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Cohomology with compact supports for real analytic spaces

by Mihnea Colţoiu

Introduction

Given X a paracompact real analytic space and $F\in Coh(X)$ it is well known that $H^q(X,F)=0$ for all $q\geq 1$. A similar statement holds if X is a complex Stein space. If X is a Stein manifold we have the following result (see ([1]) for the proof):

Theorem Let (X,O) be a complex Stein manifold of dimension n and Ω the sheaf of differential forms of type (n,o) with analytic coefficients. Then, for any coherent analytic sheaf F on X and for all $q\geq 1$, $Ext_O^{n-q}(X;F,\Omega)$ has a natural structure of Fréchet-Schwartz space and $H_c^q(X,F)$ is algebraically isomorphic to the topologic dual of $Ext_O^{n-q}(X;F,\Omega)$.

We deduce the following:

Corollary Let (X,O) be a complex Stein manifold of dimension n and F a locally free sheaf of finite rank. Then $H_c^q(X,F)=0$ if $q=n$ and $H_c^n(X,F)$ is algebraically isomorphic to the topologic dual of $Hom_O(F,\Omega)$.

The situation is entirely different for the invariants $H_c^q(X,\cdot)$ if X is a real analytic space. It will follow that $H_c^q(X,F)=0$ if $F\in Coh(X)$ and $q\geq 2$. However one doesn't generally get $H_c^1(X,F)=0$. The purpose of this paper is to give conditions on F under which $H_c^1(X,F)$ vanishes.

1. The torsion of coherent analytic sheaves

The problems discussed in this paragraph may be found in ([1]) or ([3]). Their proofs are given there in the case of complex analytic spaces but as one may easily show they also hold in the real case.

Let's resume some elementary notions:

Let A be an integral domain(commutative and with 1-element) and let M be an A-module. Let tM denote the torsion of M i.e.

$$tM:=\{m\in M \mid (\exists)a\in A\setminus\{o\} \text{ such that } am=o\}$$

If K is the field of quotients of A then tM is the kernel of the canonical map $M\to M\otimes_A K$.

Let X be a real analytic space locally irreducible(i.e. $O_{X,x}$ is integral for all $x\in X$) and let $\mathcal{M}$ denote the sheaf of germs of meromorphic sections on X. If $F\in Coh(X)$ let tF denote the torsion subsheaf of F i.e. the kernel of the canonical map $F\to F\otimes_{O_X}\mathcal{M}$. One gets $(tF)_x=tF_x$, for all $x\in X$. We say that F is torsion-free iff $tF=o$.

We shall need the following results:

a) $tF\in Coh(X)$ if $F\in Coh(X)$.

b) If $F\in Coh(X)$ is torsion-free then for all $x\in X$ there exists an open neighborhood U_x containing x and there exists a natural number $n=n(x)\in N$ such that $F|_{U_x}$ is isomorphic to a subsheaf of $O_X^n|_{U_x}$.

The properties a) and b) also follow from the third paragraph.

Definition 1. Let (X,O_X) be a real analytic space and $F\in Coh(X)$. We say that F verifies "the principle of analytic continuation" iff whenever $U\subset X$ is an open subset, $x\in U$ and $s,t\in\Gamma(U,F)$ such that $s_x\neq t_x$ it follows that $s_y\neq t_y$ for all y in a neighborhood of x.

Examples 1) If X is locally irreducible then O_X verifies "the principle of analytic continuation". Indeed, it suffices to show that, if $U\subset X$ is an open subset, $x\in U$ and $s\in\Gamma(U,O_X)$, then $s_x\neq o$ implies $s_y\neq o$ for all y in a neighborhood of x. This follows immediately by considering the morphism $O_X|_U\xrightarrow{\cdot s}O_X|_U$ (multiplication by s) and using the coherence of O_X.

2) If X is locally irreducible and $F\in Coh(X)$ is torsion-free then F verifies "the principle of analytic continuation". This statement follows from example 1) and b).

2. The cohomology groups $H^q_c(X,\cdot)$

Let X be a real analytic space and $F\in Coh(X)$. Let's denote $H^q_c(X,F)$ the cohomology groups with compact supports and coefficients in F. One may find some further details about these invariants in ([1]).

We shall need the following result:

Theorem 1. Let X be a paracompact real analytic space and $F\in Coh(X)$. Then:

A) For all $x\in X$, F_x is generated as $O_{X,x}$-module by the image of the natural map $\Gamma(X,F)\longrightarrow F_x$.

B) For all $q\geq 1$, $H^q(X,F)=o$.

The proof is based on the existence of a complexification $\tilde{X}$ and on the fact that X has a fundamental system of open neighborhoods which are Stein in $\tilde{X}$.(see ([6]) for the proof).

We remark that theorem 1.B implies the vanishing of $H^q_c(X,F)$ for all $q\geq 2$. Indeed, if K is a compact subset of X let $H^q_K(X,\cdot)$ denote the cohomology groups with supports in K. Since there exists a canonical isomorphism

$$\varinjlim_K H^q_K(X,F) \to H^q_c(X,F)$$

it suffices to prove that $H^q_K(X,F)=0$ for all compact subsets $K\subset X$ and for all $q\geq 2$. But this statement follows from the exact sequence

$$H^{q-1}(X\setminus K,F)\to H^q_K(X,F)\to H^q(X,F)$$

using theorem 1.B.

<u>Definition 2.</u> If $F\in Coh(X)$, we say that F has the property (P) iff for all $x\in X$ and for all $s\in\Gamma(X\setminus\{x\},F)$, there exists $\tilde{s}\in\Gamma(X,F)$ such that $s=\tilde{s}$ on $X\setminus K$ if K is a sufficiently large compact subset of X.

<u>Remark</u> If $H^1_c(X,F)=0$ then F has the property (P). Indeed, for all compact subsets $K\subset X$ one gets the exact sequence

$$\Gamma(X,F)\to\Gamma(X\setminus K,F)\to H^1_K(X,F)$$

Taking the direct limit as K runs over the compact subsets of X and using the fact that $H^1_c(X,F)=0$, one gets the exact sequence

$$\Gamma(X,F)\to\varinjlim_K\Gamma(X\setminus K,F)\to 0$$

which tells us that for all compact subsets $Q\subset X$ and for all $s\in\Gamma(X\setminus Q,F)$ there exists $\tilde{s}\in\Gamma(X,F)$ such that $s=\tilde{s}$ outside a sufficiently large compact subset $K\subset X$.

Taking $Q=\{x\}$ it follows that F has the property (P).

Lemma 1. Let X be a paracompact real analytic space and $x\in X$. Then there exists $s\in\Gamma(X,O_X)$ such that $s(x)=o$ and $s(y)\neq o$ for all $y\in X, y\neq x$.

Proof Let m_x be the maximal ideal of $O_{X,x}$ and $a_1,\ldots,a_n$ a set of generators for m_x. Then the sheaf of ideals I defined by:

$$I_y=\begin{cases} m_x^2 & \text{if } y=x \\ O_{X,y} & \text{if } y\neq x \end{cases}$$

is coherent and using the theorem 1.B. it follows that one may find $f_1,\ldots,f_n\in\Gamma(X,O_X)$ such that $f_{i,x}-a_i\in m_x^2$, for all $i=1,\ldots,n$. One deduces, using Nakayama's lemma, that $f_{1,x},\ldots,f_{n,x}$ generate m_x. Let $f:X\to R^n$ denote the morphism induced by the sections $f_1,\ldots,f_n$. It follows that $f_x^*:O_{n,o}\to O_{X,x}$ is surjective, hence f is an immersion in a neighborhood of x. In particular x is an isolated point of the set $Y=f^{-1}(o)=\{z\in X \mid f_1(z)=\ldots=f_n(z)=o\}$
Put $O_Y=O_X/(f_1,\ldots,f_n)\mid Y$ and let's consider the section $g_{n+1}\in\Gamma(Y,O_Y)$ defined such as follows:

$$g_{n+1,y}=\begin{cases} o & \text{if } y=x \\ 1 & \text{if } y\neq x \end{cases}$$

From theorem 1.B. it follows that the canonical map $r_Y^X:\Gamma(X,O_X)\to\Gamma(Y,O_Y)$ is surjective. Let's consider $f_{n+1}\in\Gamma(X,O_X)$ such that $r_Y^X(f_{n+1})=g_{n+1}$ and put $s=f_1^2+\ldots+f_n^2+f_{n+1}^2$. One may easily verify that s satisfies the conditions of the lemma.

<u>Proposition 1</u>. Let X be a paracompact real analytic space which is connected, noncompact and locally irreducible. Suppose $F\in Coh(X)$ such that $H^1_c(X,F)=o$. Then $F=tF$.

<u>Proof</u> Put $G=F/tF$. We must show that $G=o$. From the exact sequence $o\to tF\to F\to G\to o$, we deduce the exact sequence

$$H^1_c(X,F)\to H^1_c(X,G)\to H^2_c(X,tF)$$

Since $H^1_c(X,F)=o$ and $H^2_c(X,tF)=o$ it follows that $H^1_c(X,G)=o$. In particular G has the property (P). Suppose that there exists $x\in X$ such that $G_x\neq o$. From theorem 1.A. there exists $u\in\Gamma(X,G)$ with $u_x\neq o$. Let's consider s built as in lemma 1. For all $i\in N$, we get $(1/s)^i u\in\Gamma(X\setminus\{x\},G)$. Since G has the property (P) it follows that there exists $v=v_i\in\Gamma(X,G)$ and there exists $K=K_i\subset X$ a compact subset such that $v=(1/s)^i u$ on $X\setminus K$, which is equivalent to the fact that $s^i v=u$ on $X\setminus K$. Since X is connected, noncompact and G verifies "the principle of analytic continuation" (cf. example 2, paragraph 1) we deduce that $s^i v=u$ on X. It follows that $u_x\in m_x^i G_x$ for all $i\in N$. Since the m_x-adic topology on G_x is separated it follows that $u_x=o$. Contradiction. Hence $G=o$ and the proposition is proved.

<u>Corollary 1</u>. Let X be a real analytic space which is paracompact, connected and locally irreducible. Then:

$$X \text{ is compact} \iff H^1_c(X,O_X)=o$$

<u>Proof</u> It follows immediately from theorem 1.B. and proposition 1.

<u>Remark</u> Let X be a real analytic space and $F\in Coh(X)$. Put $X'=supp(F)$ and L=the set of all connected components of X'. It follows that:

$$H^1_c(X,F)=H^1_c(X',F)=\bigoplus_{U\in L} H^1_c(U,F)$$

From theorem 1.B. we deduce that, if every connected component of supp(F) is compact, then $H^1_c(X,F)=o$. In the following paragraph we shall discuss the conditions under which the converse of this statement holds. Therefore a generalisation of the concept of torsion will be needed.

3. Generalized torsion

Let A be a commutative ring (not necessarily integral) with 1-element and let M be an A-module. Put $M^*=\mathrm{Hom}_A(M,A)$ and $M^{**}=\mathrm{Hom}_A(M^*,A)$. Let $J:M\to M^{**}$ be the canonical map: $J(m)(m^*)=m^*(m)$. Then $tM=\ker J$ will be called the torsion submodule of M. M will be called torsion-free iff $\ker J=o$.

Remarks 1) $\ker J=o$ means that for all $m\in M\setminus\{o\}$, there exists $m^*\in M^*$ such that $m^*(m)\neq o$.

2) $\ker J=M$ means $M^*=o$.

3) M/tM is torsion-free.

4) If A is an integral domain and M is an A-module of finite type, then $\ker J=\{m\in M \mid (\exists)\ a\in A\setminus\{o\}$ such that $am=o\}$

The remarks 1), 2), 3) are obvious. The remark 4) shows that the torsion defined above generalizes the concept of torsion defined in the paragraph 1. Let's prove it. Put $N=\{m\in M \mid (\exists)\ a\in A\setminus\{o\}$ such that $am=o\}$. Obviously we get $N\subset \ker J$. Conversely, take $m\notin N$ and let's find $m^*\in M^*$ such that $m^*(m)\neq o$. If K denotes the field of quotients of A then $N=\ker(M\xrightarrow{u}M\otimes_A K)$. Since $m\notin N$, it follows that

$m\otimes 1\neq o$. Hence there exists $f\in Hom_K(M\otimes_A K,K)$ with $f(m\otimes 1)\neq o$. Let $m_1,\dots,m_k$ be a set of generators of M as A-module. There exists $a\in A\setminus\{o\}$ such that $af(m_i\otimes 1)\in A$ for all $i=1,\dots k$. We may take $m^*=a(f\circ u)$.

We shall need the following theorem:

Theorem 2. Let A be a noetherian ring (commutative and with 1-element), E an A-module of finite type which is faithful and R an A-module. The following conditions are equivalent:

1. $R=o$

2. $Hom_A(E,R)=o$

See ([5]) for the proof.

Let X be a real analytic space and $F\in Coh(X)$. We define F^*, F^{**} and tF in the same manner. Obviously they are coherent analytic sheaves.

Remarks 1) One gets locally an exact sequence $O_X^p\to O_X^q\to F^*\to o$ and then the exact sequence $o\to F^{**}\to O_X^q\to O_X^p$. It follows that F^{**} is locally isomorphic to a subsheaf of O_X^q and this property holds for F also if F is torsion-free. We deduce that, if O_X verifies "the principle of analytic continuation", then every torsion-free sheaf does.

2) F/tF is torsion-free.

Proposition 1' Let X be a real analytic space which is paracompact, connected and noncompact, such that the structure sheaf O_X verifies "the principle of analytic continuation". Take $F\in Coh(X)$ such that $H_c^1(X,F)=o$. Then $F=tF$ (i.e. $F^*=o$). The proof is essentially the same with the proof of proposition 1 (using generalized torsion). Hence one may deduce the following

<u>Corollary 1'</u> Let X be a real analytic space, which is paracompact, connected and such that O_X verifies "the principle of analytic continuation". Then:

$$X \text{ is compact} \Longleftrightarrow H^1_c(X, O_X) = o$$

<u>Proposition 2.</u> Let X be a paracompact real analytic space and $F \in Coh(X)$. Put $X' = supp(F)$ and $O_{X'} = (O_X / Ann(F))\big|_{X'}$. Suppose that $(X', O_{X'})$ verifies "the principle of analytic continuation". Then:

$H^1_c(X, F) = o \Leftrightarrow$ every connected component of X' is compact.

<u>Proof</u> „$\Leftarrow$" follows from the remark made in the end of the second paragraph. Let's prove „$\Rightarrow$". Let V be a connected component of X'. From $H^1_c(X, F) = H^1_c(X', F) = \bigoplus_{U \in L} H^1_c(U, F)$ we deduce that $H^1_c(V, F) = o$. The statement of the proposition follows now using proposition 1' and theorem 2. (F_x is an $O_{X,x}/Ann(F_x)$-faithful module).

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