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1. Introduction

We shall start by considering a singularly perturbed system

$$\varepsilon_k \frac{dy_k}{dt} = \sum_{j=0}^{N} A_{kj}(t) y_j \quad , \quad k=0,1,\ldots,N$$

where $\varepsilon_0 = 1$, and ε_k , $k \geq 1$ are functions of a parameter ε such that $\varepsilon_k > 0$ and $\lim_{\varepsilon \to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0$. We shall adopt the following main assumptions.

a) $A_{kj} : I \subset R \to \mathcal{L}(R^{n_j}; R^{n_k})$ are uniformly Lipschitz: there exists $\lambda > 0$ such that $|A_{kj}(t) - A_{kj}(s)| \leq \lambda |t-s|$ for all $t,s \in I$. Moreover A_{kj} are uniformly bounded: there exists $\mu > 0$ such that $|A_{kj}(t)| \leq \mu$ for all $t \in I$.

In what follows the interesting cases will be $I = (t_0, \infty)$ and $I = R$.

b) We assume $A_{NN}(t)$ invertible for all $t \in I$; denote $A_{kj} = A_{kj}^N$ and define $A_{kj}^{N-1} = A_{kj}^N - A_{kN}^N (A_{NN}^N)^{-1} A_{Nj}^N$, $k,j = 0,\ldots,N-1$.
We assume inductively that $A_{\ell\ell}^{\ell}(t)$ are invertible, with uniformly bounded and uniformly Lipschitz inverse , and define

$$A_{kj}^{\ell-1} = A_{kj}^{\ell} - A_{k\ell}^{\ell} (A_{\ell\ell}^{\ell})^{-1} A_{\ell j}^{\ell} \quad , k,j=0,\ldots,\ell-1$$

c) We assume finally that $A_{\ell\ell}^{\ell}$ for $1 \leq \ell \leq N$ are uniformly Hurwitz on I: there exists $\alpha_\ell > 0$ such that $\mathrm{Re}\, \sigma(A_{\ell\ell}^{\ell}(t)) \leq -2\alpha_\ell$ for all $t \in I$; here $\mathrm{Re}\, \sigma(A)$ means the real parts of the eigenvalues of A.

Under such assumption we shall obtain estimates for the fundamental matrix of the system.

These estimates will allow us to obtain a direct proof for the extension to this case of the Klimušev-Krasovski [1] stability theorem, some assymptotic expansions for the solution of the Cauchy problem , including the case of „impulsive" initial conditions, the extension to this case of the results of Flatto and Levinson and Hale and Seifert concerning periodic, bounded and almost periodic solutions.

Coresponding results will be derived for nonlinear situations.

The results reported are refinements of the ones obtained by F.Hoppensteadt with Lyapunov functions ;we point out that the method of Hoppensteadt requires more smoothness and gives less informations on the fundamental matrix . The problems of almost periodic solutions and impulsive initial conditions were not considered by F. Hoppensteadt.

2. Main estimates

Let $C_\ell(t,s,\varepsilon_\ell)$ be the fundamental matrix associated to $\frac{1}{\varepsilon_\ell}A^\ell_{\ell\ell}$; since $A^\ell_{\ell\ell}$ is uniformly Hurwitz ,uniformly bounded and uniformly Lipschitz we have

$$|C_\ell(t,s,\varepsilon)| \leq ce^{-\frac{\alpha_\ell}{\varepsilon_\ell}(t-s)} \quad ,\ell \geq 1.$$

uniformly for $s \leq t$, $s\in I$, $t\in I$.

We shall write also

$$|C_0(t,s,\varepsilon)| \leq ce^{-\alpha_0(t-s)}$$

since the most interesting results will be obtained for $\alpha_0>0$, that is assuming that the reduced model also exhibits asymptotic stability .

We shall start with the case N=1 since we intend to obtain our estimates by induction.

A. Consider the sistem

$$\dot{y}_0 = A_{00}(t)y_0 + A_{01}(t)y_1$$

$$\varepsilon\dot{y}_1 = A_{10}(t)y_0 + A_{11}(t)y_1$$

In this case $A^0_{00} = A_{00}-A_{01}A^{-1}_{11}A_{10}$; we may write

$$\dot{y}_0 = A^0_{00}(t)y_0 + A_{01}(t)A^{-1}_{11}(t)[A_{10}(t)y_0 + A_{11}(t)y_1]$$

hence, by the variation of constants formula

$$y_0(t,s,\varepsilon)=C_0(t,s)\hat{y}_0 + \int_s^t C_0(t,\sigma)\varepsilon_1 A_{01}(\sigma)A^{-1}_{11}(\sigma)\frac{dy_1}{d\sigma}(\sigma,s,\varepsilon)d\sigma$$

On the other hand ,again by the variation of constants formula

$$y_1(t,s,\varepsilon)=C_1(t,s,\varepsilon)\hat{y}_1 + \frac{1}{\varepsilon_1}\int_s^t C_1(t,\sigma,\varepsilon)A_{10}(\sigma)y_0(\sigma,s,\varepsilon)d\sigma$$

and ,by computing $\varepsilon_1 A^{-1}_{11}(\sigma)\frac{dy_1}{d\sigma}(\sigma,s,\varepsilon)$ and replacing into the formula for y_0 , we get

$$y_0(t,s,\varepsilon)= C_0(t,s)\hat{y}_0 + \int_s^t C_0(t,\sigma)A_{01}(\sigma)C_1(\sigma,s,\varepsilon)d\sigma\,\hat{y}_1 +$$

$$+\int_s^t C_0(t,\sigma)A_{01}(\sigma)A^{-1}_{11}(\sigma)A_{10}(\sigma)y_0(\sigma,s,\varepsilon)d\sigma +$$

$$+\frac{1}{\varepsilon_1}\int_s^t C_0(t,\sigma)A_{01}(\sigma)\Big[\int_s^\sigma C_1(\sigma,\tau,\varepsilon)A_{10}(\tau)y_0(\tau,s,\varepsilon)d\tau\Big]d\sigma$$

In the last integral we change the order of integration and then we calculate

$$\frac{1}{\varepsilon_1}\int_{\tau}^{t} C_0(t,\sigma)A_{01}(\sigma)C_1(\sigma,\tau,\varepsilon)\,d\sigma =$$

$$=\frac{1}{\varepsilon_1}\int_{\tau}^{t} C_0(t,\sigma)A_{01}(\tau)\;A_{11}^{-1}(\tau)\,A_{11}(\sigma)\;C_1(\sigma,\tau,\varepsilon)\;d\sigma +$$

$$+\frac{1}{\varepsilon_1}\int_{\tau}^{t} C_0(t,\sigma)\left[A_{01}(\sigma)\;A_{11}^{-1}(\sigma)\;-A_{01}(\tau)\;A_{11}^{-1}(\tau)\right]A_{11}(\sigma)\,C_1(\sigma,\tau,\varepsilon)\,d\sigma$$

In the first integral we perform an integration by parts and then substitute into the integral equation for y_0.

We get

$$y_0(t,s,\varepsilon)=C_0(t,s)\hat{y}_0+\int_s^t C_0(t,\sigma)A_{01}(\sigma)\,C_1(\sigma,s,\varepsilon)\;d\sigma\;\hat{y}_1+$$
$$+\int_s^t K_{00}(t,\tau,\varepsilon)\;\;y_0(\tau,s,\varepsilon)\,d\tau$$

where

$$K_{00}(t,\tau,\varepsilon)=\Big\{A_{01}(\tau)\;A_{11}^{-1}(\tau)\;C_1(t,\tau,\varepsilon)+\int_{\tau}^{t}C_0(t,\sigma)\,A_{00}(\sigma)\;\;A_{01}(\tau)\;A_{11}^{-1}(\tau)C_1(\sigma,\tau,\varepsilon)\,d\sigma$$
$$+\frac{1}{\varepsilon_1}\int_{\tau}^{t}C_0(t,\sigma)\left[A_{01}(\sigma)\;A_{11}^{-1}(\sigma)\;-A_{01}(\tau)\;\;A_{11}^{-1}(\tau)\right]\;A_{11}(\sigma)\;\;C_1(\sigma,\tau,\varepsilon)\,d\sigma\Big\}\;A_{10}(\tau)$$

Using the estimates for the fundamental matrix we obtain the estimate

$$|K_{00}(t,\tau,\varepsilon)|\le c\;(\varepsilon_1\,e^{-\alpha_0(t-\tau)}+e^{-\frac{\alpha_1}{\varepsilon_1}(t-\tau)})$$

and from here

$$|y_0(t,s,\varepsilon)|\le c\left(e^{-\alpha_0(t-s)}|\hat{y}_0|\;\;+\varepsilon_1 e^{-\alpha_0(t-s)}|\hat{y}_1|\right)+$$
$$+\;c\int_s^t\left[\varepsilon_1\;\;e^{-\alpha_0(t-\tau)}+e^{-\frac{\alpha_1}{\varepsilon_1}(t-\tau)}\right]|y_0(\tau,s,\varepsilon)|\;\,d\tau$$

We choose $0<\theta_0<1$ and define $\rho^0=\sup\limits_{t_0\le s\le t\le T} e^{\alpha_0\theta_0(t-s)}\;|y_0(t,s,\varepsilon)|$

$[t_0,T]\subset I$; we have

$$|y_0(t,s,\varepsilon)|\cdot e^{\alpha_0\theta_0(t-s)}\le c\;\;e^{-(1-\theta_0)\alpha_0(t-s)}(|\hat{y}_0|\;+\;\varepsilon_1\,|\hat{y}_1|)\;+$$
$$+c\int_s^t\;(\varepsilon_1\;\;e^{-\alpha_0(t-\tau)}\;\;\;+e^{-\frac{\alpha_1}{\varepsilon_1}(t-\tau)})\;e^{\alpha_0\theta_0(t-\tau)}\;d\tau\,\rho^0$$

hence $\rho^0\le c\,(|\hat{y}_0|+\varepsilon_1|\,\hat{y}_1|)+\varepsilon_1 c\,\rho^0$

the estimates for the constants denoted by c depending on $[t_0,T]$ if $\alpha_0\le 0$ and being uniform in I if $\alpha_0>0$.

We deduce

$$|y_0(t,s,\varepsilon)|\;\le c\;\;e^{-\alpha_0\theta_0(t-s)}(|\hat{y}_0|\;\;+\varepsilon_1|\hat{y}_1|)$$

and then

$$|y_1\;(t,s,\varepsilon)|\;\;\le c\;\;e^{-\alpha_0\theta_0(t-s)}(|\hat{y}_0|\;+\varepsilon_1\,|\;\;\hat{y}_1|)\;\;+c\;\;e^{-\frac{\alpha_1}{\varepsilon_1}(t-s)}\;|\hat{y}_1|$$

From these estimates we deduce the ones for the fundamental matrix ; if we denote by Γ_0^1 the first block column and by Γ_1^1 the second block column we have $\Gamma_0^1(s,s,\varepsilon)=\begin{pmatrix}E\\0\end{pmatrix}$, $\Gamma_1^1(s,s,\varepsilon)=\begin{pmatrix}0\\E\end{pmatrix}$.

where E is the identity matrix of the coresponding dimension, hence

$$\left|\Gamma_0^1(t,s,\varepsilon)\right| \le c\, e^{-\theta_0 \alpha_0 (t-s)}$$

$$\left|\Gamma_1^1(t,s,\varepsilon)\right| \le c\left[\varepsilon_1 e^{-\theta_0\alpha_0(t-s)} + e^{-\frac{\alpha_1}{\varepsilon_1}(t-s)}\right)] .$$

These are the estimates we will prove by induction in the general case $N > 1$.Remark now that for $\alpha_0 > 0$ we have from these estimates the result of Klimušev and Krasovski, but with more useful information on the behavior of the fundamental matrix.

B. In the general case we write the system in the form

$$\varepsilon_k \frac{dy_k}{dt} = \sum_{j=0}^{N-1} A_{kj}^{N-1}(t) y_j + \varepsilon_N A_{kN}(t) A_{NN}^{-1}(t) \frac{dy_N}{dt}(t) \quad , \; 0 \le k \le N-1$$

$$\varepsilon_N \frac{dy_N(t)}{dt} = \sum_{j=0}^{N-1} A_{Nj}(t)\, y_j + A_{NN}(t)\, y_N(t)$$

and with $y^{N-1} = \mathrm{col}(y_k)_{k=0,1,\dots N-1}$

$$y^{N-1}(t,s,\varepsilon) = \sum_{k=0}^{N-1} \Gamma_k^{N-1}(t,s,\varepsilon)\hat{y}_k +$$

$$+ \varepsilon_N \int_s^t \sum_{k=0}^{N-1} \frac{1}{\varepsilon_k} \Gamma_k^{N-1}(t,\sigma,\varepsilon) A_{kN}(\sigma) A_{NN}^{-1}(\sigma) \frac{dy_N}{d\sigma}(\sigma,s,\varepsilon)\, d\sigma$$

$$y_N(t,s,\varepsilon) = C_N(t,s,\varepsilon)\hat{y}_N + \frac{1}{\varepsilon_N}\int_s^t C_N(t,\sigma,\varepsilon) \sum_{j=0}^{N-1} A_{Nj}(\sigma)\, y_j(\sigma,s,\varepsilon)\, d\sigma$$

where Γ_k^{N-1} is the k^{th} block column of the fundamental matrix of the system

$$\varepsilon_k \frac{dy_k}{dt} = \sum_{j=0}^{N-1} A_{kj}^{N-1}(t)\, y_j \qquad k=0,\dots,N-1$$

(obtained after the reduction with respect to the fastest variables y_N).

As above we compute $\varepsilon_N A_{NN}^{-1}(t) \frac{dy_N}{dt}(t,s,\varepsilon)$ replace in the formula for y^{N-1} and perform a change of the order of integration.

We write further

$$\frac{1}{\varepsilon_N}\int_\tau^t \sum_{k=0}^{N-1} \frac{1}{\varepsilon_k}\Gamma_k^{N-1}(t,\sigma,\varepsilon)\, A_{kN}(\sigma)\, C_N(\sigma,\tau,\varepsilon)\, d\sigma =$$

$$= \int_\tau^t \sum_{k=0}^{N-1} \frac{1}{\varepsilon_k}\Gamma_k^{N-1}(t,\sigma,\varepsilon) A_{kN}(\tau) A_{NN}^{-1}(\tau) \frac{d}{d\sigma} C_N(\sigma,\tau,\varepsilon)\, d\sigma +$$

$$+ \frac{1}{\varepsilon_N}\int_\tau^t \sum_{k=0}^{N-1} \frac{1}{\varepsilon_k}\Gamma_k^{N-1}(t,\sigma,\varepsilon)\left[A_{kN}(\sigma) A_{NN}^{-1}(\sigma) - A_{kN}(\tau) A_{NN}^{-1}(\tau)\right] A_{NN}(\sigma)\, C_N(\sigma,\tau,\varepsilon)\, d\sigma$$

By performing an integration by parts , replacing in the formula for y^{N-1} and making obvious reductions , we get

$$y^{N-1}(t,s,\varepsilon) = \sum_{k=0}^{N-1} \Gamma_k^{N-1}(t,s,\varepsilon)\hat{y}_k + \sum_{k=0}^{N-1} \frac{1}{\varepsilon_k}\int_s^t \Gamma_k^{N-1}(t,\sigma,\varepsilon) A_{kN}(\sigma)\, C_N(\sigma,s,\varepsilon)\, d\sigma\, \hat{y}_N +$$

$$+ \int_s^t K^{N-1}(t,\tau;\varepsilon)\, y^{N-1}(\tau,s,\varepsilon)\, d\tau$$

where

$$K^{N-1}(t,\tau,\varepsilon)=\sum_{k=0}^{N-1}\frac{1}{\varepsilon_k}\Big\{\Gamma_k^{N-1}(t,t,\varepsilon)\,A_{kN}(\tau)\,A_{NN}^{-1}(\tau)\,C_N(t,\tau,\varepsilon)\,+$$

$$+\sum_{\ell=0}^{N-1}\frac{1}{\varepsilon_\ell}\int_\tau^t\Gamma_\ell^{N-1}(t,\sigma,\varepsilon)\;A_{\ell k}(\sigma)\;A_{kN}(\tau)\;A_{NN}^{-1}(\tau)\;C_N(\sigma,\tau,\varepsilon)\;d\sigma\,+$$

$$+\frac{1}{\varepsilon_N}\int_\tau^t\Gamma_k^{N-1}(t,\sigma,\varepsilon)\Big[A_{kN}(\sigma)\;A_{NN}^{-1}(\sigma)-A_{kN}(\tau)\;A_{NN}^{-1}(\tau)\Big]A_{NN}(\sigma)\,C_N(\sigma,\tau,\varepsilon)\;d\sigma\Big\}\mathcal{A}^{N-1}(\tau)$$

We introduce now the induction assumption

$$|\Gamma_k^{N-1}(t,s,\varepsilon)|\le c\Big[e^{-\theta_{N-1}\alpha_0(t-s)}\varepsilon_k+\sum_{\ell=1}^{k}\frac{\varepsilon_k}{\varepsilon_\ell}\,e^{-\theta_{N-1}\frac{\alpha_\ell}{\varepsilon_\ell}(t-s)}\Big].$$

with $0<\theta_{N-1}<1$.

By using this assumption we get

$$|K^{N-1}(t,\tau,\varepsilon)|\le c\,\frac{\varepsilon_N}{\varepsilon_{N-1}}\Big(e^{-\theta_{N-1}\alpha_0(t-\tau)}+\sum_{j=1}^{N}\frac{1}{\varepsilon_j}\,e^{-\theta_{N-1}\frac{\alpha_j}{\varepsilon_j}(t-\tau)}\Big)$$

and we deduce

$$|y^{N-1}(t,s,\varepsilon)|\le c\sum_{j=0}^{N-1}\frac{1}{\varepsilon_j}\,e^{-\theta_{N-1}\frac{\alpha_j}{\varepsilon_j}(t-s)}\sum_{k=j}^{N}\varepsilon_k|\hat{y}_k|\,+$$

$$+\,c\,\frac{\varepsilon_N}{\varepsilon_{N-1}}\int_s^t\Big[e^{-\theta_{N-1}\alpha_0(t-\tau)}+\sum_{j=1}^{N}\frac{1}{\varepsilon_j}\,e^{-\theta_{N-1}\frac{\alpha_j}{\varepsilon_j}(t-\tau)}\Big]|y^{N-1}(\tau,s,\varepsilon)|\;d\tau$$

Denote

$$\xi^{N-1}(t,\varepsilon)=|\,y^{N-1}(\,t,s,\varepsilon)|-\sum_{\ell=0}^{r(\varepsilon)}\Big(\frac{\varepsilon_N}{\varepsilon_{N-1}}\Big)^{\ell}\delta_\ell\Big[\sum_{j=1}^{N-1}\frac{1}{\varepsilon_j}e^{-\theta_N\frac{\alpha_j}{\varepsilon_j}(t-s)}\sum_{k=j}^{N}\varepsilon_k|\hat{y}_k|\Big]$$

where δ_ℓ , $r(\varepsilon)$, θ_N will be correspondingly chosen.

After some simple trasformations we get

$$\xi^{N-1}(t,\varepsilon)+\Big[\sum_{\ell=0}^{r(\varepsilon)}\Big(\frac{\varepsilon_N}{\varepsilon_{N-1}}\Big)^{\ell}\delta_\ell\Big]\Big[\sum_{j=1}^{N}\frac{1}{\varepsilon_j}e^{-\theta_N\frac{\alpha_j}{\varepsilon_j}(t-s)}\sum_{k=j}^{N}\varepsilon_k|\hat{y}_k|\Big]\le$$

$$\le c\,e^{-\theta_{N-1}\alpha_0(t-s)}\sum_{k=0}^{N}\varepsilon_k|\hat{y}_k|+c\sum_{j=1}^{N-1}\frac{1}{\varepsilon_j}e^{-\theta_{N-1}\frac{\alpha_j}{\varepsilon_j}(t-s)}\sum_{k=j}^{N}\varepsilon_k|\hat{y}_k|\,+$$

$$+\,\hat{c}\sum_{\ell=1}^{r(\varepsilon)+1}\Big(\frac{\varepsilon_N}{\varepsilon_{N-1}}\Big)^{\ell}\delta_{\ell-1}\Big[e^{-\theta_{N-1}\alpha_0(t-s)}\sum_{k=1}^{N-1}\varepsilon_k|\hat{y}_k|+\sum_{j=1}^{N-1}\frac{1}{\varepsilon_j}e^{-\theta_N\frac{\alpha_j}{\varepsilon_j}(t-s)}\sum_{k=j}^{N}\varepsilon_k|\hat{y}_k|\Big]+$$

$$+\,c\,\frac{\varepsilon_N}{\varepsilon_{N-1}}\int_s^t\Big[e^{-\theta_{N-1}\alpha_0(t-\tau)}+\sum_{j=1}^{N}\frac{1}{\varepsilon_j}\,e^{-\theta_{N-1}\frac{\alpha_j}{\varepsilon_j}(t-\tau)}\Big]\xi^{N-1}(\tau,\varepsilon)\,d\tau.$$

we take now $\delta_0=c$, $\delta_\ell=\hat{c}\,\delta_{\ell-1}$, $r(\varepsilon)$ such that $\Big(\hat{c}\,\frac{\varepsilon_N}{\varepsilon_{N-1}}\Big)^{r(\varepsilon)+1}<\varepsilon_N$;
for $\varepsilon>0$ small enough but fixed we have $\hat{c}\,\frac{\varepsilon_N}{\varepsilon_{N-1}}<1$ hence
$\lim\limits_{r\to\infty}\Big(\hat{c}\,\frac{\varepsilon_N}{\varepsilon_{N-1}}\Big)^{r}=0$ and we can find $r(\varepsilon)$ such that the inequality required holds.

Then by simply looking at resulting terms the inequality may be written

$$\xi^{N-1}(t,\varepsilon) < c\, e^{-\theta_{N-1}\alpha_0(t-s)} \sum_{k=0}^{N} \varepsilon_k |\hat{y}_k| +$$
$$+ c\,\frac{\varepsilon_N}{\varepsilon_{N-1}}\int_s^t \Big[e^{-\theta_{N-1}\alpha_0(t-\tau)} + \sum_{j=1}^{N}\frac{1}{\varepsilon_j}\, e^{-\theta_{N-1}\frac{\alpha_j}{\varepsilon_j}(t-\tau)}\Big]\,\xi^{N-1}(\tau,\varepsilon)\,d\tau$$

Define $\rho^{N-1} = \sup_{t_0 \le s \le t \le T_N} e^{\theta_N\alpha_0(t-s)}\,\xi^{N-1}(t,\varepsilon)$; with $\theta_N < \theta_{N-1}$ we get as above $\rho^{N-1} \le c\sum_{k=0}^{N}\varepsilon_k|\hat{y}_k|$, where the estimate for c is independent of $[t_0,T]$ if $\alpha_0 > 0$.

We deduce the estimates

$$|y^{N-1}(t,s,\varepsilon)| \le c\Big\{ e^{-\theta_N\alpha_0(t-s)}\sum_{k=0}^{N}\varepsilon_k|\hat{y}_k| + \sum_{j=1}^{N-1}\frac{1}{\varepsilon_j}\, e^{-\theta_N\frac{\alpha_j}{\varepsilon_j}(t-s)}\sum_{k=j}^{N}\varepsilon_k|\hat{y}_k|\Big\}.$$
$$|y_N(t,s,\varepsilon)| \le c\Big\{ e^{-\theta_N\alpha_0(t-s)}\sum_{k=0}^{N}\varepsilon_k|\hat{y}_k| + \sum_{j=1}^{N}\frac{1}{\varepsilon_j}\, e^{-\theta_N\frac{\alpha_j}{\varepsilon_j}(t-s)}\sum_{k=j}^{N}\varepsilon_k|\hat{y}_k|\Big\}.$$

which conform the induction assumption for the block columns of the fundamental matrix .

These estimates give also the Klimušev-Krasovski type result for the case of several parameters as in the case of a simple parameter.

3. Asymptotic expansions

We shall use the estimates for the fundamental matrix in order to deduce the behaviour of the solution of the Cauchy problem.

Define $\tilde{y}_j$ from the Cauchy problem

$$\varepsilon_k\frac{d\tilde{y}_k}{dt} = \sum_{j=0}^{k} A^k_{kj}(t)\,\tilde{y}_j \qquad \tilde{y}_k(t_0,\varepsilon) = y_k^0$$

and z_j by $y_j(t;t_0,y_0^0\dots y_N^0,\varepsilon) = \tilde{y}_j(t,\varepsilon) + \eta z_j(t,\varepsilon)$

where $\eta = \max_k \frac{\varepsilon_{k+1}}{\varepsilon_k}$.

We shall show that z_j are bounded , hence $\tilde{y}_j$ will represent the principal part of the solution of the Cauchy problem.

We have directly

$$\varepsilon_k\frac{dz_k(t,\varepsilon)}{dt} = \sum_{j=0}^{N} A_{kj}(t)\,z_j(t,\varepsilon) + \frac{1}{\eta}\Big(\sum_{j=0}^{N} A_{kj}(t)\,\tilde{y}_j(t,\varepsilon) - \sum_{j=0}^{k} A^k_{kj}(t)\,\tilde{y}_j(t,\varepsilon)\Big)$$
$$z_k(t_0,\varepsilon) = 0$$

We may write ,for $j \ge 1$,

$$\tilde{y}_j(t,\varepsilon) = C_j(t,t_0,\varepsilon)y_j^0 + \frac{1}{\varepsilon_j}\int_{t_0}^{t} C_j(t,\sigma,\varepsilon)\sum_{\ell=0}^{j-1}A^j_{j\ell}(\sigma)\,\tilde{y}_\ell(\sigma,\varepsilon)\,d\sigma =$$
$$= C_j(t,t_0,\varepsilon)y_j^0 + \frac{1}{\varepsilon_j}\sum_{\ell=0}^{j-1}\int_{t_0}^{t} C_j(t,\sigma,\varepsilon)A^j_{jj}(\sigma)\Big\{\big[A^j_{jj}(\sigma)\big]^{-1}A^j_{j\ell}(\sigma) -$$

$$- [A^j_{jj}(t)]^{-1} A^j_{j\ell}(t)\} \quad \tilde{y}_\ell(\sigma,\varepsilon)\, d\sigma + C_j(t, t_0,\varepsilon) \sum_{\ell=0}^{j-1} [A^j_{jj}(t)]^{-1} A^j_{j\ell}(t)\, y^0_\ell \; -$$

$$- \sum_{\ell=0}^{j-1} [A^j_{jj}(t)]^{-1} A^j_{j\ell}(t)\, \tilde{y}_\ell(t,\varepsilon) + \int_{t_0}^{t} C_j(t,\sigma,\varepsilon) \sum_{\ell=0}^{j-1} [A^j_{jj}(t)]^{-1} A^j_{j\ell}(t) \frac{d\tilde{y}_\ell}{d\sigma}(\sigma,\varepsilon)\, d\sigma$$

hence

$$\tilde{y}_j(t,\varepsilon) = - \sum_{\ell=0}^{j-1} [A^j_{jj}(t)]^{-1} A^j_{j\ell}(t)\, \tilde{y}_\ell(t,\varepsilon) + \omega_j(t,\varepsilon),$$

where

$|\omega_j(t,\varepsilon)| \le c \left(e^{-\frac{\alpha_j}{\varepsilon_j}(t-t_0)} + \frac{\varepsilon_j}{\varepsilon_{j-1}} \right)$; here c will depend upon the initial conditions.

We deduce

$$\left| \sum_{\ell=0}^{j} A^j_{j\ell}(t)\, \tilde{y}_\ell(t,\varepsilon) \right| \le c \left[\frac{\varepsilon_j}{\varepsilon_{j-1}} + e^{-\frac{\alpha_j}{\varepsilon_j}(t-t_0)} \right]$$

For $k \le N-1$ we replace in the system for z_k the formula obtained for $\tilde{y}_N$, and we see that

$$\frac{1}{\eta}\left(\sum_{j=0}^{N} A_{kj}(t)\, \tilde{y}_j(t,\varepsilon) - \sum_{j=0}^{k} A^k_{kj}(t)\, \tilde{y}_j(t,\varepsilon) \right) =$$

$$= \frac{1}{\eta} \Big[\sum_{j=0}^{N-1} A_{kj}(t)\, \tilde{y}_j(t,\varepsilon) - \sum_{j=0}^{k} A^k_{kj}(t)\, \tilde{y}_j(t,\varepsilon) \; -$$

$$- \sum_{\ell=0}^{N-1} A_{kN}(t)\, A^{-1}_{NN}(t)\, A_{N\ell}(t)\, \tilde{y}_\ell(t,\varepsilon) + A_{kN}(t)\, \omega_N(t,\varepsilon) \Big] =$$

$$= \frac{1}{\eta} \left[\sum_{j=0}^{N-1} A^{N-1}_{kj}(t)\, \tilde{y}_j(t,\varepsilon) - \sum_{j=0}^{k} A^k_{kj}(t)\, \tilde{y}_j(t,\varepsilon) + A_{kN}(t)\, \omega_N(t,\varepsilon) \right]$$

For $k \le N-2$ we may now replace $\tilde{y}_{N-1}$, and in the same way ,we get

$$\frac{1}{\eta} \left[\sum_{j=0}^{N} A_{kj}(t)\, \tilde{y}_j(t,\varepsilon) - \sum_{j=0}^{k} A^k_{kj}(t)\, \tilde{y}_j(t,\varepsilon) \right] =$$

$$= \frac{1}{\eta} \sum_{j=k+1}^{N} A^j_{kj}(t)\, \omega_j(t,\varepsilon)$$

hence

$$z(t,\varepsilon) = \sum_{k=0}^{N-1} \frac{1}{\varepsilon_k} \int_{t_0}^{t} \Gamma_k(t,\sigma,\varepsilon) \frac{1}{\eta} \sum_{j=k+1}^{N} A^j_{kj}(\sigma)\, \omega_j(\sigma,\varepsilon)\, d\sigma$$

The estimates for Γ_k together with the ones for ω_j show that z is bounded.

Remark that if $\alpha_0 > 0$, z and ω_j admit uniform exponential estimates.

Remark also that $\tilde{y}_0$ is the solution of the reduced problem and that the other components satisfy

$$\sum_{\ell=0}^{j} A^j_{j\ell}(t)\, \tilde{y}_\ell(t,\varepsilon) = A^j_{jj}(t)\, \omega_j(t,\varepsilon)$$

and the estimates for ω_j show that their principal parts satisfy the algebraic equations associated with the problem, for $t > t_0$. Remark also in ω_j the boundary-layer term $C_j(t,t_0,\varepsilon)\left[y_j^0 + \sum_{\ell=0}^{j-1}[A_{jj}^j(t)]^{-1}A_{j\ell}^j(t)\,y_\ell^0\right]$.

4. Almost periodic solutions

Consider now the case $\alpha_0 > 0$ and the system

$$\varepsilon_k \frac{dy_k}{dt} = \sum_{j=0}^{N} A_{kj}(t)\,y_j + f_k(t).$$

where f_k are bounded on the whole axis (here $I=R$) and uniformly Lipschitz. From the stability assumption and taking into account the estimates for the fundamental matrix we see that the solution defined by

$$y(t,\varepsilon) = \int_{-\infty}^{t}\sum_{k=0}^{N}\frac{1}{\varepsilon_k}\Gamma_k(t,s,\varepsilon)\,f_k(s)\,ds$$

is bounded on the whole axis (and it is the unique one with this property) and moreover it is uniformly bounded with respect to ε.

If the coefficients are almost periodic ,this solution is almost periodic . We study closer this solution. Define $\tilde{y}_k$ to be the unique bounded on the whole axis solution of the system

$$\varepsilon_k \frac{d\tilde{y}_k}{dt} = \sum_{j=0}^{k} A_{kj}^k(t)\,\tilde{y}_j + f_k^k(t) \qquad k = 0,1,\dots,N$$

where

$$f_j^{\ell-1}(t) = f_j^\ell(t) - A_{j\ell}^\ell(t)\left[A_{\ell\ell}^\ell(t)\right]^{-1} f_\ell^\ell(t)\quad ;\quad f_j^N(t) = f_j(t)$$

Define z_k by the relations

$$y_k = \tilde{y}_k + \eta\, z_k$$ where y_k are the components of the unique bounded on the whole axis solution of the given system . We get for z_k the system

$$\varepsilon_k \frac{dz_k}{dt} = \sum_{j=0}^{N} A_{kj}(t)\,z_j + \frac{1}{\eta}\Big[\sum_{j=0}^{N} A_{kj}(t)\,\tilde{y}_j(t) - \sum_{j=0}^{k} A_{kj}^k(t)\tilde{y}_j(t) + f_k(t) - f_k^k(t)\Big]$$

As in the preceding section we may write

$$\tilde{y}_k(t,\varepsilon) = -\sum_{\ell=0}^{k-1}\left[A_{kk}^k(t)\right]^{-1}A_{k\ell}^k(t)\,\tilde{y}_\ell(t,\varepsilon) - \left[A_{kk}^k(t)\right]^{-1} f_k^k(t) + \omega_k(t,\varepsilon)$$

where

$$\omega_k(t,\varepsilon) = \int_{-\infty}^{t} C_k(t,\sigma,\varepsilon)\sum_{\ell=0}^{k-1}\left[A_{kk}^k(t)\right]^{-1}A_{k\ell}^k(t)\,\frac{1}{\varepsilon_\ell}\Big[\sum_{j=0}^{\ell} A_{\ell j}^\ell(\sigma)\,\tilde{y}_\ell(\sigma,\varepsilon) + f_\ell^\ell(\sigma)\Big]d\sigma$$

$$+\frac{1}{\varepsilon_k}\int_{-\infty}^{t} C_k(t,\sigma,\varepsilon)\,A_{kk}^k(\sigma)\sum_{\ell=0}^{k-1}\Big[\left(A_{kk}^k(\sigma)\right)^{-1}A_{k\ell}^k(\sigma) - \left(A_{kk}^k(t)\right)^{-1}A_{k\ell}^k(t)\Big]\tilde{y}_\ell(\sigma,\varepsilon)\,d\sigma$$

$$+\frac{1}{\varepsilon_k}\int_{-\infty}^{t} C_k(t,\sigma,\varepsilon)\,A_{kk}^k(\sigma)\Big[\left(A_{kk}^k(\sigma)\right)^{-1} f_k^k(\sigma) - \left(A_{kk}^k(t)\right)^{-1} f_k^k(t)\Big]d\sigma$$

hence $|\omega_k(t,\varepsilon)| \le c\eta$.

By the same procedure as in the preceding section we have

$$\varepsilon_k \frac{dz_k}{dt} = \sum_{j=0}^{N} A_{kj}(t)\, z_j + \frac{1}{\eta}\sum_{j=k+1}^{N} A^j_{kj}(t)\,\omega_j(t,\varepsilon).$$

The estimates for the fundamental matrix and the ones for ω_j allow us to deduce that z_k are bounded uniformly, hence $\tilde{y}_k$ give the principal part of the bounded on the whole axis solution.

Let us remark that $\tilde{y}_k$ are the bounded on the whole axis (periodic, almost periodic respectively in the corresponding cases) of the systems

$$\varepsilon_k \frac{dy_k}{dt} = \sum_{j=0}^{k-1} A_{kj}(t)\,\tilde{y}_j(t,\varepsilon) + \sum_{j=k}^{N} A_{kj}(t)\, y_j + f_k(t)$$

$$0 = \sum_{j=0}^{k-1} A_{\ell j}(t)\,\tilde{y}_j(t,\varepsilon) + \sum_{j=k}^{N} A_{\ell j}(t)\, y_j + f_\ell(t), \quad \ell \geq k+1;$$

namely $\tilde{y}_0$ is the solution of the reduced system

$$\frac{dy_0}{dt} = \sum_{j=0}^{N} A_{0j}(t)\, y_j + f_0(t)$$

$$0 = \sum_{j=0}^{N} A_{kj}(t)\, y_j + f_k(t), \quad k \geq 1.$$

Let $\hat{y}_\ell$ be the functions defined by the equations

$$0 = \sum_{j=1}^{N} A_{kj}(t)\,\hat{y}_j + f_k(t) + A_{k0}(t)\,\tilde{y}_0(t), \; k \geq 1.$$

If in these equations we perform the succesive elimination of $\hat{y}_N$, $\hat{y}_{N-1}$, ... we get

$$\sum_{\ell=1}^{k} A^k_{k\ell}(t)\,\hat{y}_\ell + f^k_k(t) + A_{k0}(t)\,\tilde{y}_0(t) = 0$$

and it follows that we may write

$$\left|\sum_{\ell=1}^{k} A^k_{k\ell}(t)\left[\tilde{y}_\ell(t,\varepsilon) - \hat{y}_\ell(t)\right]\right| \le c\,\frac{\varepsilon_k}{\varepsilon_{k-1}}$$

and from here we get

$$|\tilde{y}_\ell(t,\varepsilon) - \hat{y}_\ell(t)| \le c\eta.$$

We see in this way that

$$|y_\ell(t,\varepsilon) - \hat{y}_\ell(t)| \le c\eta, \quad |y_0(t,\varepsilon) - \tilde{y}_0(t)| \le c\eta$$

and $\tilde{y}_0$, $\hat{y}_1$, ..., $\hat{y}_N$ represent the first approximation for the unique bounded on the whole axis solution of the affine system.

This result is the extension and in some sense a refinement of the essential part of the theorems of Levinson-Flatto [2] and Hale - Seifert [3]

5. Impulsive initial conditions

We consider now the Cauchy problem

$$\varepsilon_k \frac{dy_k}{dt} = \sum_{j=0}^{N} A_{kj}(t)\, y_j \quad , \quad y_k(t_o,\varepsilon) = \frac{1}{\varepsilon_k}\, y_k^0$$

and look for the behavior of the solutions for $\varepsilon \to 0$.

We shall proceed by induction .

A, Write the system in the form

$$\varepsilon_k \frac{dy_k}{dt} = \sum_{j=0}^{N-1} A_{kj}(t)\, y_j + A_{kN}(t)\, y_N$$

$$\varepsilon_N \frac{dy_N}{dt} = \sum_{j=0}^{N-1} A_{Nj}(t)\, y_j + A_{NN}(t)\, y_N \, .$$

DEnote

$$\mathcal{A}_N^{N-1}(t,\varepsilon) = \mathrm{col}\left(\frac{1}{\varepsilon_k} A_{kN}(t) \, , \; 0 \leq k \leq N-1\right)$$

$$\mathcal{A}_{N-1}^{N}(t) = \left(A_{N0}(t) \, \ldots\ldots \, A_{N,N-1}(t)\right)$$

$$\mathcal{A}^{N-1}(t) = \left(\frac{1}{\varepsilon_k} A_{kj}(t) \, , \; k,j = 0, \ldots, N-1\right)$$

$$y^{N-1} = \mathrm{col}\left(y_k \, , \; 0 \leq k \leq N-1\right.$$

The system is written

$$\frac{dy^{N-1}}{dt} = \mathcal{A}^{N-1}(t,\varepsilon)\, y^{N-1} + \mathcal{A}_N^{N-1}(t,\varepsilon)\, y_N$$

$$\varepsilon_N \frac{dy_N}{dt} = \mathcal{A}_{N-1}^{N}(t)\, y^{N-1} + A_{NN}(t)\, y_N$$

DEnote correspondingly the fundamental matrix

$$\begin{pmatrix} \Gamma^{N-1}(t,s,\varepsilon) & \Gamma_N^{N-1}(t,s,\varepsilon) \\ \Gamma_{N-1}^{N}(t,s,\varepsilon) & \Gamma_N(t,s,\varepsilon) \end{pmatrix}$$

Define $\tilde{\Gamma}^{N-1}$, as the fundamental matrix of the system obtained after a first reduction $\varepsilon_N = 0$; define $\tilde{\Gamma}_{N-1}^{N}$ from the Cauchy problem

$$\varepsilon_N \frac{d}{dt} \tilde{\Gamma}_{N-1}^{N} = \mathcal{A}_{N-1}^{N}(t)\, \tilde{\Gamma}^{N-1}(t,s,\varepsilon) + A_{NN}(t)\, \tilde{\Gamma}_{N-1}^{N}$$

$$\tilde{\Gamma}_{N-1}^{N}(s,s,\varepsilon) = 0$$

We write for the fundamental matrix the expansion

$$\begin{pmatrix} \Gamma^{N-1}(t,s,\varepsilon) & \Gamma_N^{N-1}(t,s,\varepsilon) \\ \Gamma_{N-1}^{N}(t,s,\varepsilon) & \Gamma_N(t,s,\varepsilon) \end{pmatrix} = \begin{pmatrix} \tilde{\Gamma}^{N-1}(t,s,\varepsilon) & 0 \\ \tilde{\Gamma}_{N-1}^{N}(t,s,\varepsilon) & C_N(t,s,\varepsilon) \end{pmatrix} +$$

$$+\varepsilon_N \begin{pmatrix} \hat{\Gamma}^{N-1}(t,s,\varepsilon) & \hat{\Gamma}^{N-1}_N(t,s,\varepsilon) \\ \hat{\Gamma}^N_{N-1}(t,s,\varepsilon) & \hat{\Gamma}_N(t,s,\varepsilon) \end{pmatrix} + \varepsilon_N^2 \begin{pmatrix} \check{\Gamma}^{N-1}(t,s,\varepsilon) & \check{\Gamma}^{N-1}_N(t,s,\varepsilon) \\ \check{\Gamma}^N_{N-1}(t,s,\varepsilon) & \check{\Gamma}_N(t,s,\varepsilon) \end{pmatrix}$$

In what follows we will obtain the principal part in $\tilde{\Gamma}^N_{N-1}$, we shall estimate $\hat{\Gamma}$ and $\check{\Gamma}$ and obtain the principal parts in $\hat{\Gamma}^{N-1}_N$ and $\hat{\Gamma}_N$ whe~~n~~re we shall need to study the behavior of the solutions with impulsive initial conditions.

Denote $y^{N-1,0}(\varepsilon) = \mathrm{col}\left(\frac{1}{\varepsilon_k}\ y_k^0\ ,\ 0 \le k \le N-1\right)$.

The solution of the Cauchy problem will be

$$\begin{pmatrix} \Gamma^{N-1}(t,t_0,\varepsilon) & \Gamma^{N-1}_N(t,t_0,\varepsilon) \\ \Gamma^N_{N-1}(t,t_0,\varepsilon) & \Gamma_N(t,t_0,\varepsilon) \end{pmatrix} \begin{pmatrix} y^{N-1,0}(\varepsilon) \\ \frac{1}{\varepsilon_N} y_N^0 \end{pmatrix}$$

and we see that

$$y^{N-1}(t,t_0,\varepsilon) = \tilde{\Gamma}^{N-1}(t,t_0,\varepsilon)\,y^{N-1,0}(\varepsilon) + \varepsilon_N \hat{\Gamma}^{N-1}(t,t_0,\varepsilon)\,y^{N-1,0}(\varepsilon) +$$

$$+ \hat{\Gamma}^{N-1}_N(t,t_0,\varepsilon)\,y_N^0 + \varepsilon_N^2 \check{\Gamma}^{N-1}(t,t_0,\varepsilon)\,y^{N-1,0}(\varepsilon) + \varepsilon_N \check{\Gamma}^{N-1}_N(t,t_0,\varepsilon)\,y_N^0$$

$$y_N(t,t_0,\varepsilon) = \tilde{\Gamma}^N_{N-1}(t,t_0,\varepsilon)\ y^{N-1,0}(\varepsilon) + \frac{1}{\varepsilon_N} C_N(t,t_0,\varepsilon)\ y_N^0 +$$

$$+ \varepsilon_N \hat{\Gamma}^N_{N-1}(t,t_0,\varepsilon)\,y^{N-1,0}(\varepsilon) + \hat{\Gamma}_N(t,t_0,\varepsilon)\,y_N^0 +$$

$$+ \varepsilon_N^2 \check{\Gamma}^N_{N-1}(t,t_0,\varepsilon)\,y^{N-1,0}(\varepsilon) + \varepsilon_N \check{\Gamma}_N(t,t_0,\varepsilon)\ y_N^0 .$$

In the same way as we did it before we obtain

$$\tilde{\Gamma}^N_{N-1}(t,s,\varepsilon) = -A^{-1}_{NN}(t)\,\mathcal{A}^N_{N-1}(t)\,\tilde{\Gamma}^{N-1}(t,s,\varepsilon) +$$

$$+ C_N(t,s,\varepsilon)\ A^{-1}_{NN}(t)\,\mathcal{A}^N_{N-1}(t) + \tilde{\omega}^N_{N-1}(t,s,\varepsilon)$$

$$\tilde{\omega}^N_{N-1}(t,s,\varepsilon) = \frac{1}{\varepsilon_N}\int_s^t C_N(t,\sigma,\varepsilon)\ A_{NN}(\sigma)\left[\ A^{-1}_{NN}(\sigma)\,\mathcal{A}^N_{N-1}(\sigma) - \right.$$

$$\left. - A^{-1}_{NN}(t)\,\mathcal{A}^N_{N-1}(t)\right]\tilde{\Gamma}^{N-1}(\sigma,s,\varepsilon)\,d\sigma +$$

$$+ \int_s^t C_N(t,\sigma,\varepsilon)\ A^{-1}_{NN}(t)\,\mathcal{A}^N_{N-1}(t)\,\tilde{\mathcal{A}}^{N-1}(\sigma,\varepsilon)\,\tilde{\Gamma}^{N-1}(\sigma,s,\varepsilon)\,d\sigma$$

where $\tilde{\mathcal{A}}^{N-1}(t,\varepsilon)$ is the matrix of the system obtained after the first reduction. Taking into account the main estimates for the columns of the fundamental matrix $\tilde{\Gamma}^{N-1}$ as obtained in section 2 we see that

$$\left|\tilde{\omega}^{N}_{N-1}(t,t_0,\varepsilon)\, y^{N-1,0}(\varepsilon)\right| \le \frac{c\,\varepsilon_N}{\varepsilon_{N-1}}\, e^{-\alpha_0(t-t_0)}$$

We define $\hat{\Gamma}$ from the system

$$\frac{d}{dt}\hat{\Gamma}^{N-1}(t,s,\varepsilon) = \tilde{\mathcal{A}}^{N-1}(t,\varepsilon)\,\hat{\Gamma}^{N-1}(t,s,\varepsilon) +$$

$$+\frac{1}{\varepsilon_N}\mathcal{A}^{N-1}_{N}(t,\varepsilon)\left[\tilde{\Gamma}^{N}_{N-1}(t,s,\varepsilon) + A^{-1}_{NN}(t)\,\mathcal{A}^{N}_{N-1}(t)\,\tilde{\Gamma}^{N-1}(t,s,\varepsilon)\right]$$

$$\varepsilon_N\frac{d}{dt}\hat{\Gamma}^{N}_{N-1}(t,s,\varepsilon) = \mathcal{A}^{N}_{N-1}(t)\,\hat{\Gamma}^{N-1}(t,s,\varepsilon) + A_{NN}(t)\,\hat{\Gamma}^{N}_{N-1}(t,s,\varepsilon)$$

$$\frac{d}{dt}\hat{\Gamma}^{N-1}_{N}(t,s,\varepsilon) = \tilde{\mathcal{A}}^{N-1}(t,\varepsilon)\,\hat{\Gamma}^{N-1}_{N}(t,s,\varepsilon) + \frac{1}{\varepsilon_N}\mathcal{A}^{N-1}_{N}(t,\varepsilon)\, C_N(t,s,\varepsilon)$$

$$\varepsilon_N\frac{d}{dt}\hat{\Gamma}_{N}(t,s,\varepsilon) = \mathcal{A}^{N}_{N-1}(t)\,\Gamma^{N-1}_{N}(t,s,\varepsilon) + A_{NN}(t)\,\hat{\Gamma}_{N}(t,s,\varepsilon)$$

with zero initial ~~conditions~~ values.

Since

$$\tilde{\mathcal{A}}^{N-1}(t,\varepsilon) = \mathcal{A}^{N-1}(t,\varepsilon) - \mathcal{A}^{N-1}_{N}(t,\varepsilon)\,A^{-1}_{NN}(t)\,\mathcal{A}^{N}_{N-1}(t)$$

we get by direct cheking, that $\check{\Gamma}$ are defined by the system

$$\frac{d}{dt}\check{\Gamma}^{N-1}(t,s,\varepsilon) = \mathcal{A}^{N-1}(t,\varepsilon)\,\check{\Gamma}^{N-1}(t,s,\varepsilon) + \mathcal{A}^{N-1}_{N}(t,\varepsilon)\,\check{\Gamma}^{N}_{N-1}(t,s,\varepsilon) +$$

$$+\frac{1}{\varepsilon_N}\mathcal{A}^{N-1}_{N}(t,\varepsilon)\,A^{-1}_{NN}(t)\left[\mathcal{A}^{N}_{N-1}(t)\,\hat{\Gamma}^{N-1}(t,s,\varepsilon) + A_{NN}(t)\,\hat{\Gamma}^{N}_{N-1}(t,s,\varepsilon)\right]$$

$$\varepsilon_N\frac{d}{dt}\check{\Gamma}^{N}_{N-1}(t,s,\varepsilon) = \mathcal{A}^{N}_{N-1}(t)\,\check{\Gamma}^{N-1}(t,s,\varepsilon) + A_{NN}(t)\,\check{\Gamma}^{N}_{N-1}(t,s,\varepsilon)$$

$$\frac{d}{dt}\check{\Gamma}^{N-1}_{N}(t,s,\varepsilon) = \mathcal{A}^{N-1}(t,\varepsilon)\,\check{\Gamma}^{N-1}_{N}(t,s,\varepsilon) + \mathcal{A}^{N-1}_{N}(t,\varepsilon)\,\check{\Gamma}_{N}(t,s,\varepsilon) +$$

$$+\frac{1}{\varepsilon_N}\mathcal{A}^{N-1}_{N}(t,\varepsilon)\,A^{-1}_{NN}(t)\left[\mathcal{A}^{N}_{N-1}(t)\,\hat{\Gamma}^{N-1}_{N}(t,s,\varepsilon) + A_{NN}(t)\,\hat{\Gamma}_{N}(t,s,\varepsilon)\right]$$

$$\varepsilon_N\frac{d}{dt}\check{\Gamma}_{N}(t,s,\varepsilon) = \mathcal{A}^{N}_{N-1}(t)\,\check{\Gamma}^{N-1}_{N}(t,s,\varepsilon) + A_{NN}(t)\,\check{\Gamma}^{N}_{N-1}(t,s,\varepsilon)$$

with zero initial values .

We see that

$$\hat{\Gamma}^{N-1}(t,s,\varepsilon) = \frac{1}{\varepsilon_N}\int_s^t \tilde{\Gamma}^{N-1}(t,\sigma,\varepsilon)\,\mathcal{A}^{N-1}_{N}(\sigma,\varepsilon)\left[\tilde{\omega}^{N}_{N-1}(\sigma,s,\varepsilon) + \right.$$

$$\left. + C_N(\sigma,s,\varepsilon)\, A^{-1}_{NN}(\sigma)\,\mathcal{A}^{N}_{N-1}(\sigma)\right]d\sigma$$

hence

$$\left|\hat{\Gamma}^{N-1}(t,t_0,\varepsilon)\, y^{N-1,0}(\varepsilon)\right| \le \frac{c}{\varepsilon_{N-1}}\, e^{-\alpha_0(t-t_0)}$$

To obtain this estimate we have used again the main estimates for the columns of the fundamental matrix Γ^{N-1}.

In the usual way we have further

$$\Gamma^{N}_{N-1}(t,s,\varepsilon) = - A^{-1}_{NN}(t)\mathcal{A}^{N}_{N-1}(t)\hat{\Gamma}^{N-1}(t,s,\varepsilon) + \hat{\omega}^{N}_{N-1}(t,s,\varepsilon)$$

with

$$\hat{\omega}^{N}_{N-1}(t,s,\varepsilon) = \frac{1}{\varepsilon_N}\int_s^t C_N(t,\sigma,\varepsilon) A_{NN}(\sigma)[A^{-1}_{NN}(\sigma)\mathcal{A}^{N}_{N-1}(\sigma) - A^{-1}_{NN}(t)\mathcal{A}^{N}_{N-1}(t)]\hat{\Gamma}^{N-1}(\sigma,s,\varepsilon)\,d\sigma + \int_s^t C_N(t,\sigma,\varepsilon) A^{-1}_{NN}(t)\mathcal{A}^{N}_{N-1}(t)\frac{d}{d\sigma}\hat{\Gamma}^{N-1}(\sigma,s,\varepsilon)\,d\sigma$$

By taking into account the preceding estimates we deduce that

$$|\hat{\omega}^{N}_{N-1}(t,t_0,\varepsilon)\, y^{N-1,0}(\varepsilon)| \leq c\,\frac{\varepsilon_N}{\varepsilon_{N-1}}\, e^{-\alpha_0(t-t_0)}.$$

Since

$$\begin{pmatrix} \check{\Gamma}^{N-1}(t,s,\varepsilon) \\ \check{\Gamma}^{N}_{N-1}(t,s,\varepsilon)\end{pmatrix} = \frac{1}{\varepsilon_N}\int_s^t \begin{pmatrix} \Gamma^{N-1}(t,s,\varepsilon) \\ \Gamma^{N}_{N-1}(t,s,\varepsilon)\end{pmatrix}\mathcal{A}^{N-1}_{N}(\sigma,\varepsilon)\,\hat{\omega}^{N}_{N-1}(\sigma,s,\varepsilon)\,d\sigma$$

we get for $\check{\Gamma}^{N-1}(t,t_0,\varepsilon)\,y^{N-1,0}(\varepsilon)$ and $\check{\Gamma}^{N}_{N-1}(t,t_0,\varepsilon)\,y^{N-1,0}(\varepsilon)$ estimates of the form $\frac{c}{\varepsilon_{N-1}} e^{-\alpha_0(t-t_0)}$.

We have further

$$\hat{\Gamma}^{N-1}_{N}(t,s,\varepsilon) = -\tilde{\Gamma}^{N-1}(t,s,\varepsilon)\mathcal{A}^{N-1}_{N}(s,\varepsilon)A^{-1}_{NN}(s) + \mathcal{A}^{N-1}_{N}(s,\varepsilon)A^{-1}_{NN}(s)C_N(t,s,\varepsilon) + \hat{\omega}^{N-1}_{N}(t,s,\varepsilon)$$

$$\hat{\omega}^{N-1}_{N}(t,s,\varepsilon) = \frac{1}{\varepsilon_N}\int_s^t \tilde{\Gamma}^{N-1}(t,\sigma,\varepsilon)[\mathcal{A}^{N-1}_{N}(\sigma,\varepsilon)A^{-1}_{NN}(\sigma) - \mathcal{A}^{N-1}_{N}(s,\varepsilon)A^{-1}_{NN}(s)]A_{NN}(\sigma)C_N(\sigma,s,\varepsilon)\,d\sigma - \int_s^t [\frac{d}{d\sigma}\tilde{\Gamma}^{N-1}(t,\sigma,\varepsilon)]\mathcal{A}^{N-1}_{N}(\sigma,\varepsilon)A^{-1}_{NN}(s)C_N(\sigma,s,\varepsilon)\,d\sigma$$

and $\quad |\hat{\omega}^{N-1}_{N}(t,s,\varepsilon)| \leq c\,\frac{\varepsilon_N}{\varepsilon_{N-1}} e^{-\alpha_0(t-s)}$

$$\hat{\Gamma}_N(t,s,\varepsilon) = \frac{1}{\varepsilon_N}\int_s^t C_N(t,\sigma,\varepsilon)\mathcal{A}^{N}_{N-1}(\sigma)\hat{\Gamma}^{N-1}_{N}(\sigma,s,\varepsilon)\,d\sigma =$$

$$= -A^{-1}_{NN}(t)\mathcal{A}^{N}_{N-1}(t)[-\tilde{\Gamma}^{N-1}(t,s,\varepsilon)\mathcal{A}^{N-1}_{N}(s,\varepsilon)A^{-1}_{NN}(s) + \mathcal{A}^{N-1}_{N}(s,\varepsilon)A^{-1}_{NN}(s)C_N(t,s,\varepsilon) + \hat{\omega}^{N-1}_{N}(t,s,\varepsilon)].$$

with $\quad |\hat{\omega}_N(t,s,\varepsilon)| \leq \frac{c\,\varepsilon_N}{\varepsilon_{N-1}}\, e^{-\alpha_0(t-s)}$

Since

$$\begin{pmatrix} \check{\Gamma}_N^{N-1}(t,s,\varepsilon) \\ \check{\Gamma}_N(t,s,\varepsilon) \end{pmatrix} = \frac{1}{\varepsilon_N} \int_s^t \begin{pmatrix} \Gamma^{N-1}(t,\sigma,\varepsilon) \\ \Gamma_{N-1}^{N}(t,\sigma,\varepsilon) \end{pmatrix} \mathcal{A}_N^{N-1}(\sigma,\varepsilon)\, \hat{\omega}_N(\sigma,s,\varepsilon)\, d\sigma$$

we get again an estimate of the form $\frac{c}{\varepsilon_{N-1}} e^{-\alpha_0(t-s)}$ . .

By summingup all estimates we see that

$$y^{N-1}(t,t_0,\varepsilon) = \tilde{\Gamma}^{N-1}(t,t_0,\varepsilon)\left[y^{N-1,0}(\varepsilon) - \mathcal{A}_N^{N-1}(t_0,\varepsilon) A_{NN}^{-1}(t_0) y_N^0\right] + \\ + \mathcal{A}_N^{N-1}(t_0,\varepsilon) A_{NN}^{-1}(t_0)\, C_N(t,t_0,\varepsilon)\, y_N^0 + O(\eta)$$

$$y_N(t,t_0,\varepsilon) = -A_{NN}^{-1}(t)\mathcal{A}_{N-1}^{N}(t)\tilde{\Gamma}^{N-1}(t,t_0,\varepsilon)\left[y^{N-1,0}(\varepsilon) - \mathcal{A}_N^{N-1}(t_0,\varepsilon)A_{NN}^{-1}(t_0)y_N^0\right] + \\ + \frac{1}{\varepsilon_N} C_N(t,t_0,\varepsilon)y_N^0 - A_{NN}^{-1}(t)\mathcal{A}_{N-1}^{N}(t)\mathcal{A}_N^{N-1}(t_0,\varepsilon)A_{NN}^{-1}(t_0)\, C_N(t,t_0,\varepsilon)y_N^0 + \\ + C_N(t,t_0,\varepsilon)A_{NN}^{-1}(t)\mathcal{A}_{N-1}^{N}(t)\, y^{N-1,0}(\varepsilon) + O(\eta).$$

$$A_{NN}(t)y_N(t,t_0,\varepsilon) + \sum_{j=0}^{N-1} A_{Nj}(t)y_j(t,t_0,\varepsilon) = A_{NN}(t)\, C_N(t,t_0,\varepsilon)A_{NN}^{-1}(t)\sum_{j=0}^{N} A_{Nj}(t)\frac{1}{\varepsilon_j} y_j^0 + O(\eta)$$

B. Denote $y_k^{N,0} = y_k^0$, $y_k^{\ell-1,0} = y_k^{\ell,0} - A_{k\ell}^{\ell}(t_0)\left[A_{\ell\ell}^{\ell}(t_0)\right]^{-1} y_\ell^{\ell,0}$;

the above formulae may be written as

$$y_j(t,t_0,\varepsilon) = \sum_{k=0}^{N-1} \tilde{\gamma}_{jk}^{N-1}(t,t_0,\varepsilon)\frac{1}{\varepsilon_k} y_k^{N-1,0} + \frac{1}{\varepsilon_j} A_{jN}(t_0)\, A_{NN}^{-1}(t_0)\, C_N(t,t_0,\varepsilon)y_N^0 + O(\eta)$$

$0 \le j \le N-1$

$$\sum_{j=0}^{N} A_{Nj}(t)\, y_j(t,t_0,\varepsilon) = A_{NN}(t)\, C_N(t,t_0,\varepsilon)\, A_{NN}^{-1}(t)\sum_{j=0}^{N} A_{Nj}(t)\frac{1}{\varepsilon_j} y_j^0 + O(\eta)$$

To perform an induction assume

$$y_j(t,t_0,\varepsilon) = \sum_{k=0}^{p} \tilde{\gamma}_{jk}^{p}(t,t_0,\varepsilon)\frac{1}{\varepsilon_k} y_k^{p,0} + \frac{1}{\varepsilon_j}\sum_{\ell=p+1}^{N} A_{j\ell}^{\ell}(t_0)\left[A_{\ell\ell}^{\ell}(t_0)\right]^{-1} C_\ell(t,t_0,\varepsilon)\, y_\ell^{\ell,0} + O(\eta)$$

$0 \le j \le p$,

The induction will consist in developing $\tilde{\gamma}_{jk}^{p}$; since

$\sum_{k=0}^{p} \tilde{\gamma}_{jk}^{p}(t,t_0,\varepsilon)\frac{1}{\varepsilon_k} y_k^{p,0}$ is now in the same assumptions as we worked with in A . we may write

$$\sum_{k=0}^{p} \tilde{\gamma}_{jk}^{p}(t,t_0,\varepsilon)\frac{1}{\varepsilon_k} y_k^{p,0} = \sum_{k=0}^{p-1} \tilde{\gamma}_{jk}^{p-1}(t,t_0,\varepsilon)\frac{1}{\varepsilon_k} y_k^{p-1,0} + \\ + \frac{1}{\varepsilon_j} A_{jp}^{p}(t_0)\left[A_{pp}^{p}(t_0)\right]^{-1} C_p(t,t_0,\varepsilon)\, y_p^{p,0} + O(\eta),$$

$0 \le j \le p-1$

$$\sum_{j=0}^{p} A^p_{pj}(t)\sum_{k=0}^{p} \tilde{\gamma}^p_{jk}(t,t_o,\varepsilon)\frac{1}{\varepsilon_k} y_k^{p,o} = A^p_{pp}(t)\, C_p(t,t_o,\varepsilon)\left[A^p_{pp}(t)\right]^{-1}\sum_{j=0}^{p} A^p_{pj}(t)\frac{1}{\varepsilon_j}\, y_j^{p,o} +$$

$$+ O(\eta);$$

we deduce

$$y_j(t,t_o,\varepsilon) = \sum_{k=0}^{p-1} \tilde{\gamma}^{p-1}_{jk}(t,t_o,\varepsilon)\frac{1}{\varepsilon_k} y_k^{p-1,o} + \sum_{\ell=p}^{N} A^\ell_{j\ell}(t_o)\left[A^\ell_{\ell\ell}(t_o)\right]^{-1} C_\ell(t,t_o,\varepsilon)\, y_\ell^{\ell,o} + O(\eta).$$

for $0 \leq j \leq p-1$

and the induction assumption is confermed ; we have further

$$\sum_{j=0}^{p} A^p_{pj}(t)\, y_j(t,t_o,\varepsilon) = \sum_{j=0}^{p} A^p_{pj}(t)\Big[\sum_{k=0}^{p} \tilde{\gamma}^p_{jk}(t,t_o,\varepsilon)\frac{1}{\varepsilon_k}\, y_k^{p,o} +$$

$$+ \frac{1}{\varepsilon_j}\sum_{\ell=p+1}^{N} A^\ell_{j\ell}(t_o)\left[A^\ell_{\ell\ell}(t_o)\right]^{-1} C_\ell(t,t_o,\varepsilon)\, y_\ell^{\ell,o} + O(\eta)\Big] = A^p_{pp}(t)\, C_p(t,t_o,\varepsilon)\left[A^p_{pp}(t_o)\right]^{-1}\cdot$$

$$\cdot\sum_{j=0}^{p} A^p_{pj}(t)\frac{1}{\varepsilon_j} y_j^{p,o} + O(\eta) + \sum_{j=0}^{p} A^p_{pj}(t)\frac{1}{\varepsilon_j}\sum_{\ell=p+1}^{N} A^\ell_{j\ell}(t_o)\left[A^\ell_{\ell\ell}(t_o)\right]^{-1} C_\ell(t,t_o,\varepsilon)\, y_\ell^{\ell,o}$$

Since $y_j^o = y_j^{p,o} + \sum_{\ell=p+1}^{N} A^\ell_{j\ell}(t_o)\left[A^\ell_{\ell\ell}(t_o)\right]^{-1} y_\ell^{\ell,o}$

we may write also

$$\sum_{j=0}^{p} A^p_{pj}(t)\, y_j(t,t_o,\varepsilon) = A^p_{pp}(t) C_p(t,t_o,\varepsilon)\left[A^p_{pp}(t)\right]^{-1}\sum_{j=0}^{p} A^p_{pj}(t)\frac{1}{\varepsilon_j} y_j^o +$$

$$+ \sum_{j=0}^{p} A^p_{pj}(t)\frac{1}{\varepsilon_j}\sum_{\ell=p+1}^{N} A^\ell_{j\ell}(t_o)\left[A^\ell_{\ell\ell}(t_o)\right]^{-1} C_\ell(t,t_o,\varepsilon)\, y_\ell^{\ell,o} -$$

$$- A^p_{pp}(t)\, C_p(t,t_o,\varepsilon)\left[A^p_{pp}(t)\right]^{-1}\sum_{j=0}^{p} A^p_{pj}(t)\frac{1}{\varepsilon_j}\sum_{\ell=p+1}^{N} A^\ell_{j\ell}(t_o)\left[A^\ell_{\ell\ell}(t_o)\right]^{-1} y_\ell^{\ell,o} + O(\eta);$$

we deduce

$$y_o(t,t_o,\varepsilon) = C^o_{oo}(t,t_o)\, y_o^{o,o} + \sum_{\ell=1}^{N} A^\ell_{o\ell}(t_o)\left[A^\ell_{\ell\ell}(t_o)\right]^{-1} C_\ell(t,t_o,\varepsilon)\, y_\ell^{\ell,o} + O(\eta)$$

and the other components of the solution can be written step by step ; for instance

$$A^1_{10}(t)\, y_o(t,t_o,\varepsilon) + A^1_{11}(t)\, y_1(t,t_o,\varepsilon) = A^1_{11}(t)\, C_1(t,t_o,\varepsilon)\left[A^1_{11}(t)\right]^{-1}\Big(A^1_{10}(t)\, y_o^o +$$

$$+ \frac{1}{\varepsilon_1} A^1_{11}(t)\, y_1^o\Big] + \sum_{j=0}^{1} A^1_{1j}(t)\frac{1}{\varepsilon_j}\sum_{\ell=2}^{N} A^\ell_{j\ell}(t_o)\left[A^\ell_{\ell\ell}(t_o)\right]^{-1} C_\ell(t,t_o,\varepsilon)\, y_\ell^{\ell,o} -$$

$$- A^o_{11}(t)\, C_1(t,t_o,\varepsilon)\left[A^1_{11}(t)\right]^{-1}\sum_{j=0}^{1} A^1_{1j}(t)\frac{1}{\varepsilon_j}\sum_{\ell=2}^{N} A^\ell_{j\ell}(t_o)\left[A^\ell_{\ell\ell}(t_o)\right]^{-1} y_\ell^{\ell,o} + O(\eta).$$

and the equation gives information concerning $y_1(t,t_o,\varepsilon)$.

We use the same relation for p =2 to get informations concerning $y_2(t,t_o,\varepsilon)$ and so on .

Consider the equations obtained for $\varepsilon = 0$, $k=1,\ldots,N$ written in the triangular form corresponding to step by step elimination ,

$$\sum_{j=0}^{p} A^{r}_{pj}(t)\, y_j = 0 \qquad p=1,2,\ldots,N$$

Denote by $\tilde{y}(t,t_o,\varepsilon)$ the solution corresponding to the slow componemts replaced by $C^{o}_{oo}(t,t_o)\, y^{o,o}_o$.

Then we have

$$A^{1}_{10}(t)\left[y_o(t,t_o,\varepsilon)-\tilde{y}_o(t,t_o)\right] + A^{1}_{11}(t)\left[y_1(t,t_o,\varepsilon)-\tilde{y}_1(t,\varepsilon\right]=$$
$$= A^{1}_{11}(t)\, C_1(t,t_o,\varepsilon)\left[A^{1}_{11}(t)\right]^{-1}\left(A^{1}_{10}(t)\, y^{o}_o + \frac{1}{\varepsilon_1} A^{1}_{11}(t)\, y^{o}_1\right) + \ldots$$
$$\ldots\ldots + O(\eta).$$

and we deduce $y_1(t,t_o,\varepsilon)-\tilde{y}_1(t,t_o,\varepsilon) \to 0$ for $\varepsilon \to 0$ $t > t_o$, since all the terms are $O(\eta)$ or of boundary-layer type . In the same way we can use all other relations .

Remark finally that the problem with impulsive initial conditions can be generated from the problem

$$\varepsilon_N \frac{dx_k}{dt} = \sum_{j=0}^{N} \frac{\varepsilon_N}{\varepsilon_j} A_{kj}(t)\, x_j \qquad x_j(t_o) = y^{o}_j$$

by letting $y_j = \frac{1}{\varepsilon_j} x_j$.

This type of problems corresponds in the case of one parameter to the ones considered (in nonlinear setting) by A . B.Vasilieva in 1975 at the conference on nonlinear oscillations in Berlin [6] .

6. Nonlinear systems

A. Consider a system of the form

$$\varepsilon_k \frac{dy_k}{dt} = g\,(t, y_o, y_1, \ldots y_N, \varepsilon)$$

We shall start by studying the stability of a given solution $\check{y}_k$; for $z_k = y_k - \check{y}_k$ we get

$$\varepsilon_k \frac{dz_k}{dt}(t,\varepsilon) = \sum_{j=0}^{N} \frac{\partial g_k}{\partial y_j}(t, \check{y}_o(t,\varepsilon), \ldots\ldots, \check{y}_N(t,\varepsilon),\varepsilon)\; z_j(t,\varepsilon) +$$
$$+ G_k(t, z_o(t,\varepsilon), \ldots\ldots, z_N(t,\varepsilon), \varepsilon) .$$

We shall assume that

$$A_{kj}(t,\varepsilon) = \frac{\partial g_k}{\partial y_j}(t, \check{y}_o(t,\varepsilon), \ldots\ldots\, \check{y}_N(t,\varepsilon), \varepsilon)$$

satisfy all the assumptions in the introduction uniformly with respect to $0<\varepsilon\leq\varepsilon_0$.

We shall discuss this question in more detail later.

We may write

$$z(t,\varepsilon)=\sum_{k=0}^{N}\Gamma_k(t,t_0,\varepsilon)\,z_k^0+\sum_{k=0}^{N}\frac{1}{\varepsilon_k}\int_{t_0}^{t}\Gamma_k(t,s,\varepsilon)G_k(s,z(s,\varepsilon),\varepsilon)\,ds$$

Denote

$$w(t,\varepsilon)=z(t,\varepsilon)-\sum_{k=0}^{N}\Gamma_k(t,t_0,\varepsilon)\,z_k^0.$$

Then

$$w(t,\varepsilon)=\sum_{k=0}^{N}\frac{1}{\varepsilon_k}\int_{t_0}^{t_0+\alpha(\varepsilon)}\Gamma_k(t,s,\varepsilon)\,G_k\Big(s,w(s,\varepsilon)+\sum_j\Gamma_j(s,t_0,\varepsilon)\,z_j^0,\varepsilon\Big)ds+$$

$$+\frac{1}{\varepsilon_k}\int_{t_0+\alpha(\varepsilon)}^{t}\Gamma_k(t,s,\varepsilon)G_k\Big(s,w(s,\varepsilon)+\sum_j\Gamma_j(s,t_0,\varepsilon)\,z_j^0,\varepsilon\Big)ds$$

where $\alpha(\varepsilon)$ will be chosen further on.

As usually $G(s,z,\varepsilon)$ is such that for any $\gamma>0$ there exists $\delta(\gamma)$ such that if $|z|<\delta(\gamma)$ it follows $|G(s,z,\varepsilon)|\leq\gamma|z|$ for all $s\geq t_0$, $0<\varepsilon\leq\varepsilon_0$.

Chose γ depending on the stability properties of the linear problem ; chose $\alpha(\varepsilon)$ such that for $t\geq t_0+\alpha(\varepsilon)$ $|\sum_j\Gamma_j(t,t_0,\varepsilon)z_j^0|<\frac{1}{2}\delta(\gamma)$; by using the main estimates obtained in section 2 for the fundamental matrix we see that if $|z_0^0|$ is small enough, then $\lim_{\varepsilon\to 0}\alpha(\varepsilon)=0$. As long as w does not leave a compact we shall have $|G(s,w(s,\varepsilon)+\sum_j\Gamma_j(s,t_0,\varepsilon)z_j^0,\varepsilon)|\leq M$ for $s\in[t_0,t_0+\alpha(\varepsilon)]$.

Taking into account the main estimates for the fundamental matrixx we get

$$|w(t,\varepsilon)|\leq M\sum_k\frac{1}{\varepsilon_k}\int_{t_0}^{t_0+\alpha(\varepsilon)}\sum_{j=0}^{k}\frac{\varepsilon_k}{\varepsilon_j}e^{-\frac{\alpha_j}{\varepsilon_j}(t-s)}ds+$$

$$+\sum_k\frac{1}{\varepsilon_k}\int_{t_0+\alpha(\varepsilon)}^{t}|\Gamma_k(t,s,\varepsilon)||G_k(s,w(s,\varepsilon)+\sum_j\Gamma_j(s,t,\varepsilon)z_j^0,\varepsilon)|ds\leq$$

$$\leq e^{-\alpha_0(t-t_0)}\beta(\varepsilon)+\sum_k\frac{1}{\varepsilon_k}\int_{t_0+\alpha(\varepsilon)}^{t}|\Gamma_k(t,s,\varepsilon)||G_k(s,w(s,\varepsilon)+\sum_j\Gamma_j(s,t,\varepsilon)z_j^0,\varepsilon)|ds$$

In the usual way we deduce

$$|w(t,\varepsilon)|\leq\tilde\beta(\varepsilon)e^{-\frac{\alpha_0}{2}(t-t_0)}\quad\text{for}\quad t\geq t_0+\alpha(\varepsilon).$$

Then for $t\geq t_0$ we get a general estimate of the form

$$|w(t,\varepsilon)|\leq C(1-e^{-\alpha_0(t-t_0)}).$$

which shows that if $\alpha(\varepsilon)$ is small enough (that is if $\varepsilon>0$ is small enough) we can find an uniform bound for $|G_k(\cdot)|$.

In this way we have finally

$$y(t,\varepsilon)=\check y(t,\varepsilon)+\sum_{k=0}^{N}\Gamma_k(t,t_0,\varepsilon)(y_k^0-\check y_k(t_0,\varepsilon))+w(t,\varepsilon)$$

with $|w(t,\epsilon)| \leq \tilde{\beta}(\varepsilon) e^{-\frac{\alpha_0}{2}(t-t_0)}$ for $t \geqslant t_0$.

and $\lim_{\varepsilon \to 0} \tilde{\beta}(\varepsilon) = 0$, if $|y_0^0 - \check{y}_0(t_0,\epsilon)|$ is sufficiently small and $\varepsilon > 0$ is sufficiently small.

We point out the fact that the formulae shows we have much liberty for the perturbations of the fast variables $y_k^0 - \check{y}_k(t_0,\varepsilon)$ $k \geqslant 1$, because of the estimates valid for the fundamental matrix B. Consider now the main problem of approximating solutions by using the reduced ~~problem~~ system.

Let $\hat{y}$ be a solution of the reduced system

$$\frac{d\hat{y}_0(t)}{dt} = g_0(t, \hat{y}_0(t), \ldots \hat{y}_N(t), 0) \qquad \hat{y}_0(t_0) = y_0^0$$

$$0 = g_k(t, \hat{y}_0(t), \ldots \hat{y}_N(t), 0) \quad , k \geqslant 1.$$

Denote $z_k(t,\varepsilon) = y_k(t,\varepsilon) - \hat{y}_k(t)$.

It follows that

$$\varepsilon_k \frac{dz_k}{dt}(t,\varepsilon) = \sum_{j=0}^{N} A_{kj}(t) z_j(t,\varepsilon) + f_k(t,\varepsilon) + G_k(t, z_0(t,\varepsilon), \ldots z_N(t_0,\varepsilon), \varepsilon)$$

with $|f_k(t,\varepsilon)| \leq \alpha(\varepsilon)$

$$|G_k(t,z,\varepsilon)| \leq \gamma(|z| + \varepsilon) \quad \text{if } |z| + \varepsilon < \delta(\gamma) ;$$

we write again

$$z(t,\varepsilon) = \sum_{k=0}^{N} \Gamma_k(t,t_0,\varepsilon) z_k^0 + \sum_{k=0}^{N} \frac{1}{\varepsilon_k} \int_{t_0}^{t} \Gamma_k(t,s,\varepsilon) f_k(s,\varepsilon)\, ds +$$
$$+ \sum_{k=0}^{N} \frac{1}{\varepsilon_k} \int_{t_0}^{t} \Gamma_k(t,s,\varepsilon) G_k(s, z(s,\varepsilon), \varepsilon)\, ds$$

For $|z_k^0|$ small enough

$$|z(t,\epsilon)| \leq \sum_{k=0}^{N} |\Gamma_k(t,t_0,\varepsilon)| |z_k^0| + \tilde{\alpha}(\varepsilon) +$$
$$+ \tilde{c}\gamma \sum_{j=0}^{N} \frac{1}{\varepsilon_j} \int_{t_0}^{t} e^{-\frac{\alpha_j}{\varepsilon_j}(t-s)} |z(s,\varepsilon)|\, ds$$

Let $\varphi(t,\varepsilon) = \sum_{k=1}^{N} |\Gamma_k(t,t_0,\varepsilon)| |z_k^0|$

$$W(t,\varepsilon) = |z(t,\varepsilon)| - \sum_{p=0}^{\infty} \delta_p \varphi_p(t,\varepsilon)$$

with $\delta_0 = 1$, $\delta_{p+1} = (\hat{c}\gamma)\delta_p$, $\varphi_0(t,\varepsilon) = \varphi(t,\varepsilon)$

$$\varphi_{p+1}(t,\varepsilon) = \tilde{e}\gamma \sum_{j=0}^{N} \frac{1}{\varepsilon_j} \int_{t_0}^{t} e^{-\frac{\alpha_j}{\varepsilon_j}(t-s)} \varphi_p(s,\varepsilon)\, ds$$

We have

$$\varphi(t,\varepsilon) \leq k(\eta + e^{-2\frac{\alpha_1}{\varepsilon_1}(t-t_0)}) \sum_{q=1}^{N} |z_q^0|$$

Assume $\varphi_p(t,\varepsilon) \leq k(\eta + e^{-\frac{\alpha_1}{2\varepsilon_1}(t-t_0)}) \sum_{\varrho=1}^{N} |z_\varrho^0| (\hat{c}\gamma)^p$

Then $\varphi_{p+1}(t,\varepsilon) \leq \tilde{c}\gamma_k \sum_{\varrho=1}^{N} |z_\varrho^0| (\hat{c}\gamma)^p (\eta + e^{-\frac{\alpha_1}{2\varepsilon_1}(t-t_0)})\, c$

and the estimates is confermed with $\hat{c} = c\tilde{c}$

The same $\hat{c}$ has to be used to construct δ_p ; then if $\gamma < \frac{1}{\hat{c}}$

we have

$$\sum_{p=0}^{\infty} \delta_p \varphi_p(t,\varepsilon) \leq \hat{k}(\eta + e^{-\frac{\alpha_1}{2\varepsilon_1}(t-t_0)}) \sum_{\varrho=1}^{N} |z_\varrho^0|$$

we deduce

$$w(t,\varepsilon) \leq \tilde{\alpha}(\varepsilon) + \tilde{c}\gamma \sum_{j=0}^{N} \frac{1}{\varepsilon_j} \int_{t_0}^{t} e^{-\frac{\alpha_j}{\varepsilon_j}(t-s)} w(s,\varepsilon)\, ds$$

and from here $w(t,\varepsilon) \leq \beta(\varepsilon)$, hence finally

$$|z(t,\varepsilon)| \leq \beta(\varepsilon) + \hat{k}(\eta + e^{-\frac{\alpha_1}{2\varepsilon_1}(t-t_0)}) \sum_{\varrho=1}^{N} |z_\varrho^0|.$$

which shows that for convenient initial conditions the solution is uniformly with respect to t close to the one of the reduced system .

This is essentially the type of result obtained by F . Hoppensteadt [4] under more smoothness.

The uniformity of the aproximation shows that if $\check{y}$ above has good initial conditions the stability property in the first part of this section is obtained by checking the assumptions in the introduction for the matrices

$$\hat{A}_{kj}(t) = \frac{\partial g_k}{\partial y_j}(t, \hat{y}_0(t), \ldots, \hat{y}_N(t), 0) .$$

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