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INTRODUCTION. Let X be a complex Banach space, and let $\underset{\sim}{C}$ be the complex field. If $A \subset \underset{\sim}{C}$, then A^c denotes the complement of A in $\underset{\sim}{C}$, and cl A denotes the closure of A in $\underset{\sim}{C}$. In 1963, Ciprian Foiaş [9] introduced the class of decomposable bounded linear operators that has proved to be of considerable importance. He defined a bounded operator on X to be decomposable if for any open cover $\{G_1, \ldots, G_n\}$ of its spectrum $\sigma(T)$ there exists a family of (closed) subspaces $\{Y_1, \ldots, Y_n\}$ that are "spectral maximal" subspaces for T such that $X = Y_1 + \ldots + Y_n$ and $\sigma(T \mid Y_j) \subset G_j$ (or equivalently, $\sigma(T \mid Y_j) \subset \text{cl } G_j$) for $j = 1, \ldots, n$. Here the term "spectral maximal" subspace for T means a subspace Y of X that is T-invariant and such that if Z is any other T-invariant subspace with $\sigma(T \mid Z) \subset \sigma(T \mid Y)$, then $Z \subset Y$. The "classical theory" of decomposable operators is presented in the volume of Colojoară and Foiaş [5] and recent summaries of newer developments have been given by Erdelyi and Lange [8] and by Radjabalipour [14]. Unfortunately, both of these summaries were written before the following remarkable characterization of decomposable operators was established by E. Albrecht [2] (and, independently, by R. Lange [10] and B. Nagy [11,12]) : An operator T on X is decomposable if and only if for every 2-cover $\{G_1, G_2\}$ of $\sigma(T)$ there are T-invariant subspaces $\{Y_1, Y_2\}$ such that $X = Y_1 + Y_2$ and $\sigma(T|Y_j) \subset G_j$ (or equivalently, $\sigma(T|Y_j) \subset \text{cl}\, G_j$) for $j = 1,2$. This result also uses a theorem of Radjabalipour [13] to pass from 2-covers to arbitrary finite covers of $\sigma(T)$.

The class of decomposable operators contains the operators that are "spectral" in the sense of Dunford (see [6]) as well as the compact operators. It is an open question whether the sum and product of two commuting decomposable operators is decomposable. Partial results in this direction have been established by Colojoară and Foiaş and by Apostol and will be mentioned later. It is proved here that the sum and product are decomposable, provided the restriction of one of the operators to every spectral maximal subspace of the other is decomposable. This is a strong and somewhat unpleasant hypothesis; unfortunately we are unable to eliminate it, or to show that it is necessary.

We remark that the case of non-commuting decomposable operators is much more difficult, although some results in this direction for sums of very special operators are known (see [14;Sect. 4]).

PRELIMINARIES. In the following S and T denote commuting decomposable operators on a complex Banach space X. It is a well-known fact from the Gelfand theory of commutative Banach algebras that

$$\sigma(S+T) \subset \sigma(S)+\sigma(T) = \{ s+t : s\in\sigma(S),\ t\in\sigma(T) \},$$
$$\sigma(ST) \subset \sigma(S)\cdot\sigma(T) = \{ st : s\in\sigma(S),\ t\in\sigma(T) \}.$$

If F is a closed subset of $\mathbb{C}$, we use the notation

$$X_T(F) := \{ x\in X : \sigma_T(x)\subset F \},$$

where $\sigma_T(x)$ denotes the "local spectrum" of x corresponding to T. It is well known (see [5; pp.18,19,31]) that if T is decomposable, then the set $X_T(F)$ is a closed T-hyperinvariant spectral maximal subspace for T and that $\sigma(T|X_T(F)) \subset F\cap\sigma(T)$. Moreover, every spectral maximal subspace for T has the form $X_T(F)$ for some F.

1. LEMMA. If S and T are commuting decomposable operators and $F\subset \mathbb{C}$ is closed, then $\sigma(S|X_T(F)) \subset\sigma(S)$.

<u>Proof</u>. Since $X_T(F)$ is T-hyperinvariant, it is invariant under both S and $R(\lambda;S)$ for λ in $\rho(S)$. It is easily seen that $R(\lambda;S|X_T(F)) = R(\lambda;S)|X_T(F)$ for λ in $\rho(S)$.

2. LEMMA. If S and T are commuting decomposable operators and $E, F\subset\mathbb{C}$ are closed, then the space $Y := X_S(E)\cap X_T(F)$ is invariant under $S+T$ and ST. Moreover $\sigma(S+T|Y)\subset\sigma(S+T)$ and $\sigma(ST|Y)\subset\sigma(ST)$.

<u>Proof</u>. The operators $S+T$, $R(\lambda;S+T)$ (for λ in $\rho(S+T)$), ST, and $R(\alpha;ST)$ (for α in $\rho(ST)$) commute with

both S and T. Hence Y is invariant under these operators and the assertions follow readily.

3. LEMMA. If $K \subset \mathbb{C}$ is a bounded convex set and if $\sigma(T) \subset K$, then $\sigma(T|Y) \subset K$ for any T-invariant subspace Y of X.

Indeed, it is well known that the spectrum $\sigma(T|Y)$ can grow only by filling in some of the "holes" in $\sigma(T)$. Since K is convex, the assertion follows.

C. Apostol [3] defined a decomposable operator T to be <u>strongly decomposable</u> if its restriction to every subspace $X_T(F)$ ($F \subset \mathbb{C}$ closed) is also decomposable. Recently, E. Albrecht [1] showed that not every decomposable operator is strongly decomposable. The condition that we will require is similar to that of strong decomposability.

4. DEFINITION. Let T be decomposable and let S commute with T. Then S is said to be T-<u>strongly decomposable</u> (or to be <u>strongly decomposable with respect to</u> T) if the restriction $S|X_T(F)$ is decomposable for each closed set F in $\mathbb{C}$.

WARNING. If S is a subset of $\mathbb{C}$, F.-H. Vasilescu and I. Bacalu have studied the notion of an S-decomposable operator and there is a corresponding notion of "strong S-decomposability", which is very different from the one just introduced.

It is clear that if S is T-strongly decomposable, then S is decomposable. Also T is strongly decomposable if and only if it is T-strongly decomposable.

There are some cases in which T-strong decomposability is automatic.

5. LEMMA. If T is a decomposable operator and S is a compact [resp., quasinilpotent] operator that commutes with T, then S is T-strongly decomposable.

Proof. Each restriction $S|X_T(F)$ is compact [resp., quasinilpotent] and hence is decomposable.

6. LEMMA. If T is a spectral operator and S is a decomposable operator that commutes with T, then S is T-strongly decomposable.

Proof. If E is a bounded projection operator that commutes with a decomposable operator S, then it is easy to see that $S|EX$ is decomposable. Now if T is a spectral operator and if $F \subset \mathbb{C}$ is closed, then $X_T(F) = E(F)X$, where E is the spectral measure for T. Therefore $S|X_T(F)$ is decomposable for each closed F.

THE MAIN RESULT. We now show that the sum and product of two commuting decomposable operators are decomposable provided one of the operators is strongly decomposable with respect to the other one.

7. THEOREM. Let S and T be commuting decomposable operators on X and suppose that S is T-strongly decomposable. Then the operators $S + T$ and ST are decomposable on X.

Proof. We will treat the case of $S + T$ in detail. Let $\{G_1, G_2\}$ be an open 2-cover of $\sigma(S + T)$ and let $\{H_1, H_2\}$

be an open 2-cover of $\sigma(S + T)$ such that $\text{cl}\, H_j \subset G_j$ and H_j is bounded by $|\sigma(S)| + |\sigma(T)| + 1$, where $|\sigma(S)|$ denotes the spectral radius of S. Now choose $d > 0$ such that

$$(\alpha) \qquad d < \frac{1}{2} \min \{ \text{dist}[\sigma(S+T),(H_1 \cup H_2)^c] ,\ \text{dist}[H_1,G_1{}^c] ,\ \text{dist}[H_2,G_2{}^c] \} .$$

For each point s in $\sigma(S)$, take an open ball $U(s)$ with radius less than d and center s, and let $\{U(s_1),\dots,U(s_N)\}$ be a finite cover for $\sigma(S)$. Similarly, let $\{V(t_1),\dots,V(t_M)\}$ be a finite cover for $\sigma(T)$ consisting of open balls with radii less than d and centers t_k. For convenience, we denote $U(s_j)$ by U_j and $V(t_k)$ by V_k.

We note that the pairs of indices (j,k) $(j = 1,\dots,N;$ $k= 1,\dots,M)$ fall into three disjoint cases.

Case 0. Here $s_j + t_k \notin H_1 \cup H_2$. In this case $\text{cl}\, U_j + \text{cl}\, V_k$ is disjoint from $\sigma(S + T)$, for if $|s - s_j| < d$ and $|t - t_k| < d$, then $|(s + t) - (s_j + t_k)| < 2d$ $< \text{dist}[\sigma(S + T), (H_1 \cup H_2)^c]$ whence it follows that $s + t \notin \sigma(S + T)$.

Case I. Here $s_j + t_k \in H_1$. In this case $\text{cl}\, U_j + \text{cl}\, V_k \subset G_1$ for if $|s - s_j| < d$ and $|t - t_k| < d$, then $|(s + t) - (s_j + t_k)| < 2d < \text{dist}[H_1,G_1{}^c]$ whence it follows that $s + t \in G_1$.

Case II. Here $s_k + t_k \in H_2 - H_1$. In this case we have $\text{cl}\, U_j + \text{cl}\, V_k \subset G_2$.

Since T is decomposable and $\{V_1, \ldots, V_M\}$ is an open cover of $\sigma(T)$, then

$$(\beta) \qquad X = X_T(\mathrm{cl}\ V_1) + \ldots + X_T(\mathrm{cl}\ V_M) \ ,$$

where of course $\sigma(T|X_T(\mathrm{cl}\ V_k)) \subset \mathrm{cl}\ V_k$ for $k = 1,\ldots,M$. We let $S_k := S|X_T(\mathrm{cl}\ V_k))$ for $k = 1,\ldots,M$; hence, by Lemma 1, we have $\sigma(S_k) \subset \sigma(S)$. By hypothesis, S is T-decomposable; hence S_k is decomposable on the space $X_T(\mathrm{cl}\ V_k)$. Since $\{U_1,\ldots,U_N\}$ is an open cover of $\sigma(S_k)$, there exist subspaces Y_{jk} $(j = 1,\ldots,N)$ of $X_T(\mathrm{cl}\ V_k)$ such that

$$(\gamma) \qquad X_T(\mathrm{cl}\ V_k) = Y_{1k} + \ldots + Y_{Nk}$$

having the property that $\sigma(S_k|Y_{jk}) \subset \mathrm{cl}\ U_j$. In fact, we may take

$$Y_{jk} := X_S(\mathrm{cl}\ U_j) \cap X_T(\mathrm{cl}\ V_k) \ ,$$

because $\sigma(S|X_S(\mathrm{cl}\ U_j)) \subset \mathrm{cl}\ U_j$ and, since $\mathrm{cl}\ U_j$ is a convex set, it follows from Lemma 3 that $\sigma(S_k|Y_{jk}) = \sigma(S|Y_{jk}) \subset \mathrm{cl}\ U_j$.

On the other hand, since $\sigma(T|X_T(\mathrm{cl}\ V_k)) \subset \mathrm{cl}\ V_k$, another application of Lemma 3 proves that $\sigma(T|Y_{jk}) \subset \mathrm{cl}\ V_k$. Since the restrictions of S and T to Y_{jk} commute, it follows that

$$\begin{aligned}\sigma(S + T|Y_{jk}) &\subset \sigma(S|Y_{jk}) + \sigma(T|Y_{jk}) \\ &\subset \mathrm{cl}\ U_j + \mathrm{cl}\ V_k \ ,\end{aligned}$$

for $j = 1,\ldots,N$; $k = 1,\ldots,M$.

Now suppose that the pair (j,k) satisfies Case 0, Since, by Lemma 2, we have $\sigma(S + T|Y_{jk}) \subset \sigma(S + T)$, it follows that we must have $\sigma(S + T|Y_{jk}) = \emptyset$, whence it follows that $Y_{jk} = (0)$. Similarly, if Case I holds, then

$\sigma(S + T|Y_{jk}) \subset G_1$ and if Case II holds, then $\sigma(S + T|Y_{jk}) \subset G_2$.

Now let Z_1 [resp., Z_2] be the linear span of the Y_{jk} for (j,k) satisfying Case I [resp., II]. If $z \in Z_1$ then

$$z = \sum_{(j,k)\in I} y_{jk} \qquad \text{with } y_{jk} \in Y_{jk},$$

so that we have

$$\sigma_{S+T}(z) \subset \bigcup \sigma_{S+T}(y_{jk}) \subset \bigcup \sigma(S + T|Y_{jk}) \subset G_1 .$$

Consequently we have that

$$\sigma(S + T|Z_1) = \bigcup_{z\in Z_1} \sigma_{S+T}(z) \subset G_1 ;$$

similarly, we have $\sigma(S + T|Z_2) \subset G_2$. It is clear from (β) and (γ) that $X = Z_1 + Z_2$.

Since $\{G_1, G_2\}$ is an arbitrary open 2-cover of $\sigma(S + T)$, it follows from the theorem of Albrecht cited earlier that $S + T$ is decomposable.

To treat the case of the product ST, we argue as above, with the following minor modification. If $A := |\sigma(S)| + |\sigma(T)| + 1$, and if $d > 0$ is chosen as in (α), then we take the radii of the balls $U(s)$ and $V(t)$ to be less than $d' := d/2A$. The remaining details are entirely similar.

It is known that if T is a decomposable operator and if Q is a quasinilpotent operator that commutes with T, then $T + Q$ and T are quasinilpotent equivalent and hence $T + Q$ is decomposable (see [5; pp. 11, 40]). Similarly it was proved by Apostol [4; p. 1498] that the sum and product of a spectral operator and a commuting decomposable operator are decomposable. These results follow from Theorem 7 and Lemmas 5 and 6.

Of course, the product of a compact operator and any bounded operator is compact and hence decomposable. However the following corollary of Lemma 5 and Theorem 7 does not appear to be obvious.

8. COROLLARY. If T is a decomposable operator and K is a compact operator that commutes with T, then $T + K$ is decomposable.

R E F E R E N C E S

1. Albrecht, Ernst, On two questions of I. Colojoară and C. Foiaş, Manuscripta Math. 25(1978), 1-15.

2. ------- , On decomposable operators, Integral Equations and Operator Theory 2(1979), 1-10.

3. Apostol, Constantin, Restrictions and quotients of decomposable operators on a Banach space, Rev. Roumaine Math. Pures Appl. 13(1968), no.1, 147-150.

4. ------- , Spectral decompositions and functional calculus, Rev. Roumaine Math. Pures Appl. 13(1968), no. 10, 1481-1528.

5. Colojoară, Ion, and Foiaş, Ciprian, Theory of generalized spectral operators, Gordon and Breach, New York-London, 1968.

6. Dunford, Nelson, and Schwartz, Jacob T., Linear operators. Part III. Spectral Operators, Wiley-Interscience, New York, 1971.

7. Erdelyi, Ivan, and Lange, Ridgley, Operators with spectral decomposition properties, J. Math. Anal. Appl. 66(1978), 1-19.

8. ------- , Spectral decompositions on Banach spaces, Lecture Notes in Math., no. 623, Springer-Verlag, Berlin-New York, 1977.

9. Foiaş, Ciprian, Spectral maximal spaces and decomposable operators in Banach spaces, Archiv der Math. (Basel) 14(1963), 341-349.

10. Lange, Ridgley, On generalization of decomposability, Glasgow Math. J. (to appear).

11. Nagy, B., Operators with the spectral decomposition property are decomposable (preprint).

12. ------- , A spectral residuum for each closed operator (preprint).

13. Radjabalipour, Mehdi, Equivalence of decomposable and 2-decomposable operators, Pacific J. Math. 77(1978), 243-247.

14. ------- , Decomposable operators, Bull. Iranian Math. Soc., no.9(1978), 1L-49L.

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