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II) THE SUPERPOSITION PROPERTY FOR TAYLOR'S FUNCTIONAL CALCULUS

by

Mihai PUTINAR

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7

I SOME INVARIANTS FOR SEMI-FREDHOLM SYSTEMS OF ESSENTIALLY COMMUTING OPERATORS

by Mihai Putinar

The aim of this paper is to give a sequence of entire numbers associated in a natural way to a matrix (T_{ij}) of essentially commuting operators on a Banach space,which is semi-Fredholm and satisfies some condition (e.g. if it has a right or left essential inverse matrix S_{ji} ,such that all the commutators $[S_{ji},S_{kl}]$, $[S_{ji},T_{\ell k}]$ are compact).These numbers are invariant to compact or small norm perturbations of T_{ij}.

Let H be a Hilbert space and $\mathcal{A} \subset \mathcal{L}(H)$ an essential commutative subalgebra of linear bounded operators.Let $\mathcal{T} = (T_{ij})$ be a matrix with elements of $\mathcal{A}$ which is right essential invertible :

$$\mathcal{T} \in \mathcal{M}(m,n;\mathcal{A}) \quad , \quad \mathcal{S} \in \mathcal{M}(n,m;\mathcal{A})$$

$$\mathcal{T}\mathcal{S} = I + \text{compact}.$$

Then the operator $\mathcal{T}: H^n \longrightarrow H^m$ is semi-Fredholm on the right,so that the equation

(1) $\mathcal{T}x = y$

has solution for almost all $y \in H^m$,in the sense that Im $\mathcal{T}$ has finite codimension in H^m.We shall proof that the solution of (1) is uniquely determined,if it exists,modulo a finite dimensional space and by a finite set of canonical commutation relations,or compact perturbations of them in the following sense :

A canonical commutation relation for $\mathcal{T}$ is a system $\mathcal{R} \in \mathcal{M}(n,1;\ \mathcal{A}_0)$ with elements in the algebra $\mathcal{A}_0$ generated by all T_{ij}, such that

$$\mathcal{T}\mathcal{R} \in \mathcal{A}_0[\mathcal{A}_0,\mathcal{A}_0]\mathcal{A}_0$$

and it is universal in the following sense : if one denotes by (X_{ij}), $1 \leq i \leq m, 1 \leq j \leq n$, a system of indeterminates, there exists a polynomial $R \in \mathbb{C}[X_{ij}] \otimes \mathbb{C}^n$, such that

$$\mathcal{R} = R(T_{ij}) \quad \text{and}$$

$$\begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \cdots & & \cdots \\ X_{m1} & \cdots & X_{mn} \end{pmatrix} R = 0.$$

To conclude, we have a system of canonical commutation relations $\mathcal{T}_1$ which defines an operator, such that

$$H^p \xrightarrow{\mathcal{T}_1} H^n \xrightarrow{\mathcal{T}} H^m \longrightarrow 0$$

is an exact sequence, modulo compacts.

If we repeat this, covering the kernel of relations by corresponding canonical commutation relations, the procedure ends after finite number of steps, thus we obtain an essentially exact sequence of Hilbert spaces (Fredholm complex in another terminology). Such an essentially exact sequence has a entire number invariant which is stable under compact or small perturbations, invariant which generalizes the Euler characteristic of a complex.

Translations in a certain sense of the above procedure give a sequence of Fredholm complexes and corresponding invariants $\mathrm{ind}_p \mathcal{T}$, $p \in \mathbb{Z}$, which will be the indices of $\mathcal{T}$.

The construction of the essential complexes will be made in an algebraic context, by using exterior and symmetric algebras and the particularisation to the essentially commutative case still works in the Banach space context.

The first section contains the algebraic preliminaries, namely the construction of the complexes and trivial homotopies for them. These complexes are obtained by mixing Koszul complexes in a manner which remind the Spencer complexes.

In the second part we use a notion of essential Fredholm complex of Banach spaces, which is a particular case of the notion of Fredholm complex of vector bundles of Segal [14]. With a natural definition for the numerical index of such a complex, we prove some properties of this invariant. This index and its properties agree in particular cases with the indices of [2] or [17].

The third section contains the definitions of the indices associated to a semi-Fredholm system of operators, the stability theorem for this invariants and some properties of them.

In the fourth part a K-theoretic interpretation of the indices is made, with advantages in proving other properties of the indices. The indices are surely not independents, at least in some special cases, but we don't know the relations which they satisfy. Also we formulate this problem in terms of K-theory.

The last part contains applications to elliptic (on the right or on the left) systems of pseudodifferential operators on a compact manifold. The sequence of indices which we define for such a system generalizes the analytic index of an elliptic system. Also systems with right or left invertible symbol of Toeplitz operators, make clearer, by means of examples, our constructions.

I would like to thank Professor D.Voiculescu for his useful remarks. I thank also Professor V.Iftimie for his commentaries on the applications to pseudodifferential operators.

1. ALGEBRAIC PRELIMINAIRES

Let A be a commutative, unital, $\mathbb{C}$-algebra and $a = (a_1, \dots, a_n)$ a n-tuple of elements of A. The Koszul complex $K_\circ(a)$ associated to a can be defined as follows:

$$K_p(a) = \wedge^p[Y,A] \quad , \; p \in \mathbb{Z}$$

$$(1.1) \quad \delta_p : K_p(a) \longrightarrow K_{p-1}(a)$$

$$\delta_p(aY_{i_1} \wedge \dots \wedge Y_{i_p}) = \sum_{j=1}^{p} (-1)^j \, a_{i_j} aY_{i_1} \wedge \dots \wedge \widehat{Y_{i_j}} \wedge \dots \wedge Y_{i_p}$$

where we make the convention $1 \leq i_1 < \dots < i_p \leq n$.

If there are elements $b_1, \dots, b_n \in A$ such that $a_1 b_1 + \dots + a_n b_n = 1$, then the maps

$$\varepsilon^p : K_p(a) \longrightarrow K_{p+1}(a) \, , \; p \in \mathbb{Z}$$

$$(1.2) \quad \varepsilon^p(aY_{i_1} \wedge \dots \wedge Y_{i_p}) = \sum_{k=1}^{n} b_k aY_k \wedge Y_{i_1} \wedge \dots \wedge Y_{i_p}$$

give a trivial homotopy for $K_\circ(a)$.

We have denoted by $\wedge^\cdot[Y,A]$ the exterior algebra with n generators $Y_1, \dots, Y_n$ and with coefficients in A.

There is a dual notion, that of cochains Koszul complex:

$$K^p(a) = \wedge^p[Y,A] \quad , \quad \delta^p : K^p(a) \longrightarrow K^{p+1}(a) \; ,$$

$$\delta^p(aY_{i_1} \wedge \dots \wedge Y_{i_p}) = \sum_{k=1}^{n} a_k aY_k \wedge Y_{i_1} \wedge \dots \wedge Y_{i_p} \, .$$

If the ideal generated by $a_1, \dots, a_n$ coincides with A, then there exists a trivial homotopy for $K^\circ(a)$, similar with (1.1).

Let now $K_{i\circ}$ be m complexes, $1 \leq i \leq m$, of vector spaces, which differ only by their boundary operators $\partial_{i\circ} =: \partial_i$.
We shall suppose

$$(1.3) \qquad \partial_i \partial_j + \partial_j \partial_i = 0 , \qquad 1 \leq i,j \leq m$$

the compositions being made in all possible combinations.

Let $S^{\bullet}$ be the symmetric algebra with m independent generators $X_1, \dots, X_m$,with coefficients in $\mathbb{C}$ .We shall identify S^p with the set of homogenous polynomials of degree p.

We shall define the complex $K_{\bullet} = K_{\bullet}(K_{1\bullet}, \dots, K_{m\bullet})$:

$$1.4 \qquad \dots \xrightarrow{D} K_1 \otimes S^1 \xrightarrow{D} K_0 \otimes S^0 \xrightarrow{\partial_1 \dots \partial_m} K_{-m} \otimes S^0 \xrightarrow{D'} K_{-m-1} \otimes S^1 \xrightarrow{D'} \dots$$

where the tensor products are on $\mathbb{C}$ and the operators D, D' work as follows :

$$(1.5) \qquad D\left(xX_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}}\right) = \sum_{j=1}^{p} \partial_{i_j} xX_{i_1}^{\alpha_{i_1}} \dots X_{i_j}^{\alpha_{i_j}-1} \dots X_{i_p}^{\alpha_{i_p}} ,$$

$x \in K_h$, $1 \leq i_1 < \dots < i_p \leq m$, $\alpha_{i_1} + \dots + \alpha_{i_p} = h$ and all $\alpha_{i_j} > 0$,

$$(1.5)' \qquad D'\left(xX_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}}\right) = \sum_{k=1}^{m} \partial_k x X_k X_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}} ,$$

where $x \in K_{-h-m}$, $1 \leq i_1 < \dots < i_p \leq m$, $\alpha_{i_1} + \dots + \alpha_{i_p} = h$, $\alpha_{i_j} > 0$.

It is easy to prove that $K_{\bullet}$ is a complex.

PROPOSITION 1.1 <u>Assume, with the above notations, that there are the homotopy operators e_i on the complexes K_i, respectively, with the following properties</u> :

$$e_i \partial_i + \partial_i e_i = 1 , \qquad 1 \leq i \leq m ,$$

$$(1.6) \qquad e_i \partial_j + \partial_j e_i = 0 , \qquad 1 \leq i,j \leq m , \ i \neq j ,$$

$$e_i e_j + e_j e_i = 0 , \qquad 1 \leq i,j \leq m .$$

<u>Then the complex $K_{\bullet}$ is homotopiqually trivial.</u>

<u>Proof</u> : Let us define the maps, also of degree +1 :

$$\tilde{e}_1 = e_1 e_2 \partial_2 \cdots e_m \partial_m \ , \ \tilde{e}_2 = e_2 e_3 \partial_3 \cdots e_m \partial_m \ , \ldots , \ \tilde{e}_m = e_m$$

$$\tilde{e}'_1 = e_1 \partial_2 e_2 \cdots \partial_m e_m \ , \ \tilde{e}'_2 = e_2 \partial_3 e_3 \cdots \partial_m e_m \ , \ldots , \ \tilde{e}'_m = e_m .$$

Whith this one we shall define the homotopy operators for K. :

$$\begin{tikzcd}
\cdots \arrow[r] & S^1 \otimes K_1 \arrow[r, "D"] & S^0 \otimes K_0 \arrow[r, "\partial_1 \cdots \partial_m"] \arrow[dl, "E"'] & S^0 \otimes K_{-m} \arrow[r, "D'"] \arrow[dl, "e_m \cdots e_1"'] & S^1 \otimes K_{-m-1} \cdots \arrow[dl, "E'"'] \\
\cdots \arrow[r] & S^1 \otimes K_1 \arrow[r, "D"] & S^0 \otimes K_0 \arrow[r, "\partial_1 \cdots \partial_m"] & S^0 \otimes K_{-m} \arrow[r, "D'"] & S^1 \otimes K_{-m-1} \cdots
\end{tikzcd}$$

$$(1.7) \qquad E\left(xX_{i_1}^{\alpha_{i_1}} \ldots \ X_{i_p}^{\alpha_{i_p}}\right) = \sum_{k \geq i_p} \tilde{e}_k x X_{i_1}^{\alpha_{i_1}} \ldots \ X_{i_p}^{\alpha_{i_p}} X_k ,$$

the conventions for the indices being that of (1.5),

$$(1.7)' \qquad E'\left(xX_{i_1}^{\alpha_{i_1}} \ldots \ X_{i_p}^{\alpha_{i_p}}\right) = \tilde{e}'_{i_p} x X_{i_1}^{\alpha_{i_1}} \ \ldots \ \ X_{i_p}^{\alpha_{i_p}} ,$$

the indices being like of (1.5)' ,except the case $x \in K_{-m}$.

First we verify the homotopy relations on the terms $S^0 \otimes K_0$ and $S^0 \otimes K_{-m}$:

$$\left(e_m \cdots e_1 \partial_1 \cdots \partial_m + DE\right)x = e_1 \partial_1 e_2 \partial_2 \cdots e_m \partial_m x + D\left(\sum_{k=1}^{m} e_k x X_k\right) =$$

$$e_1 \partial_1 \ e_2 \partial_2 \cdots e_m \partial_m x + \sum_k \partial_k e_k x = e_1 \partial_1 e_2 \partial_2 \cdots e_m \partial_m x + \partial_1 e_1 e_2 \partial_2 \cdots e_m \partial_m x +$$

$$+ \partial_2 e_2 e_3 \partial_3 \cdots e_m \partial_m x + \cdots + \partial_m e_m x = \left(e_1 \partial_1 + \partial_1 e_1\right) e_2 \partial_2 \cdots \ e_m \partial_m x + \cdots +$$

$$+ \partial_m e_m x = \left(e_2 \partial_2 + \partial_2 e_2\right) e_3 \partial_3 \cdots \ e_m \partial_m x + \cdots + \partial_m e_m x = \left(e_m \partial_m + \partial_m e_m\right) x = x ,$$

$$\left(\partial_1 \cdots \partial_m e_m \cdots e_1 + E'D'\right) x = \partial_1 e_1 \partial_2 e_2 \cdots \partial_m e_m x +$$

$$+ E'\left(\sum_{k=1}^{m} \partial_k x X_k\right) = \partial_1 e_1 \partial_2 e_2 \cdots \partial_m e_m x + \cdots + e_m \partial_m x =$$

$$(\partial_1 e_1 + e_1 \partial_1)\partial_2 e_2 \cdots \partial_m e_m x + \cdots + e_m \partial_m x =$$

$$(\partial_2 e_2 + e_2 \partial_2)\partial_3 e_3 \cdots \partial_m e_m x + \cdots + e_m \partial_m x = \cdots = (\partial_m e_m + e_m \partial_m) x = x.$$

For the terms on the left of $S^o \otimes K_o$ one verify the homotopy relations as follows :

$$(1.8) \quad ED\left(xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}}\right) = E\left(\sum_{j=1}^{p} \partial_{i_j} xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_j}^{\alpha_{i_j}-1} \cdots X_{i_p}^{\alpha_{i_p}}\right) =$$

$$\sum_{j=1}^{p-1} \sum_{k \geq i_p} \tilde{e}_k \partial_{i_j} xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_j}^{\alpha_{i_j}-1} \cdots X_{i_p}^{\alpha_{i_p}} X_k + \tilde{e}_{i_p} \partial_{i_p} xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}} +$$

$$+ \sum_{k > i_p} \tilde{e}_k \partial_{i_p} xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}} X_k .$$

The term $\tilde{e}_{i_{p-1}} \partial_{i_p} xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_{p-1}}^{\alpha_{i_{p-1}}+1}$, if it appears, changes not the sum, because $\tilde{e}_{i_{p-1}} \partial_{i_p} = 0$.

$$(1.8)' \quad DE\left(xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}}\right) = D\left(\sum_{k \geq i_p} \tilde{e}_k xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}} X_k\right) =$$

$$= \sum_{k \geq i_p} \sum_{j=1}^{p-1} \partial_{i_j} \tilde{e}_k xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_j}^{\alpha_{i_j}-1} \cdots X_{i_p}^{\alpha_{i_p}} X_k + \sum_{k > i_p} \partial_{i_p} \tilde{e}_k xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}-1} X_k$$

$$+ \sum_{k \geq i_p} \partial_k \tilde{e}_k xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}} .$$

From (1.3) and (1.6), $\partial_j \tilde{e}_k + \tilde{e}_k \partial_j = \partial_j e_k e_{k+1} \partial_{k+1} \cdots e_m \partial_m +$

$+ e_k \partial_j e_{k+1} \partial_{k+1} \cdots e_m \partial_m = 0$ for $1 \leq j < k \leq m$, and

$$\tilde{e}_j \partial_j + \sum_{k \geq j} \partial_k \tilde{e}_k = 1 \quad , \quad 1 \leq j \leq m .$$

Indeed $\tilde{e}_j \partial_j + \sum_{k \geq j} \partial_k e_k = e_j \partial_j e_{j+1} \partial_{j+1} \cdots e_m \partial_m +$

$$+ \partial_j e_j e_{j+1} \partial_{j+1} \cdots \quad e_m \partial_m + \partial_{j+1} e_{j+1} e_{j+2} \partial_{j+2} \cdots \quad e_m \partial_m + \cdots + \partial_m e_m =$$

$$(e_j \partial_j + \partial_j e_j) e_{j+1} \partial_{j+1} \cdots \quad e_m \partial_m + \cdots + \partial_m e_m = \cdots = e_m \partial_m + \partial_m e_m = I.$$

Finally by adding (1.8) to (1.8)' one obtains $ED + DE = I$.

On the right of $S^o \otimes K_{-m}$ one compute in the same way :

$$(1.9) \qquad E'D'\left(xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}}\right) = E'\left(\sum_{k<i_p} \partial_k xX_k X_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}} + \right.$$

$$\left. + \sum_{k \geq i_p} \partial_k xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}} X_k\right) = \sum_{k<i_p} \tilde{e}'_{i_p} \partial_k xX_k X_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}-1} +$$

$$+ \sum_{k \geq i_p} \tilde{e}'_k \partial_k xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}},$$

$$(1.9)' \qquad D'E'\left(xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}}\right) = D'\left(\tilde{e}'_{i_p} xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}-1}\right) =$$

$$= \sum_{k<i_p} \partial_k \tilde{e}'_{i_p} xX_k X_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}-1} + \partial_{i_p} \tilde{e}'_{i_p} xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}} +$$

$$+ \sum_{k>i_p} \partial_k \tilde{e}'_{i_p} xX_{i_1}^{\alpha_{i_1}} \cdots X_{i_p}^{\alpha_{i_p}} X_k .$$

The last term of (1.9)' is zero because $\partial_k \tilde{e}'_{i_p} = 0$.

Remarking that $\partial_k \tilde{e}'_j + \tilde{e}'_j \partial_k = 0$ for $k < j$ and that $\partial_j \tilde{e}'_j + \sum_{k \geq j} \tilde{e}'_k \partial_k = 1$, $1 \leq j \leq m$, the equality $E'D' + D'E' = I$ holds, q.e.d.

Let $\mathcal{A}$ be a (m,n) matrix (a_{ij}) with the elements in a commutative algebra A. We shall denote by $a_i = (a_{i1}, \ldots, a_{in})$ the lines of $\mathcal{A}$ and by $a^j = (a_{1j}, \ldots, a_{mj})'$ the columns of $\mathcal{A}$. We shall apply the above proposition to the Koszul complexes $K_i = K_\circ(a_i)$ or apart to $K^j = K^\circ(a^j)$. If we denote by ∂_i and ∂^j the corresponding bounary operators, the relations (1.3) holds.

We shall denote by $C_\circ(p)$ the translations of a complex $C_\circ$: $C_q(p) = C_{p+q}$, $\partial_q(p) = \partial_{p+q}$, $p, q \in \mathbb{Z}$.

COROLLARY 1.2 If the matrix $\mathcal{A}$ is left or right invertible, then the complexes $K_\circ(K^1(p),\dots,K^n(p))\otimes_A M$, respectively $K_\circ(K_1(p),\dots,K_m(p))\otimes_A M$ are exact, for each $p\in\mathbb{Z}$ and A-module M.

Proof : Let $\mathcal{B}=(b_{ji})$ be a left inverse for $\mathcal{A}$. Then $m\geqslant n$ and if one denote by b_j the lines of $\mathcal{B}$:

$$\text{1.10}\qquad \begin{aligned} b_j a^j &= \sum_i b_{ji}a_{ij} = 1 \quad, 1\leq j\leq n,\\ b_j a^k &= \sum_i b_{ji}a_{ik} = 0 \quad, 1\leq j,k\leq n \;,\; j\neq k. \end{aligned}$$

Then one define with the elements of $\mathcal{B}$, in analogy with (1.1), the operators of degree -1, $e_k\colon K^j \longrightarrow K^j$,

$$e_k(xY_{i_1}\wedge\dots\wedge\widehat{Y}_{i_p}) = \sum_{h=1}^{p}(-1)^h b_{ki_h}xY_{i_1}\wedge\dots\wedge\widehat{Y}_{i_h}\wedge\dots\wedge Y_{i_p} \;,$$

and the relations (1.6) follows from (1.10).

For a right invertible matrix the procedure is dual.

By the Proposition (1.1) the complexes $K_\circ(K^1(p),\dots,K^n(p))$, respectively $K_\circ(K_1(p),\dots,K_m(p))$ are homotopiqually trivial, which is enough for the proof of the corollary.

2. ESSENTIAL FREDHOLM COMPLEXES

There are many directions to extend the theory of Fredholm operators to complexes of operators for example the Hilbert space context [2] where there exists a good notion of Fredholm complex with nice stability properties or the Banach space context and unbounded operators on them [17], where there are still problems. We shall relate these ways by a notion of essential Fredholm complex of Banach spaces and associate an index with good stability properties.

We denote by $X, Y, \ldots; \delta, \varepsilon, \ldots$ Banach spaces and bounded linear operators on them, $\mathcal{L}(X,Y)$ the set of linear bounded operators between X and Y and by $\mathcal{K}(X,Y)$ the set of compact operators.

DEFINITION 2.1 <u>An essential complex of Banach spaces is a sequence of Banach spaces</u> X^p <u>and operators</u> δ^p :

$$0 \longrightarrow X^0 \xrightarrow{\delta^0} X^1 \xrightarrow{\delta^1} \cdots \longrightarrow X^{n-1} \xrightarrow{\delta^{n-1}} X^n \longrightarrow 0$$

<u>such that</u> $\delta^p \delta^{p-1}$ <u>is a compact operator for every</u> $p \in \mathbb{Z}$.

We shall use only finite complexes, i.e. $X^p = 0$ for $p < 0$ or for large p.

The Fredholm property for such an essential complex will mean the exactness modulo compacts :

DEFINITION 2.2 <u>An essential complex</u> $X^\bullet$ <u>is Fredholm iff for any Banach space</u> Y, <u>the complex</u>

$$\mathcal{L}(Y, X^\bullet) / \mathcal{K}(Y, X^\bullet)$$

<u>is exact</u>.

This is equivalent with the existence of a homotopy between the identity of the complex and the zero map, modulo compacts :

PROPOSITION 2.3 <u>An essential complex</u> $X^\bullet$ <u>is Fredholm if and only if there exist operators</u> $\varepsilon^p : X^p \longrightarrow X^{p-1}$ <u>such that</u> :

$$(2.1) \qquad \delta^{p-1}\varepsilon^p + \varepsilon^{p+1}\delta^p = I + \underline{\text{compact}} \; , \; p \in \mathbb{Z} .$$

<u>Moreover if</u> $\varepsilon_1^\bullet$ <u>and</u> $\varepsilon_2^\bullet$ <u>satisfy</u> (2.1), <u>then they are homotopic in the class of essentially trivial homotopies</u>.

This caracterisation of Fredholm complexes agrees with a particular case of the definition of Fredholm complexes of vector bundles, given by G. Segal [14].

<u>Proof</u> : The sufficiency is clear because for each Banach space Y, the complex $\mathcal{L}(Y, X^\bullet)/\mathcal{K}(Y, X^\bullet)$ is homotopiqually trivial.

To prove the necessity, we shall construct by decreasing induction on p the operators ε^p. The first step is clear because the complex is finite. Suppose that there exists the $\varepsilon^{q,s}$ for $q \geqslant p+1$. Then the complex

$$\mathcal{L}(X^p,X^{p-1})/\mathcal{K}(X^p,X^{p-1}) \xrightarrow{\delta_*^{p-1}} \mathcal{L}(X^p,X^p)/\mathcal{K}(X^p,X^p) \xrightarrow{\delta_*^p}$$

$$\longrightarrow \mathcal{L}(X^p,X^{p+1})/\mathcal{K}(X^p,X^{p+1})$$

is exact, so that if we choose the class of $I-\varepsilon^{p+1}\delta^p$ in the middle term, there is $\varepsilon^p \in \mathcal{L}(X^p,X^{p-1})$ such that

$$I-\varepsilon^{p+1}\delta^p = \delta^{p-1}\varepsilon^p + \text{compact}.$$

Indeed, $\delta^p(I-\varepsilon^{p+1}\delta^p) = \delta^p - \delta^p\varepsilon^{p+1}\delta^p =$

$= \delta^p - (I-\varepsilon^{p+2}\delta^{p+1})\delta^p + \text{compact} = \text{compact}.$

Suppose that we have two essential trivial homotopies $\varepsilon_1^{\cdot}$ and $\varepsilon_2^{\cdot}$ of $X^{\cdot}$. Then, for each $\lambda \in [0,1]$, $\lambda\varepsilon_1^{\cdot} + (1-\lambda)\varepsilon_2^{\cdot}$ is still an essential trivial homotopy, which proves the last affirmation of the proposition.

Let us remark that if we replace $\varepsilon^{\cdot}$ by $\varepsilon^{\cdot}\delta^{\cdot}\varepsilon^{\cdot}$, then we obtain an essential complex

$$(2.2)\qquad 0 \longleftarrow X^0 \xleftarrow{\varepsilon^1} X^1 \xleftarrow{\varepsilon^2} \cdots \longleftarrow X^{n-1} \xleftarrow{\varepsilon^n} X^n \longleftarrow 0$$

which is Fredholm by the symmetry of the relations (2.1).

Let $X^{\cdot}$ be an essential Fredholm complex, homotopiqually trivial by an essential homotopy $\varepsilon^{\cdot}$. Then the operator

$$(2.3)\qquad T = \begin{pmatrix} \delta^0 & \varepsilon^2 & & 0 \\ & \delta^2 & \varepsilon^4 & \\ 0 & & \delta^4 & \ddots \end{pmatrix} : \bigoplus_p X^{2p} \longrightarrow \bigoplus_p X^{2p+1}$$

is a Fredholm operator. Indeed, let us define

$$S = \begin{pmatrix} \varepsilon^1 & & & 0 \\ \delta^1 & \varepsilon^3 & & \\ & \delta^3 & \varepsilon^5 & \\ 0 & & & \ddots \end{pmatrix} \quad : \quad \bigoplus_p X^{2p+1} \longrightarrow \bigoplus_p X^{2p} .$$

Then the operators TS and ST are essential invertible :

$$ST = \begin{pmatrix} I & \varepsilon^1\varepsilon^2 & & 0 \\ & I & \varepsilon^3\varepsilon^4 & \\ & & I & \\ 0 & & & \ddots \end{pmatrix} + \text{compact} ,$$

$$TS = \begin{pmatrix} I & \varepsilon^2\varepsilon^3 & & 0 \\ & I & \varepsilon^4\varepsilon^5 & \\ & & I & \\ 0 & & & \ddots \end{pmatrix} + \text{compact} .$$

By the last assertion of Proposition 2.3, the index of T does not depend on the choice of the essentialy trivial homotopy $\varepsilon^{\cdot}$, so we give the following

DEFINITION 2.4 Let X° be an essential Fredholm complex and let T be the operator defined by 2.3 .Then the index of X° is :

$$\text{ind } X^{\circ} = \text{ind } T .$$

The notion of essential Fredholm complex and associated index are obviously stable under compact perturbations of the coboundary operators $\delta^{\cdot}$.

Now we shall prove the stability under small perturbations. Let $(X^\bullet, \delta^\bullet)$ be an essential Fredholm complex and let $\tilde{\delta}^\bullet$ be a sufficiently small norm perturbation of $\delta^\bullet$, such that $\tilde{\delta}^{\bullet 2}$ = compact. Then for each Banach space Y, the complex

$$\left(\mathcal{L}(Y,X^\bullet)/\mathcal{K}(Y,X^\bullet)\ ,\ \tilde{\delta}^\bullet_*\right)$$

is exact. Indeed this holds by the stability theorem for exact complexes (for example [17, Thm. 2.11]).

In order to prove the invariance of the index, let $\tilde{\varepsilon}^\bullet$ be a trivial essential homotopy for $\tilde{\delta}^\bullet$. By the classical theorem of stability for Fredholm operators, the operator

$$T_0 = \begin{pmatrix} \tilde{\delta}^0 & \varepsilon^2 & & 0 \\ & \tilde{\delta}^2 & \varepsilon^4 & \\ & & \tilde{\delta}^4 & \\ 0 & & & \ddots \end{pmatrix}$$

is Fredholm as a perturbation of T, and ind T_0 = ind T.

Let us remark that for each $\lambda \in [0,1]$, the operator

$$T = \begin{pmatrix} \tilde{\delta}^0 & \lambda\tilde{\varepsilon}^2 + (1-\lambda)\varepsilon^2 & & 0 \\ & \tilde{\delta}^2 & \lambda\tilde{\varepsilon}^4 + (1-\lambda)\varepsilon^4 & \\ & & \tilde{\delta}^4 & \\ 0 & & & \ddots \end{pmatrix}$$

is Fredholm. Indeed, if we take

$$S_\lambda = \begin{pmatrix} \lambda\tilde{\varepsilon}^1 + (1-\lambda)\varepsilon^1 & & 0 \\ \tilde{\delta}^1 & \lambda\tilde{\varepsilon}^3 + (1-\lambda)\varepsilon^3 & \\ & \tilde{\delta}^3 & \\ 0 & & \ddots \end{pmatrix}$$

then

$$T_\lambda S_\lambda = \begin{pmatrix} \lambda + (1-\lambda)(\tilde{\delta}^0 \varepsilon^1 + \varepsilon^2 \tilde{\delta}^1) & & * \\ & \lambda + (1-\lambda)(\tilde{\delta}^2 \varepsilon^3 + \varepsilon^4 \tilde{\delta}^3) & \\ 0 & & \ddots \end{pmatrix}$$

and the diagonal blocks are essentially invertible, because $\tilde{\delta}^0 \varepsilon^1 + \varepsilon^2 \tilde{\delta}^1, \dots$ are essentially small perturbations of the identity, hence $-\lambda(1-\lambda)^{-1}$ is not an element of the essential spectrum of them.

Similarely one prove that $S_\lambda T_\lambda$ is essentially invertible.

By the homotopical invariance of the index, we obtain :

$$\text{ind } T_0 = \text{ind } T_1 = \text{ind } T ,$$

hence

$$\text{ind } X^\circ = \text{ind } T = \text{ind } T_0 = \text{ind } T = \text{ind } X^\circ .$$

Concluding, we have proved the following

THEOREM 2.5 <u>The Fredholm property and the associated index of an essential complex of Banach spaces are invariant under small norm or compact perturbations of the boundary operators</u>.

The next proposition relates our index to the Euler characteristic $\chi(X^\circ)$ of a complex X°.

PROPOSITION 2.6 <u>Let</u> X° <u>be a Fredholm complex</u> (<u>i.e. essential Fredholm complex and</u> $\delta^p \delta^{p-1} = 0$, $p \in \mathbb{Z}$).

<u>Then</u> $H^p(X^\circ)$, $p \in \mathbb{Z}$, <u>are finite dimensional spaces and</u>

$$\text{ind } X^\circ = \chi(X^\circ) \quad . \tag{2.4}$$

<u>Proof</u> : Let ε° be a homotopy of X° and $k^p \in \mathcal{K}(X^p)$, such that

$$\varepsilon^{p+1} \delta^p + \delta^{p-1} \varepsilon^p = I + k^p \quad , p \in \mathbb{Z} .$$

Then $\delta^{p-1}\varepsilon^p \big|_{\mathrm{Ker}\,\delta^p} = (I + k^p)\big|_{\mathrm{Ker}\,\delta^p} : \mathrm{Ker}\,\delta^p \longrightarrow \mathrm{Ker}\,\delta^p$,
so that $k^p(\mathrm{Ker}\,\delta^p) \subset \mathrm{Ker}\,\delta^p$ .But $(I + k^p)$ $\mathrm{Ker}\,\delta^p$ is a Fredholm operator on $\mathrm{Ker}\,\delta^p$,hence

$$\dim H^p(X^\circ) = \dim \mathrm{Ker}\,\delta^p/\mathrm{Im}\,\delta^{p-1} \leqslant \dim \mathrm{Coker}\,(I + k^p)\big|_{\mathrm{Ker}\,\delta^p}$$

and the last number is finite.As a consequence,the ranges of operators δ^p are closed.

We shall prove the equality (2.4) by induction on the length of the complex X° .If the length of the complex is one,then (2.4) is the definition of the index.Suppose the affirmation is true for complexes of the length n-1.

Let X° be a Fredholm complex of the length n :

$$0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots \longrightarrow X^n \longrightarrow 0 .$$

Let $\varepsilon^\cdot$ be a homotopy of the complex $X^\cdot$ and let denote $\tilde{X}^\cdot$ the complex :

$$0 \longrightarrow X^1/\mathrm{Im}\,\delta^0 \longrightarrow X^2 \longrightarrow X^3 \longrightarrow \cdots \longrightarrow X^n \longrightarrow 0 .$$

Then $\chi(X^\circ) = \dim \mathrm{Ker}\,\delta^0 - \mathrm{ind}\,\tilde{X}^\circ$ and $\varepsilon^\cdot$ induces a homotopy $\tilde{\varepsilon}^\cdot$ for $\tilde{X}^\cdot$: $\tilde{\varepsilon}^2 = \pi \circ \varepsilon^2$, $\tilde{\varepsilon}^q = \varepsilon^q$ for $q \geqslant 3$,where π is the projection map $\pi : X^1 \longrightarrow X^1/\mathrm{Im}\,\delta^0$.

A short computation shows that

$$\mathrm{ind}\,T = \dim \mathrm{Ker}\,\delta^0 + \mathrm{ind} \begin{pmatrix} \tilde{\varepsilon}^2 & & 0 \\ \delta^2 & \varepsilon^4 & \\ 0 & \delta^4 & \ddots \end{pmatrix} =$$

$$= \dim \mathrm{Ker}\,\delta^0 - \mathrm{ind} \begin{pmatrix} \delta^1 & \varepsilon^3 & & 0 \\ & \delta^3 & \varepsilon^5 & \\ 0 & & & \ddots \end{pmatrix} =$$

$= \dim \operatorname{Ker} \delta^0 - \operatorname{ind} \widetilde{X}^\circ$,

because the second matrix operator is the essential inverse of the first, hence by the induction hypothesis $\operatorname{ind} T = \chi(X^\circ)$.

In the Hilbert spaces case one can prove the following :

PROPOSITION 2.6 Let H° be an essential complex of Hilbert spaces. The following assertions are equivalent :

i) H° is Fredholm ,

ii) $(\delta^{\cdot *})$ is almost an essential trivial homotopy for H°, namely the operator $\delta\delta^* + \delta^*\delta$ is Fredholm.

iii) There exist compact modifications of the boundaries such that H° becomes a complex with cohomology spaces of finite dimension.

COROLLARY 2.7 If H_1° and H_2° are two essential Fredholm complexes of Hilbert spaces and if $f^\circ : H_1^\circ \longrightarrow H_2^\circ$ is, modulo compacts, a morphism of complexes which is linear isomorphism, then $\operatorname{ind} H_1^\circ = \operatorname{ind} H_2^\circ$.

Proof : Let $\widetilde{H}_1^\circ$ be a complex which is a compact modification of H_1° . Then transporting by f° the boundaries of H_1° , one obtain a compact perturbation of H_2° into an essential Fredholm complex $\widetilde{H}_2^\circ$. Then

$$\operatorname{ind} H_1^\circ = \operatorname{ind} \widetilde{H}_1^\circ = \chi(\widetilde{H}_1^\circ) = \chi(\widetilde{H}_2^\circ) = \operatorname{ind} \widetilde{H}_2^\circ = \operatorname{ind} H_2^\circ \ .$$

The property of fredholmicity for an essetial complex is stable also by passing to duals :

THEOREM 2.8 Let X° be an essential Fredholm complex of Banach spaces. Then the dual essential complex $X^{\circ\prime}$ is also Fredholm and

$$\operatorname{ind} X^{\circ\prime} = \operatorname{ind} X^\circ$$

if the zero'th covariant component of $X^{\circ\prime}$ is the dual of

<u>the zero'th contravariant component of</u> X° .

<u>Proof</u> : Let ε' be an essential trivial homotopy of X°. Then the homotopy relations for X° give by duality homotopy relations for $X^{\circ\prime}$, hence the dual complex is Fredholm. The associated Fredholm operator of $X^{\circ\prime}$ is

$$S = \begin{pmatrix} \varepsilon^{1\prime} & \delta^{1\prime} & & 0 \\ & \varepsilon^{3\prime} & \delta^{3\prime} & \\ & & \varepsilon^{5\prime} & \\ 0 & & & \ddots \end{pmatrix}$$

so that $\operatorname{ind} S = -\operatorname{ind} \begin{pmatrix} \varepsilon^1 & & 0 \\ \delta^1 & \varepsilon^3 & \\ 0 & \delta^3 & \ddots \end{pmatrix} = \operatorname{ind} X^\circ$.

3. SEMI-FREDHOLM SYSTEMS OF ESSENTIALLY COMMUTING OPERATORS

Let X be a Banach space and $\mathcal{T} = (T_{ij})$, $1 \leq i \leq m$, $1 \leq j \leq n$, a system of linear bounded operators with the property :

(3.1) $[T_{ij}, T_{hk}] \in \mathcal{K}(X)$, $1 \leq i,h \leq m$, $1 \leq j,k \leq n$.

We shall denote by K_i (K^j) the essential Koszul complex of the i'th line (j'column) of $\mathcal{T}$. Also one can form the essential complexes

$$K^p(\mathcal{T},X) = K_\cdot(K^1(p),\dots,K^n(p)) \quad \text{and}$$

$$K_p(\mathcal{T},X) = K_\cdot(K_1(p),\dots,K_m(p))$$

the notations of first section being used. Indeed they are essential complexes because

$$\mathcal{L}(X,K_p(\mathcal{T},X))/\mathcal{K}(X,K_p(\mathcal{T},X)) = K_p(\mathcal{T},\mathcal{L}(X)/\mathcal{K}(X))$$

is a complex.

The component of contravariant index equall zero in $K^{p\cdot}$ (with degree p in Y and 1 in X) is that marked in the scheme

$$\cdots \longrightarrow \boxed{*} \longrightarrow * \xrightarrow{\partial_1\dots\partial_m} * \longrightarrow \boxed{*} \longrightarrow \cdots$$

when the component of covariant index equall zero in $K_{p\cdot}$ (with degree p in Y and 1 in X) is that pointed.

Let $\mathcal{S} = (S_{ji})$, $1 \leq i \leq m$, $1 \leq j \leq n$, be a system of operators, such that

$$3.2 \qquad [S_{ji},S_{kh}] \in \mathcal{K}(X), \quad [S_{ji},T_{hk}] \in \mathcal{K}(X)$$

for each i,j,k,h.

The following lemma gives a criterion in order that the essential complexes $K_p(\mathcal{T})$ be Fredholm.

LEMMA 3.1 <u>Assume that, with the above notations, the system $\mathcal{S}$ is a left or right inverse, modulo compacts, of $\mathcal{T}$. Then, for each $p \in \mathbb{Z}$, the essential complex $K^p(\mathcal{T})$, respectively $K_p(\mathcal{T})$, is Fredholm.</u>

<u>Proof</u> : Suppose $\mathcal{T}\mathcal{S}$ = I + compact. Then, by the Corollary 1.2, the complex $K_p(\mathcal{T},\mathcal{L}(X)/\mathcal{K}(X))$ is homotopiqually trivial, i.e. the essential complex $K_p(\mathcal{T})$ is Fredholm.

DEFINITION 3.2 <u>Let $\mathcal{T}$ be a system of essentially commuting operators on a Banach space.</u>

If the essential complexes $K_p(\mathcal{T})$ or $K^p(\mathcal{T})$ are Fredholm, the indices of $\mathcal{T}$ are :

$$\operatorname{ind}_p \mathcal{T} = \operatorname{ind} K_p(\mathcal{T}) \text{ , respectively } \operatorname{ind}^p \mathcal{T} = \operatorname{ind} K^p(\mathcal{T}).$$

As a consequence of the Theorem 2.5 one can prove the following

THEOREM 3.3 Let $\mathcal{T}$ be a system of essentially commuting operators on a Banach space, such that the essential complexes $K_p(\mathcal{T})$ or $K^p(\mathcal{T})$, $p \in \mathbb{Z}$, are Fredholm.

Then the indices $\operatorname{ind}_p \mathcal{T}$, respectively $\operatorname{ind}^p \mathcal{T}$, $p \in \mathbb{Z}$, are invariant by compact or small norm perturbations of $\mathcal{T}$, such that condition (3.1) holds.

If the system $\mathcal{T}$ has only one line, the essential complexes $K_p(\mathcal{T})$ are translations of the essential Koszul complex $K_\cdot(\mathcal{T})$, so that if one of these is Fredholm, then all are Fredholm, and

$$\operatorname{ind}_p \mathcal{T} = \operatorname{ind} K_\cdot(\mathcal{T})(-1)^p = (-1)^p \operatorname{ind}_0 \mathcal{T} .$$

If the system $\mathcal{T}$ is of type (n,n), i.e. a square matrix, then $K_0(\mathcal{T})$ coincides with $K^0(\mathcal{T})$ and both with the complex

$$0 \longrightarrow \bigoplus_1^n X \xrightarrow{\ \mathcal{T}\ } \bigoplus_1^n X \longrightarrow 0$$

so that

$$\operatorname{ind}^0 \mathcal{T} = -\operatorname{ind}_0 \mathcal{T} = \text{classical index of } \mathcal{T} ,$$

if there exist these indices.

Also in this case if the operator $\mathcal{T}$ is Fredholm, then there is an essential bilateral inverse of $\mathcal{T}$ which satisfies (3.2), so that all the essential complexes $K_p(\mathcal{T})$, $K^p(\mathcal{T})$ are Fredholm. A short computation shows that

$$\operatorname{ind}_{-1} \mathcal{T} = \operatorname{ind} \det \mathcal{T} .$$

The indices have a good behaviour by passing to duals :

THEOREM 3.4 Let $\mathcal{T}$ be a system of essentially commuting operators on a Banach space X, such that the essential complexes $K_p(\mathcal{T})$, $p \in \mathbb{Z}$, be Fredholm.

Then the dual system $\mathcal{T}'$ on X' has the essential complexes $K^p(\mathcal{T}')$ Fredholm and

$$\text{ind}_p \, \mathcal{T} = \text{ind}^p \, \mathcal{T}' \quad , p \in \mathbb{Z} . \tag{3.3}$$

Proof : The essential complex $K^p(\mathcal{T}')$ is the dual of $K_p(\mathcal{T})$. Indeed, the components are dual, and the boundary operators agree by the duality

$$\langle xY_{i_1} \wedge \dots \wedge Y_{i_h} \otimes X_{j_1} \dots X_{j_k} , x'Y_{i_1'} \wedge \dots \wedge Y_{i_h'} \otimes X_{j_1'} \dots X_{j_k'} \rangle$$

$= \langle x, x' \rangle \, \delta_{i_1 i_1'} \dots \delta_{i_h i_h'} \delta_{j_1 j_1'} \dots \delta_{j_k j_k'}$, where

$1 \leq i_1 < \dots < i_h \leq m$, $1 \leq i_1' < \dots < i_h' \leq m$, $1 \leq j_1 \leq \dots \leq j_k \leq n$,

$1 \leq j_1' \leq \dots \leq j_k \leq n$ and h,k depends on p and q. More precisely

$$D(\mathcal{T})' = D'(\mathcal{T}') , \ [D'(\mathcal{T})]' = D(\mathcal{T}'),$$

$$[\partial_1(\mathcal{T}) \dots \partial_m(\mathcal{T})]' = \partial^1(\mathcal{T}') \dots \partial^m(\mathcal{T}') + \text{compact} .$$

By the Theorem 2.8 and by the choice of zero'th components of K_p and K^p we obtain (3.3).

In what follows we shall give some properties of the index of a system of type (1,n) on Hilbert spaces.

THEOREM 3.5 Let $\mathcal{T}_i$ be an essential commuting system with one line, on the Hilbert space H_i, such that the essential Koszul complex $K.(\mathcal{T}_i, H_i)$ be Fredholm , i=1,2 .

Then the essential complex $K.(\mathcal{T}_1 \hat{\otimes} I, I \hat{\otimes} \mathcal{T}_2 ; H_1 \hat{\otimes} H_2)$ is Fredholm and

$$\text{ind} \left(\mathcal{T}_1 \hat{\otimes} I , I \hat{\otimes} \mathcal{T}_2 \right) = \text{ind}\, \mathcal{T}_1 \cdot \text{ind}\, \mathcal{T}_2 .$$

Proof : Let K_i be a compact modification into a Fredholm complex of the essential complex $K_\bullet(\mathcal{T}_i,H_i)$, $i=1,2$. Then by the main theorem of [6] , $K_1\hat{\otimes} K_2$ is a Fredholm complex and $\operatorname{ind} K_1\hat{\otimes} K_2 = \operatorname{ind} K_1\cdot\operatorname{ind} K_2$. But $K_1\hat{\otimes} K_2$ is a compact perturbation of $K_\bullet(\mathcal{T}_1)\hat{\otimes} K_\bullet(\mathcal{T}_2)$, so that

$$\operatorname{ind}\ (\mathcal{T}_1\hat{\otimes} I\ ,\ I\hat{\otimes}\mathcal{T}_2) = \operatorname{ind} K_1\hat{\otimes} K_2 = \operatorname{ind} K_1\cdot\operatorname{ind} K_2 =$$

$$= \operatorname{ind}\mathcal{T}_1\cdot\operatorname{ind}\mathcal{T}_2\ .$$

Another multiplicativity property which generalizes the multiplicativity of the index of Fredholm operators, is the following :

THEOREM 3.6 *Let $\mathcal{T}$ be a one line system of essentially commuting operators on a Hilbert space H, and let Q,R be operators on H which essentially commute with $\mathcal{T}$.*

If the essential Koszul complexes of $(\mathcal{T}, Q)$ and $(\mathcal{T}, R)$ are Fredholm, then the essential Koszul complex of $(\mathcal{T}, RQ)$ is also Fredholm, and

$$\operatorname{ind}\ (\mathcal{T}\ ,RQ) = \operatorname{ind}\ (\mathcal{T}\ ,R) + \operatorname{ind}\ (\mathcal{T},Q)\ .$$

Proof : Assume $\dim H = \infty$, if not all the indices are zero. There exists the decomposition

$$K_\bullet\ (\mathcal{T},Q) = K_\bullet(\mathcal{T}) \oplus K_{\bullet -1}(\mathcal{T})\wedge Y_Q$$

with respect on which the Fredholm operator associated to the system $(\mathcal{T}\ ,Q)$ has the decomposition :

$$T_Q = \begin{pmatrix} T & -Q \\ Q & T \end{pmatrix} .$$

Let us remark that T_Q coincides with $T_{L^\bullet}$ where $L^\bullet$ is the essential Fredholm complex

$$0 \longrightarrow H \xrightarrow{\binom{T}{Q}} H \oplus H \xrightarrow{(-Q\ ,\ T)} H \longrightarrow 0 \ .$$

By Proposition 2.6.iii) there exists compact operators k_1, k_2 such that

$$(-Q\ , T) \begin{pmatrix} T+k_1 \\ Q+k_2 \end{pmatrix} = 0 \ .$$

Let $\tilde{L}^{\bullet}$ be the modified complex. Then the exact sequence of complexes

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & H \arrow[r] \arrow[u] & H \arrow[r] \arrow[u] & 0 & \\
0 \arrow[r] & H \arrow[r] \arrow[u, "T"] & H \oplus H \arrow[r] \arrow[u, "{(-Q\ ,\ T)}"] & H \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[r] \arrow[u] & H \arrow[r] \arrow[u, "{\binom{T+k_1}{Q+k_2}}"] & H \arrow[r] \arrow[u, "T+k_1"] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

gives the long exact sequence

$$(3.4) \qquad 0 \longrightarrow H^0(\tilde{L}^{\bullet}) \longrightarrow \mathrm{Ker}\,(T+k_1) \xrightarrow{\partial^0} \mathrm{Ker}\ T \longrightarrow H^1(\tilde{L}^{\bullet}) \longrightarrow$$

$$\longrightarrow \mathrm{Coker}(T+k_1) \xrightarrow{\partial^1} \mathrm{Coker}\ T \longrightarrow H^2(\tilde{L}^{\bullet}) \longrightarrow 0 \ .$$

A short calculation gives $\partial^0 = Q+k_2$ and $\partial^1 = Q$,so that ∂^0 and ∂^1 have finite dimensional kernel and cokernel, and from (3.4) :

$$\operatorname{ind} T_Q = \chi(\tilde{L}^\circ) = \operatorname{ind}\left(Q+k_2 : \operatorname{Ker}(T+k_1) \longrightarrow \operatorname{Ker} T\right) -$$
$$- \operatorname{ind}\left(Q : \operatorname{Coker}(T+k_1) \longrightarrow \operatorname{Coker} T\right).$$

Similarely, modifing the other boundary operator of L_R° there exists compact operators k_3, k_4 such that $(R+k_3)T = (T+k_4)R$ and

$$\operatorname{ind} T_R = \operatorname{ind}\left(R : \operatorname{Ker} T \longrightarrow \operatorname{Ker}(T+k_4)\right) -$$
$$- \operatorname{ind}\left(R+k_3 : \operatorname{Coker} T \longrightarrow \operatorname{Coker}(T+k_4)\right).$$

Then the operator T_{RQ} associated to $(\mathcal{T}, RQ)$ coincides modulo compacts with that one associated to the complex

$$0 \longrightarrow H \xrightarrow{\begin{pmatrix} T+k_1 \\ R(Q+k_2) \end{pmatrix}} H \oplus H \xrightarrow{(-(R+k_3)Q\ ,\ T+k_4)} H \longrightarrow 0$$

Indeed this is a complex : $(R+k_3)Q(T+k_1) = (R+k_3)T(Q+k_2)$

$= (T+k_4)R(Q+k_2)$, and

$$\operatorname{ind} T_{RQ} = \operatorname{ind}\left(R(Q+k_2) : \operatorname{Ker}(T+k_1) \longrightarrow \operatorname{Ker}(T+k_4)\right) -$$
$$- \operatorname{ind}\left((R+k_3)Q : \operatorname{Coker}(T+k_1) \longrightarrow \operatorname{Coker}(T+k_4)\right),$$

where $\operatorname{ind}(\cdot) = \dim\operatorname{Ker}(\cdot) - \dim\operatorname{Coker}(\cdot)$.

The above indices exist and $\operatorname{Ind} T_{RQ} = \operatorname{ind} T_R + \operatorname{ind} T_Q$ by the following result :

LEMMA 3.7 <u>Let</u> $X \xrightarrow{T} Y \xrightarrow{S} Z$ <u>be two linear operators with Ker and Coker of finite dimension, between vector spaces. Then</u> ST <u>has the same property and</u> $\operatorname{ind} ST = \operatorname{ind} S + \operatorname{ind} T$

For the proof see the proof of 13, Thm.VII.3 .

COROLLARY 3.8 <u>If</u> $= T_1,\dots,T_n$ <u>is an essential commuting system of operators on a Hilbert space and</u> $m = m_1,\dots,m_n$ <u>, then if the essential Koszul complex of is Fredholm, then</u> $^m = T_1, \dots, T_n$ <u>has the same property and</u> ind $^m = m_1 m_2 \dots m_n$ ind .

This answers to a question raised by R.Curto in his thesis 2 ,where there is a deeper investigation of the essential commuting systems with one line,on Hilbert spaces.

4. A K-THEORETIC APPROACH

The indices defined above can be obtained in a natural way,in certain cases for example the essential normal case from some elements of K^1 of a Stiefel manifold.

For terminology and notations see [3] and [13] .

Let A be an essential commutative Banach subalgebra of $\mathcal{L}(E)$,with E denoting a Banach space.Then the unital Banach algebra $B = A/\mathcal{K} \cap A$ is commutative,and let us denote by X its maximal spectrum.By the Theorem of Novodvorski [13] , $K^1(X)$ is isomorphic with $K_1(B)$ via the Gelfand transformation $\mathcal{G} : B \longrightarrow \mathcal{C}(X)$.

Let $\mathcal{T}$ be a system of elements of A ,invertible on the right in B,also by a system with elements in A :

$$\mathcal{T} \in \mathcal{M}(m,n;\ A) \quad , \ \mathcal{S} \in \mathcal{M}(n,m;\ A)$$

$$\mathcal{T}\mathcal{S} = I_m + \text{compact} \ .$$

We shall denote by $\pi(\mathcal{T}), \pi(\mathcal{S})$ the images in B of these matrices.The system $\pi(\mathcal{T})$ determines by its Gelfand transformation a map

$$\widehat{\pi(\mathcal{T})} \quad : X \longrightarrow V_{m,n-m}(\mathbb{C}) .$$

We shall identify the Stiefel manifold $V_{m,n-m}(\mathbb{C})$ with a subset of $\mathbb{C}^{mn}$, denoting by (z_{ij}) the coordinates in $\mathbb{C}^{mn}$, with respect to the matrix representation

$$(4.1) \quad \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \cdots & & & \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{pmatrix}$$

of the points of $V_{m,n-m}(\mathbb{C})$.

Let us fix $p \in \mathbb{Z}$. The essential complex $K_p(\mathcal{T}, A)$ is Fredholm by lemma 3.1, with an essential trivial homotopy $\mathcal{E}^{\cdot}$ defined with the elements of $\mathcal{S}$. Let T be the associated Fredholm operator (2.3) . Then we have the following complexes, boundaries, trivial homotopies and associated invertible operators, respectively :

$$K_p(\pi(\mathcal{T}), B) \quad , \pi(\delta^{\cdot}) \; , \; \pi(\varepsilon^{\cdot}) \; , \; \pi(T)$$

$$K_p(\widehat{\pi(\mathcal{T})}, \mathcal{C}(X)) \quad , \widehat{\pi(\delta^{\cdot})} \; , \; \widehat{\pi(\varepsilon^{\cdot})} \; , \; \widehat{\pi(T)}$$

$$K_p(\widehat{\pi(\mathcal{T})}(x), \mathbb{C}) \quad , \widehat{\pi(\delta^{\cdot})}(x) \; , \; \widehat{\pi(\varepsilon^{\cdot})}(x) \; , \; \widehat{\pi(T)}(x)$$

for each $x \in X$.

By the homotopical unicity of the trivial homotopy for the last complex, we have $[\widehat{\pi(\varepsilon^{\cdot})}(x)] = [\widehat{\pi(\delta^{\cdot})^{*}}(x)]$, hence $[\widehat{\pi(\varepsilon^{\cdot})}] = [\widehat{\pi(\delta^{\cdot})}^{*}]$, denoting by $[\cdot]$ the K_1 equivalence class.

Therefore if we denote by $\mathcal{Z}$ the right invertible system (4.1) on $V_{m,n-m}(\mathbb{C})$ and by $T_{\mathcal{Z}}$ the associated invertible matrix to $K_p(\mathcal{Z}, \mathcal{C}(V_{m,n-m}(\mathbb{C}))$, then

$$[\widehat{\pi(T)}] = \widehat{\pi(\mathcal{T})}^{!}[T_{\mathcal{Z}}] \quad .$$

Concluding, there exist elements $\xi_p \in K^1(V_{m,n-m}(\mathbb{C}))$, $p \in \mathbb{Z}$, such that

$$\mathrm{ind}_p \mathcal{T} = \mathrm{index} \circ \xi_!^{-1} \circ \widehat{\pi(\mathcal{T})}^{!} \xi_p \quad ,$$

$$K^1(V_{m,n-m}(\mathbb{C})) \xrightarrow{\widehat{\pi(\mathcal{T})}^{!}} K^1(X) \xrightarrow{\xi_!^{-1}} K_1(B) \xrightarrow{ind} \mathbb{Z} \ ,$$

where index : $K_1(B) \longrightarrow \mathbb{Z}$ is the index map.

This way one can prove some properties of the invariants $\mathrm{ind}_p \mathcal{T}$. For example :

PROPOSITION 4.1 <u>Let</u> $\mathcal{T}$ <u>be as above. If we add a column with elements of</u> A <u>to</u> $\mathcal{T}$ <u>, then the new system</u> $\mathcal{T}'$ <u>is still right invertible in</u> B <u>and</u>

$$\mathrm{ind}_p \mathcal{T}' = \mathrm{ind}_p \mathcal{T} + \mathrm{ind}_{p-1} \mathcal{T} \quad , \ p \in \mathbb{Z} \quad .$$

<u>Proof</u> : The first affirmation is obvious. The second follows from the formula

$$i^! \xi'_p = \xi_p + \xi_{p-1} \qquad ,$$

where $i : V_{m,n-m}(\mathbb{C}) \longrightarrow V_{m,n+1-m}(\mathbb{C})$ is the natural inclusion.

Indeed, there is the following decomposition

$$K_p(\mathcal{Z}', \mathcal{C}) = K_p(\mathcal{Z}, \mathcal{C}) \oplus K_{p-1}(\mathcal{Z}, \mathcal{C}) \wedge Y_{n+1}$$

where $\mathcal{C} = \mathcal{C}(V_{m,n+1-m}(\mathbb{C}))$;with respect on this

$$T_{\mathcal{Z}'} = \begin{pmatrix} T_{\mathcal{Z}} & * \\ 0 & T_{\mathcal{Z}} \end{pmatrix}$$

so that, by some elementary computations

$$i^! \xi'_p = i^! [T_{z'} + T^*_{z'}] = i^! \left[\begin{pmatrix} T_z + T^*_z & * \\ * & T_z + T^*_z \end{pmatrix}\right] =$$

$$= \left[\begin{pmatrix} T_z + T^*_z & 0 \\ 0 & T_z + T^*_z \end{pmatrix}\right] = \xi_p + \xi_{p-1} .$$

One can prove that $K^1(V_{m,n-m}(\mathbb{C}))$ is a free abelian group with at most 2^{m-1} generators.Therefore the elements (ξ_p) $p \in \mathbb{Z}$ are not independent ,but we was not able to compute the relations which they satisfy.We hope in the aid of a topologist to solve this problem.

The full problem is the following :

<u>What relations there exist between</u> $\mathrm{ind}_p \mathcal{T}$, $p \in \mathbb{Z}$, <u>where</u> $\mathcal{T}$ <u>is as in Theorem 3.3</u> ?

For a square matrix $\mathcal{T}$ of essentially commuting operators,one can prove that $\mathrm{ind}_0 \mathcal{T}$ (= classical ind $\mathcal{T}$) and $\mathrm{ind}_{-1} \mathcal{T}$ (= ind det $\mathcal{T}$) are independent.However,Markus and Feldman have proved that

$$\mathrm{ind}_0 \mathcal{T} + \mathrm{ind}_{-1} \mathcal{T} = 0$$

if the commutators of the elements of $\mathcal{T}$ are trace class [12].

5. APPLICATIONS

The main part of this sectionis devoted to pseudodifferential operators.First some notations and terminology :

Let M be a compact manifold and $L^s_{\rho,\delta}(M)$ the set of pseudodifferential operators on M,of order $s \in \mathbb{R}$ and of type (ρ, δ) , $0 \leq \delta < \rho \leq 1-\delta$,in the sense of Hörmander (see for example [9]). An elliptic on the right system of pseudodifferential operators (in the sense of Douglis-Nirenberg) will be a matrix $\mathcal{P} = (P_{ij})$ with

$$(5.1) \qquad P_{ij} \in L^{t_j - s_i}(M) \quad \text{for } 1 \leqslant i \leqslant m\ ,\ 1 \leqslant j \leqslant n$$

and some real numbers t_j, s_i, such that for each $(x,\xi) \in T^*M$, the matrix of principal symbols

$$\sigma_o(\mathcal{P})(x,\xi) = \left(\sigma_{t_j - s_i}(P_{ij})(x,\xi) \right)$$

is invertible on the right, for ξ sufficiently large.

For such an elliptic on the right system $\mathcal{P}$, one can construct, with some modifications, essential complexes like $K_p(*)$, as follows :

Let σ be a real number and let us denote by $H^*(M)$ the Sobolev spaces of M. We shall use the indeterminates X and Y of first section only for simplifications of the boundaries formulas. The essential complex $K^{\sigma}_{p-1,\cdot}(\mathcal{P})$ will be :

$$(5.2) \quad \cdots K^{\sigma}_{p+3} \xrightarrow{D} K^{\sigma}_{p+2} \xrightarrow{D} K^{\sigma}_{p+1} \xrightarrow{\partial_1 \dots \partial_m} K^{\sigma}_{p} \xrightarrow{D'} K^{\sigma}_{p-1} \xrightarrow{D'} \cdots$$

where we omit the first index p-1 and

$$K^{\sigma}_{p-q} = \bigoplus H^{\tau(\sigma,j,i,\alpha)}(M) \otimes Y_{j_1} \wedge \dots \wedge Y_{j_{p-q}} \otimes X_{i_1}^{\alpha_{i_1}} \dots \quad X_{i_h}^{\alpha_{i_h}}$$

summing by the indices : $1 \leqslant j_1 < \dots < j_{p-q} \leqslant n\ ,\ 1 \leqslant i_1 < \dots <$

$< i_h \leqslant m\ ,\ \alpha_{i_1} + \dots + \alpha_{i_h} = q$, all α_{i_k} 0 and the order

$$\tau(\sigma, j, i, \alpha) = \sigma + t_{j_1} + \dots + t_{j_{p-q}} + \alpha_{i_1} s_{i_1} + \dots + \alpha_{i_h} s_{i_h} \quad ,$$

all this for $0 \leqslant q \leqslant p$.

$$K^{\sigma}_{p-q} = 0 \quad \text{for } q > p .$$

$$K^{\sigma}_{p+q} = \bigoplus H^{\tau(\sigma,j,i,\alpha)}(M)\otimes Y_{j_1}\wedge \dots \wedge Y_{j_{p+m+q-1}} \otimes X_{i_1}^{\alpha_{i_1}}\dots X_{i_h}^{\alpha_{i_h}}$$

the sum by : $1\leq j_1 < \dots < j_{p+m+q-1}\leq n$, $1\leq i_1 < \dots < i_h \leq m$,

$\alpha_{i_1}+\dots+\alpha_{i_h} = q-1$, all α_{i_k} 0 and $\tau(\sigma,j,i,\alpha)$ =

$= \sigma + t_{j_1}+\dots+t_{j_{p+m+q-1}} - (s_1+\dots+s_m) - (\alpha_{i_1}s_{i_1}+\dots+\alpha_{i_h}s_{i_h})$,

for $q\geqslant 1$.

The corresponding boundaries are as in the first section:

$$D(xY_{j_1}\wedge\dots\wedge Y_{j_k}\otimes X_{i_1}^{\alpha_{i_1}}\dots X_{i_h}^{\alpha_{i_h}}) =$$

$$= \sum_{s=1}^{h}\sum_{t=1}^{k}(-1)^t P_{i_s j_t} xY_{j_1}\wedge\dots\wedge \widehat{Y}_{j_t}\wedge\dots\wedge Y_{j_k}\otimes X_{i_1}^{\alpha_{i_1}}\dots X_{i_s}^{\alpha_{i_s}-1}\dots X_{i_h}^{\alpha_{i_h}}$$

$$D'(xY_{j_1}\wedge\dots\wedge Y_{j_k}\otimes X_{i_1}^{\alpha_{i_1}}\dots X_{i_h}^{\alpha_{i_h}}) =$$

$$= \sum_{s=1}^{m}\sum_{t=1}^{k}(-1)^t P_{s j_t} xY_{j_1}\wedge\dots\wedge \widehat{Y}_{j_t}\wedge\dots\wedge Y_{j_k}\otimes X_s X_{i_1}^{\alpha_{i_1}}\dots X_{i_h}^{\alpha_{i_h}}$$

and similarely $\partial_1 \dots \partial_m$.

Using the compactness of the commutator

$$[P_1,P_2] : H^{\sigma+s_1+s_2}(M) \longrightarrow H^{\sigma}(M) ,$$

where $P_i \in L^{s_i}(M)$, $i=1,2$,one can prove that $K^{\sigma}_{p-1,\cdot}(\mathcal{P})$ is an essential complex of Hilbert spaces.

One can prove that there exists for each $s\in\mathbb{R}$, operators $A_s \in L^s(M)$ such that the principal symbol $\sigma_0(A_s)$ would be invertible,real and the extensions $A_s : H^{\sigma}(M)\to H^{\sigma-s}(M)$ would be isomorphisms.Moreover,the inverse A_s^{-1} is a pseudodifferential operator, modulo compacts.

Then the elliptic on the right system $\mathcal{P}$ which satisfies (5.1) can be written like

$$\mathcal{P} = \begin{pmatrix} A_{-s_1} & & 0 \\ & \ddots & \\ 0 & & A_{-s_m} \end{pmatrix} \mathcal{P}_0 \begin{pmatrix} A_{t_1} & & 0 \\ & \ddots & \\ 0 & & A_{t_n} \end{pmatrix}$$

where $\mathcal{P}_0$ is also elliptic on the right and satisfies (5.1) and the ellipticity condition with $t_j=s_i=0$. With the operators A_s one can define an essential isomorphism in the sense of Corollary 2.7 between $K_p^{\sigma}(\mathcal{P})$ and $K_p^{\sigma}(\mathcal{P}_0)$, so that we can suppose that the ellipticity condition holds for all $t_j=s_i=0$. But in this case $K_p^{\sigma}(\mathcal{P}_0)$ coincides with $K_p^{\sigma}(\mathcal{P}_0, H^{\sigma}(M))$, because all the operators P^0_{ij} have order zero.

THEOREM 5.1 Let $\mathcal{P}$ be an elliptic on the right(left) system of pseudodifferential operators on a compact manifold M.

Then the essential complexes $K_{p_0}^{\sigma}(\mathcal{P})$, respectively $K^{p_0}_{\sigma}(\mathcal{P})$, are Fredholm for all $\sigma \in \mathbb{R}$ and $p \in \mathbb{Z}$, and ind $K_p^{\sigma}(\mathcal{P})$, respectively ind $K^{p_0}_{\sigma}(\mathcal{P})$, depends not on σ.

Proof : Assume that the components of $\mathcal{P}$ have order zero. There is by Theorem 6.3.7 of [9] a right parametrix $\mathcal{E}$ of $\mathcal{P}$, also of order zero. If σ is a real number, the extensions of $\mathcal{P}$ and $\mathcal{E}$ to $H^{\sigma}(M)$ give a system of essentially commuting operators $\mathcal{P}^{\sigma}$ with a right essential inverse $\mathcal{E}^{\sigma}$ such that all the components of $\mathcal{P}^{\sigma}, \mathcal{E}^{\sigma}$ commute modulo compacts. Then by Lemma 3.1 the essential complex $K_{p_0}^{\sigma}(\mathcal{P})$ is Fredholm.

The independence of the index of σ results by Corollary 2.7 . The left invertible case is dual.

Let $\mathcal{P}$ be a system like in Theorem 5.1 .Then the indices $\operatorname{ind}_p(\mathcal{P}) = \operatorname{ind} K_p^{\sigma}(\mathcal{P})$ for some $\sigma \in \mathbb{R}$, $p \in \mathbb{Z}$,depend only on the principal symbol of $\mathcal{P}$ and they are locally constant in the space of right elliptic symbols.Moreover, if we work only with homogenous symbols,they are also topological invariants.

For example let φ be a smooth function on $\mathbb{R}$,such that $\varphi(\xi) = 1$ for $|\xi| \geqslant 2$ and $\varphi(\xi) = 0$ for $|\xi| \leqslant 1$.The function

$$\sigma : T^*(S^1) \longrightarrow \mathbb{C},$$

$$\begin{aligned} \sigma(\xi dz) &= z\,\varphi(\xi) \quad \text{if } \xi \geqslant 0, \\ &= \varphi(\xi) \quad \text{if } \xi < 0, \end{aligned}$$

is an elliptic symbol of order zero on the sphere S^1.A pseudodifferential operator on S^1 with this symbol is the following [13,XVI.6.2] :

$$P : L^2(S^1) \longrightarrow L^2(S^1)$$

$$\begin{aligned} P(z^k) &= z^{k+1} \quad \text{for } k \geqslant 0 \ , \\ &= z^k \quad \text{for } k < 0 \ , \end{aligned}$$

where (z^k) is the natural base of $L^2(S^1)$.The index of P is -1.

Let now σ_k be the symbol $1 \otimes \ldots \otimes \overset{k\downarrow}{\sigma} \otimes \ldots \otimes 1$ on the n-dimensional torus $T^n = S^1 \times \ldots \times S^1$,and let $\mathcal{P} = (P_1, \ldots, P_n)$ be an essential commuting system of pseudodifferential operators associated to the symbol $(\sigma_1, \ldots, \sigma_n)$ .Then $\mathcal{P}$ is an elliptic to the right system, and by Corollary 3.8

$$\operatorname{ind} \mathcal{P}^m = (-1)^n m_1 m_2 \ldots m_n \ ,$$

for each $m \in \mathbb{N}^n$.

Another class of operators with compact commutators are Toeplitz operators. One can define the invariants $\mathrm{ind}_p \mathcal{T}$ for each system $\mathcal{T}$ of Toeplitz operators on S^1 (or on another manifolds like $S^{2n-1}, T^n, \ldots$) which has the matrix symbol $\Phi = (\varphi_{ij})$ invertible to the right in all the points of S^1. Indeed, if this condition holds, then $\Phi^*(\Phi\Phi^*)^{-1}$ is a right inverse of Φ in the set of continous matrix symbols on S^1.

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REFERENCES

1 COBURN L.A. Toeplitz operators an odd spheres, Lect. Notes in Math. 345 , Springer, 7-13, 1973

2 CURTO R.E. Fredholm and invertible tuples of bounded linear operators , Ph. D. Diss.,State Univ. of New York at Stony Brook, 1978

3 DOLD A. Lectures on algebraic topology, Springer , 1972

4 DOUGLIS A.-NIRENBERG L. Interior estimates for elliptic systems of partial differential equations, Comm. Pure Appl. Math. 8, 503-538 , 1955

5 GILKEY P.B. The Index-Theorem and the Heat Equation, Princeton Univ. , 1973

6 GROSU C.-VASILESCU F.-H. The Künneth formula for Hilbert complexes, Preprint Series in Math. 5/1980, INCREST

7 HELTON J.W.-HOWE R.E. Traces of commutators of integral operators, Acta Math. 130, 271-305, 1975

8 HÖRMANDER L. Linear partial differential operators, Springer, 1964

9 IFTIMIE V. Pseudodifferential and Fourier integral operators, in Romanian , Univ. of Bucarest, 1978

10 KATO T. Perturbation theory for linear operators, Springer 1963

11 MAC LANE S. Homology , Springer, 1963

12 MARKUS A.S.-FELDMAN I.A. The index of operator matrices in Russian , Funct. Analysis and Appl. t.11 ,2 ,1977, 83-84

13 PALAIS R. Seminar on the Atiyah-Singer Index Theorem, Princeton Univ. Press ,1965

14 SEGAL G. Fredholm complexes , Quart. J. Math. Oxford 21 1970 , 385-402

15 TAYLOR J.L. Banach Algebras and Topology , Univ. of Utah , 197

16 TELEMAN K. Methods and results of the modern differential geometry in Romanian , Ed. Stiinţifică , Bucarest , 1979

17 VASILESCU F.-H. Stability of the index of a complex of Banach spaces, J. Operator Theory, 2 1979 , 247-275 .

THE SUPERPOSITION PROPERTY FOR TAYLOR'S FUNCTIONAL CALCULUS

by Mihai Putinar

This note contains a proof of the superposition theorem for J.L.Taylor's functional calculus [4].The technique used in the proof is based on the Cauchy-Weil integral.

The notations and the terminology are that of [3] and [4].

THEOREM : *Let* X *be a Banach space and let* a *be a commuting n-tuple of linear bounded operators on* X.

If f *is an analytic* $\mathbb{C}^m$ *-valuated function in a neighbourghood of* Sp(a,X) *and if* g *is an analytic function in a neighbourghood of* f(Sp(a,X)), *then*

$$(g \circ f)(a) = g(f(a)) .$$

Proof : With a good choice of the domain of f, the function $g \circ f$ is analytic in a neighbourghood of Sp(a,X), so that there exists the operator $(g \circ f)(a)$. By the spectral mapping theorem [4], the function g is defined in a neighbourghood of Sp(f(a),X), hence there exists g(f(a)).

Let V be an open subset of $\mathbb{C}^m$, which contains Sp(f(a),X) and on which is defined the function g. Let also U be an open subset of $\mathbb{C}^n$ which contains Sp(a,X), on which is defined f and such that $f(U) \subset V$.

Then for each $x \in X$,

$$(g\circ f)(a)x = (2\pi i)^{-n}\int_U R_{z-a}(g\circ f)(z)x \wedge dz$$

$$= (2\pi i)^{-2n}\int_U R_{z-a}\left(\int_V R_{w-f(z)}\, g(w)x \wedge dw\right) \wedge dz$$

(1) $$= (2\pi i)^{-2n}\int_{U\times V} R_{z-a,w-f(z)}\; g(w)x \wedge dw \wedge dz$$

(2) $$= (2\pi i)^{-2n}\int_{U\times V} R_{z-a,w-f(a)}\, g(w)x \wedge dw \wedge dz$$

$$= (2\pi i)^{-n}\int_U R_{z-a}\left((2\pi i)^{-n}\int_V R_{w-f(a)}\, g(W)\, x \wedge dw\right) \wedge dz$$

$$= (2\pi i)^{-n}\int_U R_{z-a}\, g(f(a))\, x \wedge dz$$

$$= g(f(a))x \quad ,$$

where for (1) the theorem 3.6 of [4] is used and (2) is a consequence of the equality

(3) $$R_{z-a,w-f(z)} = R_{z-a,w-f(a)}$$

which we shall prove.

The resolvent map $R_{z-a,w-f(z)}$ closes the diagram D :

$$\begin{array}{ccc} H^0(\bar\partial_z,\bar\partial_w;\mathcal{E}(U\times V,X)) & \longrightarrow & H^{m+n}(\bar\partial_z,\bar\partial_w, z-a, w-f(z);\mathcal{E}(U\times V,X)) \\ & & \uparrow S \\ H^{m+n}(\bar\partial_z,\bar\partial_w;\mathcal{E}_0(U\times V,X)) & \longleftarrow & H^{m+n}(\bar\partial_z,\bar\partial_w, z-a, w-f(z);\mathcal{E}_0(U\times V,X)) \end{array}$$

and analogously $R_{z-a,w-f(a)}$ closes a diagram D. It is enough to prove that $D=\tilde{D}$ i.e. the corresponding objects and maps are equall.

Remarking that the Koszul complex $K^{\cdot}(\bar{\partial}_z, \bar{\partial}_w, z-a, w-f(z))$ is the complex associated to a bicomplex with respect to the group of variables $(\bar{\partial}_z, \bar{\partial}_w, w-f(z))$ and z-a, there exists a regular spectral sequence which converges to the cohomology of the above complex, with the first terms

$$E_1^{pq} = K^p(\bar{\partial}_z, \bar{\partial}_w, w-f(z); H^q(z-a; \mathcal{E}(U\times V, X)) \quad \text{and}$$

$$E_2^{pq} = H^p(\bar{\partial}_z, \bar{\partial}_w, w-f(z); H^q(z-a; \mathcal{E}(U\times V, X)).$$

In the same way, replacing f(z) with f(a), there is a spectal sequence $\tilde{E}_r^{pq}$ which converges to $H^{\cdot}(\bar{\partial}_z, \bar{\partial}_w, z-a, w-f(a))$.

By the Proposition 4.6 of [4], f(z) acts as f(a) on $H^{\cdot}(z-a; \mathcal{E}(U\times V, X))$, hence $E_r^{pq} = \tilde{E}_r^{pq}$ for $p,q\in\mathbb{Z}$ and r=1,2. Noticing that the boundaries of the spectral sequences $E_{\cdot}^{pq}$, $\tilde{E}_{\cdot}^{pq}$ become from the boundaries of the initial bicomplex, one prove by an inductive argument that $E_r^{pq} = \tilde{E}_r^{pq}$ as groups, for each $r\in\mathbb{N}$.

The maps of D become from maps between spectral sequenceswith respect to the filtration given by (z-a, w-f(z)). The first and the last spectral sequence are degenerated and independent of (z-a, w-f(z)) so that also the maps of D and $\tilde{D}$ agree.

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REFERENCES

[1] BOURBAKI N. Théories spectrales, Hermann, 1967

[2] GODEMENT R. Topologie algébrique et theorie des faisceaux, Hermann, 1958

[3] PUTINAR M. Three papers in several variables spectral theory, Preprint Series in Math. INCREST No 43/1979

[4] TAYLOR J.L. A functional analytic calculus for several commuting operators, Acta Math. 125,1970, 1-38.