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The aim of this paper is to give equivalent conditions for a norm on a Banach space to be rough using the sets defined by (2) below. This enable us to obtain unitary characterizations for rough and strongly rough norms(see the definitions below), using either (1) or (2) below.

We want to thank Professor V. Zizler for helpful and stimulating conversations related to the subject matter of this paper.

Let X be a real Banach space and for each $x,y \in X$, let $\|x\|'(y)$ denotes the one sided Gateaux differential of $\|.\|$ at x in the direction y, i.e.,

$$(1) \qquad \|x\|'(y) = \lim_{t\to 0^+} t^{-1}(\|x+ty\| - \|x\|)$$

We denote the unit ball of X by B_X and

$$S_X = \{x \in X : \|x\| = 1\}$$

For each $x \in X$ let us denote

$$(2) \qquad A(x) = A_X(x) = \{f \in S_{X^*} : f(x) = \|x\|\}$$

where X^* is the dual space of X.

By Ascoli-Mazur Theorem ([1]) we have

(3) $\|x\|'(y) = \max\{f(y) : f \in A(x)\}$.

<u>Definition</u> 1. ([3] , [4]) A norm of a Banach space X is called to be <u>rough</u> if there is an $\varepsilon > 0$ such that for every $x \in X$ and $\delta > 0$, there exist $x_1, x_2, u \in X$, $\|x_i - x\| < \delta$ $i = 1,2$, $u \in S_X$ with $\|x_2\|'(u) - \|x_1\|'(u) \geq \varepsilon$.

For a bounded set $A \subset X$, we denote the diameter of A by diam A, i.e., diam A = $\sup\{\|a_1 - a_2\| : a_1, a_2 \in A\}$.

<u>Proposition 1.</u> <u>The following properties of a given norm of X are equivalent.</u>

i) $\|.\|$ <u>is rough</u>.

ii) <u>There is an</u> $\varepsilon > 0$ <u>such that for every</u> $x \in S_X$ <u>and</u> $\delta > 0$, <u>there exist</u> $y_1, y_2 \in S_X$, $\|y_i - x\| < \delta$, $i = 1,2$ <u>with</u> $\mathrm{diam}(A(y_1) \cup A(y_2)) \geq \varepsilon$.

iii) <u>There is an</u> $\varepsilon > 0$ <u>such that for every</u> $x \in S_X$ <u>and</u> $\delta > 0$, <u>there exists</u> $y \in S_X$, $\|y - x\| < \delta$ <u>with</u> $\mathrm{diam}(A(x) \cup A(y)) \geq \varepsilon$

iv) <u>There is an</u> $\varepsilon > 0$ <u>such that for every</u> $x \in S_X$ <u>and</u> $\delta > 0$, $\mathrm{diam}(\bigcup\{A(y) : \|y - x\| < \delta\}) \geq \varepsilon$.

<u>Proof.</u> i)$\Rightarrow$ii). Since $\|.\|$ is rough, choose $\varepsilon > 0$ given by Definition 1, and let $x \in S_X$ and $\delta > 0$. We can suppose $\delta < 2$. By i) for this $x \in S_X$ and $\delta/2$, there exist $x_1, x_2, u \in X$, $\|x_i - x\| < \delta/2$, $i = 1,2$, $u \in S_X$ with $\|x_2\|'(u) - \|x_1\|'(u) \geq \varepsilon$. By (3) there exist $f_i \in A(x_i)$, $i = 1,2$ such that $\|x_i\|'(u) = f_i(u)$, $i = 1,2$. Then:

$$\varepsilon \leq \|x_2\|'(u) - \|x_1\|'(u) = f_2(u) - f_1(u) \leq \|f_1 - f_2\| \leq \mathrm{diam}(A(x_1) \cup A(x_2))$$

Let $y_i = x_i/\|x_i\|$, $i = 1,2$ ($x_i \neq 0$ since $\delta < 2$ and $\|x_i - x\| < \delta/2$)
We have for $i = 1,2$

$$\|x-y_i\| = \|x - x_i/\|x_i\|\| = (1/\|x_i\|)\,\|\,\|x_i\|\,x - \|x_i\|x_i + \|x_i\|x_i - x_i\| \le$$
$$\le \|x-x_i\| + |\,\|x_i\|-1| = \|x-x_i\| + |\,\|x_i\|-\|x\|\,| \le 2\,\|x-x_i\| < \delta$$

Since $A(y_i) = A(x_i)$, $i = 1,2$, the implication i) $\Rightarrow$ ii) is proved.

ii) $\Rightarrow$ iii). Let $\varepsilon > 0$ be given by ii). If iii) does not hold, then for $\varepsilon/2$ there exist $x \in S_X$ and $\delta > 0$ such that $\mathrm{diam}(A(x) \cup A(y)) < \varepsilon/2$ for each $y \in S_X$, $\|y-x\| < \delta$. By ii), for x and δ as above, there exist $y_1, y_2 \in S_X$, $\|y_i - x\| < \delta$, $i = 1,2$, such that $\mathrm{diam}(A(y_1) \cup A(y_2)) \geq \varepsilon$. Let $f_n, g_n \in A(y_1) \cup \cup A(y_2)$ such that

$$(4) \qquad \lim_n \|f_n - g_n\| \geq \varepsilon$$

Let $\alpha = \max\{\mathrm{diam}(A(x) \cup A(y_1)), \mathrm{diam}(A(x) \cup A(y_2)\}$. We have $\alpha < \varepsilon/2$. Let $f \in A(x)$. Then $\|f_n - g_n\| \le \|f_n - f\| + \|f - g_n\| \le 2\alpha < \varepsilon$ for each n, in contradiction with (4). Therefore ii) $\Rightarrow$ iii).

iii) $\Rightarrow$ iv) is obvious.

iv) $\Rightarrow$ i). Let $\varepsilon > 0$ be given by iv). If i) does not hold, then by [2] (see Proposition 1, i) $\Rightarrow$ iv)) for $\varepsilon/2$ there is an $x \in S_X$ such that whenever $f_n, g_n \in S_{X^*}$, $\lim_n f(x) = \lim_n g_n(x) = 1$, we have $\limsup_n \|f_n - g_n\| \le \varepsilon/2$. By iv), for this x and each n we have $\mathrm{diam}(\cup \{A(y) : \|y-x\| < 1/n\}) \geq \varepsilon$. Then there exist $f_n, g_n \in \cup \{A(y) : \|y-x\| < 1/n\}$ such that $\|f_n - g_n\| \geq 3\varepsilon/4$ for each n. Clearly, $\lim_n f_n(x) = \lim_n g_n(x) = 1$

and since $f_n, g_n \in S_{X^*}$, we have by the above cited result of [2] that $\limsup_n \|f_n-g_n\| < \varepsilon/2$, a contradiction with $\|f_n-g_n\| \geq 3\varepsilon/4$ for each n. This completes the proof.

Corollary 1. The following properties of a given norm of X are equivalent.

i) $\|\cdot\|$ is rough.

ii) There is an $\varepsilon > 0$ such that for every $x \in S_X$ and $\delta > 0$, there exist $x_1, x_2, u \in S_X$, $\|x_i - x\| < \delta$, $i = 1,2$, such that $\|x_2\|'(u) + \|x_1\|'(-u) \geq \varepsilon$.

iii) The same with ii) but with $x_1 = x_2$ or one of x_1, x_2 equals x.

Proof. i)$\Rightarrow$ii). This follows by Definition 1, using the well-known fact that $-\|x_1\|'(u) \leq \|x_1\|'(-u)$.

ii)$\Rightarrow$i). Using formula (3) we obtain that ii) implies condition iii) of Proposition 1 above, whence the $\|\cdot\|$ is rough.

i)$\Rightarrow$iii). By Proposition 1, i)$\Rightarrow$iii), there is an $\varepsilon' > 0$ such that for every $x \in S_X$ and $\delta > 0$, there exists $y \in S_X$, $\|y-x\| < \delta$ with $\operatorname{diam}(A(x) \cup A(y)) \geq \varepsilon'$. Let $f, g \in A(x) \cup A(y)$ such that $\|f-g\| > \varepsilon'/2$. Then there is a $u \in S_X$ such that $(f-g)(u) \geq \varepsilon'/2$. If $f \in A(x)$ and $g \in A(y)$, then by (3) we obtain $\varepsilon'/2 \leq f(u) + g(-u) \leq \|x\|'(u) + \|y\|'(-u)$. Similarly, if $f, g \in A(x)$ or $f, g \in A(y)$ we obtain respectively $\|x\|'(u) + \|x\|'(-u) \geq \varepsilon'/2$ or $\|y\|'(u) + \|y\|'(-u) \geq \varepsilon'/2$. Therefore we have iii) e.g., for $\varepsilon = \varepsilon'/2$.

Since iii)$\Rightarrow$ii) is obvious, this completes the proof of the corollary.

Definition 2.([2]) A norm of a Banach space X is said to be strongly rough if there is an $\varepsilon > 0$ such that for every $x \in S_X$ there is a $u \in S_X$ with $\|x\|'(u) + \|x\|'(-u) \geq \varepsilon$.

Remark 1. By [2] , Proposition 2, i)$\Leftrightarrow$ii) it follows that $\|\cdot\|$ is strongly rough if and only if there is an $\varepsilon > 0$ such that for every $x \in S_X$, diam $A(x) \geq \varepsilon$.

Definition 3. A norm of a Banach space X is said to be quasy strongly rough if there is an $\varepsilon > 0$ such that for every $x \in S_X$ and $\delta > 0$ there exist $y, u \in S_X$, $\|y-x\| < \delta$ with $\|y\|'(u) + \|y\|'(-u) \geq \varepsilon$.

The usual norms of $C[a,b]$ and $\ell^1(N)$ are quasy strongly rough but not strongly rough.

Proposition 2. The norm of a Banach space X is quasy strongly rough if and only if there is an $\varepsilon > 0$ such that the set $\{ x \in S_X : \text{diam } A(x) \geq \varepsilon \}$ is dense in S_X in the norm topology.

Proof. Using formula (3), the proof of this result is similar and simpler than that of Proposition 1.

If $F \in S_{X^{**}}$ (X^{**} is the second dual of X) and $\delta > 0$, then we denote

$$K(F,\delta) = \{ f \in B_{X^*} : F(f) \geq 1 - \delta \}$$

Such a set is called a "slice" of B_X [5] .

We shall consider X embedded in X^{**} , by the canonical embedding.

Remark 2. If for each $\varepsilon > 0$ there exist $x, y \in S_X$ and $\delta > 0$, $\|x-y\| < \delta$ such that $\text{diam}(K(x,\delta) \cap K(y,\delta)) < \varepsilon$ then it is easy to show that the norm of X is not quasy strongly rough (since $A(x) \cup A(y) \subset K(x,\delta) \cap K(y,\delta)$). I do not know if the converse is true. (By [2], Proposition 1 i)$\Leftrightarrow$ii) we know that the norm of X is not rough if and only if for each $\varepsilon > 0$ there exist $x \in S_X$ and $\delta > 0$ such that $\operatorname{diam} K(x,\delta) < \varepsilon$).

Clearly, one can extend the notion of a rough norm by letting in Definition 1, that one, two or all of x_1, x_2, u to belong to X^{**}. The geometrical equivalences of a rough norm given in Proposition 1, enable us to give other extensions. Let us denote for $f \in S_{X^*}$

$$D(f) = \{x \in S_X : f(x) = 1\}$$

Then for example we can call a norm of X almost rough if there is an $\varepsilon > 0$ such that for every $x \in S_X$ and $\delta > 0$, there exists $F \in S_{X^{**}}$, $\|F - x\| < \delta$ with $\operatorname{diam}(A(x) \cup D(F)) \geq \varepsilon$. Obviously, if $x \in X$, then $D(x) = A(x)$.

<u>Remark</u> 3. If for each $\varepsilon > 0$ there exist $x \in S_X$, $F \in S_{X^{**}}$ and $\delta > 0$, $\|F - x\| < \delta$ with $\operatorname{diam}(K(F,\delta) \cap K(x,\delta)) < \varepsilon$ then the norm of X is not almost rough (since $A(x) \cup D(F) \subset K(F,\delta) \cap \cap K(x,\delta)$).

For each Banach space X, let

$$\varepsilon_X = \sup\{\varepsilon : \operatorname{diam} A(x) \geq \varepsilon \text{ for each } x \in S_X\}$$

Then clearly this supremum is attained and we have $0 \leq \varepsilon_X \leq 2$. The known examples of spaces are with $\varepsilon_X = 0$ or 2 but it is not difficult to show that for each λ, $0 < \lambda < 2$, there exists a space X with $\varepsilon_X = \lambda$. Indeed, since for $X = \ell^1(\Gamma)$, Γ uncountable we have $\varepsilon_X = 2$ (see [2]), let $f_0 = (\xi_\gamma)_{\gamma\in\Gamma} \in X^*$ $\xi_{\gamma_0} = 1$, $\xi_\gamma = 0$ for $\gamma \neq \gamma_0$. Let $C = \{ f \in X^* : \|f-\alpha f_0\| \le \lambda \}$, for suitable α depending on λ . Let B^* be the (weak*) closed convex hull of B_{X^*} , C and $-C$. Then B^* induces a norm on X with the desired properties. Note that for $x \in X$, $x = (\xi_\gamma)_{\gamma\in\Gamma}$ $\xi_{\gamma_0} = \mu$ and $\xi_\gamma = 0$ for $\gamma \neq \gamma_0$ where μ is determined to have in the new norm ,norm of x equals 1,we have diam $A(x) = \lambda$.

Supposing $\varepsilon_X > 0$,(i.e., the norm of X is strongly rough), we want necessary and (or) sufficient conditions such that for each $x \in S_X$ to exist only one $u = u_X \in S_X$ with $\|x\|'(u) + \|x\|'(-u) \geq \varepsilon_X$.(see Definition 2). A necessary condition will be given by the next result:

Proposition 3. Let X be a Banach Space with $\varepsilon_X > 0$. A necessary condition that for the element $x \in S_X$ to exist a unique $u \in S_X$ with

(5) $$\|x\|'(u) + \|x\|'(-u) \geq \varepsilon_X$$

is that diam $A(x) = \varepsilon_X$. Moreover, the uniqueness of $u \in S_X$ implies equality in (5).

Proof. Let $x \in S_X$ and suppose diam $A(x) > \varepsilon_X$. Then there exist $f_1, f_2 \in A(x)$ such that $\|f_1 - f_2\| > \varepsilon_X$. Hence there exists $y \in S_X$ such that $\|f_1-f_2\| \geq (f_1-f_2)(y) > \varepsilon_X$. Then $y \neq x$. Let $X_2 = \operatorname{sp}\{x,y\}$. Then $A_{X_2}(x) = [\varphi_1, \varphi_2]$, $\varphi_i \in S_{X_2^*}$, $\varphi_1 \neq \varphi_2$. We can suppose , interchanging the indices if necessary, that $\varphi_1(y) > \varphi_2(y)$. Let $\psi_i = f_i | X_2 \in A_{X_2}(x)$. Then $\psi_i = \lambda_i \varphi_1 + (1-\lambda_i)\varphi_2$, $0 \le \lambda_i \le 1$. We have $\lambda_1 > \lambda_2$. Indeed, if $\lambda_1 \le \lambda_2$, then $\varepsilon_X<(f_1-f_2)(y)=\Psi_1(y)-\Psi_2(y)=(\lambda_1-\lambda_2)(\varphi_1-\varphi_2)(y)\leq 0$ which is impossible. Therefore $\varepsilon_X<(f_1-f_2)(y)=(\lambda_1-\lambda_2)(\varphi_1-\varphi_2)(y)\leq \leq(\varphi_1-\varphi_2)(y)$. Then by (3) we obtain $(\varphi_1-\varphi_2)(y)=\|x\|'(y)+ +\|x\|'(-y)>\varepsilon_X$. Let $H=\{z\in X_2:(\varphi_1-\varphi_2)(z)\geq\varepsilon_X\}$. We have $y\in H\cap S_{X_2}$ and since $(\varphi_1-\varphi_2)(y)>\varepsilon_X$, there exists $z\in H\cap S_{X_2}$, such that $\varepsilon_X=(\varphi_1-\varphi_2)(z)=\|x\|'(z)+\|x\|'(-z)$ and $y\neq z$, a contradiction. Therefore diam $A(x)=\varepsilon_X$. The last part of the proof shows that if $y\in S_X$ is unique such that $\|x\|'(y)+\|x\|'(-y)\geq\varepsilon_X$ then here we must have equality.

<u>Remark 4.</u> Let $x\in S_X$. We have $\{y\in S_X:\|x\|'(y)+\|x\|'(-y)\geq\varepsilon_X\}= =\{y\in S_X:\|x\|'(y)+\|x\|'(-y)=\varepsilon_X\}$ if and only if diam $A(x)=\varepsilon_X$. Indeed since for $\varepsilon_X=0$ this is obvious, let $\varepsilon_X>0$. If diam $A(x)\neq\varepsilon_X$ then as in the first part of the proof of Proposition 3 we contradict the equality of the two sets. If diam $A(x)=\varepsilon_X$ and there exists $y\in S_X$ such that $\|x\|'(y)+\|x\|'(-y)>\varepsilon_X$, then as in the proof of Proposition 3, in $X_2=sp\{x,y\}$ we have $A_{X_2}=[\varphi_1,\varphi_2],\varphi_i\in S_{X_2^*}$ and we can suppose $\varphi_1(y)>\varphi_2(y)$. Then by (3), $\varepsilon_X<\|x\|'(y)+\|x\|'(-y)=(\varphi_1-\varphi_2)(y)\leq\|\varphi_1-\varphi_2\|$. By a corollary of Hahn-Banach theorem, it follows diam $A(x)>\varepsilon_X$, contradicting the hypothesis.

There exist spaces with the property diam $A(x)=\varepsilon_X>0$ for each $x\in S_X$. For example $\ell^1(\Gamma)$, Γ uncountable has this property with $\varepsilon_X=2$ as we have mentioned above. Since for X smooth we have $\varepsilon_X=0$ and diam $A(x)=\varepsilon_X$ for each $x\in S_X$ the following questions arise: does there exists a space with diam $A(x)=\varepsilon_X$ for each $x\in S_X$, and $0<\varepsilon_X<2$? If the answer is yes, then is it true that for each λ, $0<\lambda<2$, there exists X with diam $A(x)=\varepsilon_X=\lambda$ for each $x\in S_X$?

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