamma/$ we have

$$V_1 \cdot V_2 = C + D$$

where D is some effective cycle of codimension 2 and $(D \cdot \gamma) \geq 7 - 2\sigma$ (since $\deg Y \geq 7 - 2\sigma$).

Then we get

$$(\gamma^3) = 8 - 2\sigma = (C \cdot \gamma) + (D \cdot \gamma),$$

hence $(C \cdot \gamma) = 1$

In this case $B = C$ and $\deg Y = 7 - 2\sigma$.

Now assume $\dim B = 0$. Because of $(\gamma^3) = 8 - 2\sigma$, $\deg(Y) = 7 - 2\sigma$ the base locus B must be empty or equal to a point, where any 3 generic surfaces of $/\gamma/$ intersect transversally. Especially the base point is nonsingular on any $V \in /\gamma/$, by Bertini's theorem $/\gamma/$ contains therefore smooth irreducible surfaces V, but then by lemma 5 the base locus B must be empty q.e.d.

## 9. The case $\sigma = 1$

We will prove

Theorem 3: If $X^4$ is an oriented self-dual compact Einstein manifold of positive scalar curvature and $H^2(X, \mathbb{Z}) \neq 0$, then $X^4$ is diffeomorphic to the complex projective plane $\mathbb{P}^2$ and the projective spinor bundle $P^- = P$ is analytically isomorphic to the flag manifold $\mathbb{F}(1,2) \subset \mathbb{P}^2 \times \mathbb{P}^2$. The embedding is induced by the linear system $/\gamma/$.

Proof: Consider again $P^- = P$, $/\gamma/$ and $\Phi$.

Step I: $/\gamma/$ has no base points. Assume the contrary, then $/\gamma/$ has a base curve B. In this case Y is a variety in $\mathbb{P}^7$ of dimension 3 and degree 5 = codim Y + 1. By [20] it follows that Y is one of the following varieties:

A rational scroll $\mathbb{P}(E) \longrightarrow \mathbb{P}^1$, where

a) $E = \underline{O}_{\mathbb{P}^1} \oplus \underline{O}_{\mathbb{P}^1} \oplus \underline{O}_{\mathbb{P}^1}(2)$

b) $E = \underline{O}_{\mathbb{P}^1} \oplus \underline{O}_{\mathbb{P}^1}(1) \oplus \underline{O}_{\mathbb{P}^1}(1)$

or a cone over a rational scroll $\mathbb{P}(E) \longrightarrow \mathbb{P}^1$, where

c) $E = \underline{O}_{\mathbb{P}^1} \oplus \underline{O}_{\mathbb{P}^1}(3)$ , $\mathbb{P}(E) = F_3 \subset \mathbb{P}^6$

d) $E = \underline{O}_{\mathbb{P}^1}(1) \oplus \underline{O}_{\mathbb{P}^1}(2)$ , $\mathbb{P}(E) = F_1 \subset \mathbb{P}^6$ ,

or finally

e) a cone over the curve $\mathbb{P}^1 \subset \mathbb{P}^5$
(5-fold Veronese embedding).

The embedding of the scrolls is given by the following linear system $/H/$: Let $L$ be the relative bundle of hyperplane sections of $\mathbb{P}(E) \xrightarrow{\pi} \mathbb{P}^1$ (such that $\pi^* E \rightarrow L$ is surjective), then $H = L \otimes \pi^* \underline{O}_{\mathbb{P}^1}(1)$.

But in each of these cases we could find a linear subsystem $/V'/ + V'' \subset /\gamma/$ , $V'' > 0$ , $\dim /V'/ \geqslant 4$. In the cases a) - d) we can take $/V'/ = /L/$ in the notation introduced above, in the case e) we can take the linear system corresponding to $/P_1 + P_2 + P_3 + P_4/$ on $\mathbb{P}^1$. (Observe that $H^0(Y, L) = H^0(\mathbb{P}^1, E) \simeq \mathbb{C}^5$!)

Hence $/\gamma/$ cannot have base points, by Lemma 3.

<u>Step II</u>: The morphism $\Phi$ is a closed embedding $\Phi$: $P \longrightarrow Y \subset P^7$, deg $Y = 6$.

Since for a generic hyperplane $H \subset \mathbb{P}^7$ the restriction of $\Phi$:

$V = \Phi^{-1}(H) \longrightarrow Y \cap H$ is an isomorphism (by lemma 5), the singular locus of $Y$ is finite. Furthermore $V$ is obtained by blowing up 3 non-colinear points of $\mathbb{P}^2$.

From the exact sequences

$$0 \longrightarrow \underline{O}_P((n-1)V) \longrightarrow \underline{O}_P(nV) \longrightarrow \underline{O}_V(n) \longrightarrow 0$$

$$0 \longrightarrow \underline{O}_Y(n-1) \longrightarrow \underline{O}_Y(n) \longrightarrow \underline{O}_V(n) \longrightarrow 0$$

(we consider $V$ to be embedded into $\mathbb{P}^7$) we get

$$\chi(\underline{O}_P(nV)) - \chi(\underline{O}_P((n-1)V)) = \chi(\underline{O}_Y(n)) - \chi(\underline{O}_Y(n-1))$$

$$= \chi(\underline{O}_V(n))$$

$$= 3\,n\,(n+1) + 1$$

and consequently $\chi(\underline{O}_P(nV)) = \chi(\underline{O}_Y(n)) = (n+1)^3$.

This proves that $\Phi$ is a closed embedding.

<u>Step III</u>: $\mathbb{P}$ is the intersection of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ with a linear subspace of $\mathbb{P}^8$.

Let $L_1$, $L_2$ be holomorphic line bundles on $P$ such that $c_1(L_1) = w$, $c_1(L_2) = w + x$, then

$\omega_P \simeq L_1^{-2} \otimes L_2^{-2}$.

We will show $\dim /L_1/ = \dim /L_2/ = 2$. By lemma 3 it is sufficient to show $\dim /L_1/ > 0$, $\dim /L_2/ > 0$.

By the Hirzebruch-Riemann-Roch theorem we get

$\chi(L_1) = \chi(L_2) = 3$.

If we choose $n$ big enough, the line bundles $L_1 \otimes (L_1 \otimes L_2)^{\otimes(n+2)}$ and $L_1 \otimes (L_1 \otimes L_2)^{\otimes n}$ will be ample and the linear system $/(L_1 \otimes L_2)^{\otimes n}/ = /n\gamma/$ will contain a smooth connected surface $W$ (by Bertini's theorem).

Tensorizing the exact sequence

$0 \longrightarrow \mathcal{O}_P(-W) \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_W \longrightarrow 0$ by $L_1^{-1} \otimes \omega_P$

we get an exact sequence

$$0 \longrightarrow L_1^{-\otimes(n+3)} \otimes L_2^{-\otimes(n+2)} \longrightarrow L_1^{-1} \otimes \omega_P \longrightarrow \mathcal{O}_W \otimes L_1^{-1} \otimes \omega_P \longrightarrow 0$$

and by the adjunction formula it holds that

$$\mathcal{O}_W \otimes L_1^{-1} \otimes \omega_P \simeq \omega_W \otimes \mathcal{O}_P(-W) \otimes L_1^{-1}$$
$$\simeq \omega_W \otimes L_1^{-\otimes(n+1)} \otimes L_2^{-\otimes n}$$

By Kodairas vanishing theorem it follows that

$H^q(P, L_1^{-\otimes(n+3)} \otimes L_2^{-\otimes(n+2)}) = 0$ for $q \leq 2$ and

$H^1(W, \omega_W \otimes L_1^{-\otimes(n+1)} \otimes L_2^{-\otimes n}) = 0$ by Serre duality. Consequently $H^1(P, \omega_P \otimes L_1^{-1}) = 0$ and $H^2(P, L_1) = 0$ by Serre duality, hence

$\dim H^0(P, L_1) \geq 3$ and by symmetry

$\dim H^0(P, L_2) \geq 3$.

Now we can infere (by lemma 3) that $/L_1/$ and $/L_2/$ define morphisms $\psi_1: P \longrightarrow \mathbb{P}^2$, $\psi_2: P \longrightarrow \mathbb{P}^2$ and

$\psi = (\psi_1, \psi_2) : P \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ satisfying

$$\psi^*(\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\mathbb{P}^2}(1)) = \psi^*(\mathcal{O}_{\mathbb{P}^8}(1)) = L_1 \otimes L_2$$

Because of $/L_1 \otimes L_2/ = /\gamma/$, the morphism $\psi : P \rightarrow \mathbb{P}^8$ factorizes through a linear subspace $H \cong \mathbb{P}^7$ and the embedding $\phi \; P \longrightarrow \mathbb{P}^7$.

Then $P$ considered as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$ is defined by a bilinear form $H^0(P, L_1) \otimes H^0(P, L_2) \longrightarrow \mathbb{C}$. Since $P$ is nonsingular, this bilinear form must be of rank 3, i.e. in suitable homogenous coordinates the variety $P \subset \mathbb{P}^2 \times \mathbb{P}^2$

is defined by an equation of

$$U_oV_o + U_1V_1 + U_2V_2 = 0.$$

<u>Step IV:</u> $X^4$ is diffeomorphic to $\mathbb{P}^2(\mathbb{C})$

If $H$ is a line in $\mathbb{P}^2$, then the restriction $\Psi_1 : V = \Psi_2^{-1}(H) \longrightarrow \mathbb{P}^2$ is the blowing up of a point of $\mathbb{P}^2$.

If for example $H$ is defined by the equation

$$V_o + a\,V_1 + b\,V_2 = 0,$$

then $V \subset \mathbb{P}^2 \times H = \mathbb{P}^2 \times \mathbb{P}^1$ is defined by the equation (using $V_1$, $V_2$ as homogenous coordinates on $H$) by

$$(U_1 - aU_o)\,V_1 + (U_2 - bU_o)\,V_2 = 0$$

i.e. $\Psi_1/V$ is the blowing up of the point $(U_o : U_1 : U_2) = (1 : a : b)$.

Consider a fibre $F_p$ of $P \longrightarrow X^4$ $(p \in X^4)$, then $/L_2/.F_p$ is a linear system without a base point on $F_p \cong \mathbb{P}^1$, and because of $(F_p \cdot L_2) = (-x \cdot L_2) = 1$ this linear system is of degree 1. Therefore the restriction of $\Psi_2$ to $F_p$ maps $F_p$ isomorphically onto a line $H_p \subset \mathbb{P}^2$. (The same is true for $\Psi_1$.) We claim that $H_p \neq H_q$ if $p \neq q$.

Assume $H_p = H_q = H$, then $F_p$, $F_q$ are strict transforms of lines of $\mathbb{P}^2$ under the morphism $\Psi_1 : V = \Psi_2^{-1}(H) \longrightarrow \mathbb{P}^2$.

Since they are mapped onto $H$ under $\Psi_2$, these lines cannot pass through the centre of the blowing up $\Psi_1$, consequently $F_p$, $F_q$ have a non-empty intersection, hence $p = q$.

Thus we can define an injective map

$\varphi : X^4 \longrightarrow \mathbb{P}^2$ = dual space to $\mathbb{P}^2$ by $\varphi(p) = H_p = \Psi_2(F_p)$.

To see that $\varphi$ is a diffeomorphism we express this map in coordinates.

For this purpose we choose an open set $U \subset X^4$ such that P/U is isomorphic to the trivial $\mathbb{P}^1$-bundle and a trivialization $P/U \simeq U \times \mathbb{P}^1$. Choosing homogenous coordinates on $\mathbb{P}^1$ and using the trivialization we get $\mathcal{C}^\infty$-sections of P over U, $p(x)$ corresponding to $(x, o)$ and $q(x)$ corresponding to $(x, \infty)$.

If $\psi_0, \psi_1, \psi_2$ is the base of $H^0(X, L_2)$ corresponding to the homogenous coordinates $V_0, V_1, V_2$, the map $\varphi$ on U is expressed by

$$\varphi/U = ((p^*\psi_1 \otimes q^*\psi_2 - p^*\psi_2 \otimes q^*\psi_1) : (p^*\psi_2 \otimes q^*\psi_0 - p^*\psi_0 \otimes q^*\psi_2) : (p^*\psi_0 \otimes q^*\psi_1 - p^*\psi_1 \otimes q^*\psi_0))$$

in the following sense: Consider a point $O \in U$ and the fibre $F_O$, since the restriction map $H^0(P, L_2) \longrightarrow H^0(F_O, L_2 \otimes \underline{O}_{F_O})$ is surjective (because $/L_2/ \cdot F_O$ is a full linear system), we can choose a holomorphic section $\psi$ of $L_2$ in a neighbourhood of the fibre $F_o$ such that $\psi$ generates the line bundle $L_2$ in the points $p(o)$ and $q(o)$, i.e. $\psi_i = f_i\psi$ in a neighbourhood of $p(o)$ and $q(o)$, where $f_o, f_1, f_2$ are holomorphic functions. Then $\varphi(x)$ is the line defined by the equation

$$[f_1(x,o)\, f_2(x,\infty) - f_1(x,\infty)\, f_2(x,o)]\, V_o + [f_2(x,o) f_o(x,\infty) - f_2(x,\infty)\, f_o(x,o)]\, V_1 + [f_o(x,o)\, f_1(x,\infty) - f_o(x,\infty) f_1(x,o)]\, V_2 = 0$$

for points x in a neighbourhood of O.

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