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On Noether's Theorem

by Theodor Ghinda

Let $u^1, \ldots, u^m$ be some real-valued functions depending on n variables $x^1, \ldots, x^n$. We adopt the notation $u^\alpha_i = \frac{\partial u^\alpha}{\partial x^i}$ and consider the functional $L = L(x^i, u^\alpha, u^\alpha_i)$ of class C^2 in all its n + m + nm variables. We further note (using the summation convention) :

$$\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ji} \frac{\partial}{\partial u^\alpha_j} \quad , \quad [L]_\alpha = \frac{\partial L}{\partial u^\alpha} - \frac{d}{dx^i}\left(\frac{\partial L}{\partial u^\alpha_i}\right).$$

We consider the following form of Noether's theorem :

Theorem. Let $u^1, \ldots, u^m$ be solutions of the equations $[L]_\alpha = 0$ $(\alpha = 1, \ldots, m)$ and let $\xi^i = \xi^i(x^j, u^\beta, u^\beta_j)$ $(i = 1, \ldots, n)$, $\eta^\alpha = \eta^\alpha(x^j, u^\beta, u^\beta_j)$ $(\alpha = 1, \ldots, m)$, $V^i = V^i(x^j, u^\beta, u^\beta_j)$ $(i = 1, \ldots, n)$ be some real-valued functions which satisfy the relation :

$$L\frac{d\xi^i}{dx^i} + \frac{\partial L}{\partial x^i}\xi^i + \frac{\partial L}{\partial u^\alpha}\eta^\alpha + \frac{\partial L}{\partial u^\alpha_i}\left(\frac{d\eta^\alpha}{dx^i} - u^\alpha_\ell \frac{d\xi^\ell}{dx^i}\right) + \frac{dV^i}{dx^i} = 0. \qquad (1)$$

Then $u^1, \ldots, u^m$ also verify the relation :

$$\frac{d}{dx^i}\left[L\xi^i + \frac{\partial L}{\partial u^\alpha_i}\left(\eta^\alpha - u^\alpha_\ell \xi^\ell\right) + V^i\right] = 0. \qquad (2)$$

The result may be obtained combining the statements given by D.Lovelock and H.Rund [1] and by A.Trautman [2] (see also [3]) and assuming moreover that the functions ξ^i, η^α and V^i depend on the partial derivatives u^β_j too (the proof remains the same).

The theorem is very useful for mechanics, permitting a relatively easy deduction of the main conservation laws. Usually, one considers that one single function among $\xi^1, \ldots, \xi^n$, $\eta^1, \ldots, \eta^m$ is different from zero, choses it and $V^1, \ldots, V^n$ so that the condition (1) be verified and writes the corresponding conservation law (2). Then one makes another choice, etc. The question that naturally arises is whether the possibilities of the theorem have thus been exhausted.

<u>Proposition</u>. In order to obtain all the conservation laws which can be derived using the previous theorem, it is sufficient to assume that one single function among $\xi^1, \dots, \xi^n, \eta^1, \dots, \eta^m$ is different from zero. If the non-vanishing function is ξ^K (fixed K), then every conservation law given by the general form of the condition (1) can be obtained from the following $\xi_1^K, V_1^1, \dots, V_1^n$:

$$\xi_1^K = \xi^K \tag{3}$$

$$V_1^j - \frac{\partial L}{\partial u_j^\alpha} u_K^\alpha \xi_1^K = V^j + L\xi^j + \frac{\partial L}{\partial u_j^\alpha}\left(\eta^\alpha - u_l^\alpha \xi^l\right) \qquad (j \neq K) \tag{4}$$

$$V_1^K = V^K + \frac{\partial L}{\partial u_K^\alpha}\left(\eta^\alpha - u_j^\alpha \xi^j\right) \qquad (j \neq K) \tag{5}$$

(no sum over K). If the non-vanishing function is η^γ (fixed γ), then every conservation law given by the general situation can be obtained from the following $\eta_1^\gamma, V_1^1, \dots, V_1^n$:

$$\eta_1^\gamma = \eta^\gamma \tag{6}$$

$$V_1^i + \frac{\partial L}{\partial u_i^\gamma}\eta_1^\gamma = V^i + L\xi^i + \frac{\partial L}{\partial u_i^\alpha}\left(\eta^\alpha - u_l^\alpha \xi^l\right). \tag{7}$$

<u>Proof</u>. As in the statement, the indices K and γ will be excluded from the summation convention and the index j will except the value K.

First, we must verify that the functions chosen through the relations (3) - (5) and (6) - (7) , respectively, belong to the announced class, i.e. they satisfy the condition (1). For the first choice, we have :

$$L\frac{d\xi_1^K}{dx^K} + \frac{\partial L}{\partial x^K}\xi_1^K - \frac{\partial L}{\partial u_i^\alpha}u_K^\alpha\frac{d\xi_1^K}{dx^i} + \frac{dV_1^i}{dx^i} =$$

$$= L\frac{d\xi^K}{dx^K} + \frac{\partial L}{\partial x^K}\xi^K - \frac{\partial L}{\partial u_i^\alpha}u_K^\alpha\frac{d\xi^K}{dx^i} + \frac{dV^K}{dx^K} + \frac{d}{dx^K}\left[\frac{\partial L}{\partial u_K^\alpha}\left(\eta^\alpha - u_j^\alpha\xi^j\right)\right] +$$

$$+ \frac{dV^j}{dx^j} + \frac{d}{dx^j}\left(\frac{\partial L}{\partial u_j^\alpha}u_K^\alpha\xi^K\right) + \frac{d}{dx^j}\left[L\xi^j + \frac{\partial L}{\partial u_j^\alpha}\left(\eta^\alpha - u_l^\alpha\xi^l\right)\right] =$$

$$= \frac{d}{dx^i}\left[L\xi^i + \frac{\partial L}{\partial u_i^\alpha}\left(\eta^\alpha - u_l^\alpha\xi^l\right) + V^i\right] - \xi^K\frac{dL}{dx^K} + \frac{d}{dx^K}\left(\frac{\partial L}{\partial u_K^\alpha}u_K^\alpha\xi^K\right) +$$

$$+ \frac{\partial L}{\partial x^K}\xi^K - \frac{\partial L}{\partial u_i^\alpha}u_K^\alpha\frac{d\xi^K}{dx^i} + \frac{d}{dx^j}\left(\frac{\partial L}{\partial u_j^\alpha}u_K^\alpha\xi^K\right) =$$

$$= -\xi^K\frac{dL}{dx^K} + \xi^K\frac{\partial L}{\partial x^K} - \frac{\partial L}{\partial u_i^\alpha}u_K^\alpha\frac{d\xi^K}{dx^i} + \frac{d}{dx^i}\left(\frac{\partial L}{\partial u_i^\alpha}u_K^\alpha\xi^K\right) =$$

$$= -\xi^K\frac{\partial L}{\partial u^\alpha}u_K^\alpha - \xi^K\frac{\partial L}{\partial u_i^\alpha}u_{iK}^\alpha + \xi^K\frac{d}{dx^i}\left(\frac{\partial L}{\partial u_i^\alpha}u_K^\alpha\right) = -\xi^K u_K^\alpha[L]_\alpha = 0.$$

For the functions given by (6) and (7) , we follow the same way:

$$\frac{\partial L}{\partial u^\gamma}\eta_1^\gamma+\frac{\partial L}{\partial u_i^\gamma}\frac{d\eta_1^\gamma}{dx^i}+\frac{dV_1^i}{dx^i}=\frac{\partial L}{\partial u^\gamma}\eta^\gamma+\frac{\partial L}{\partial u_i^\gamma}\frac{d\eta^\gamma}{dx^i}+$$

$$+\frac{d}{dx^i}\left[L\xi^i+\frac{\partial L}{\partial u_i^\alpha}(\eta^\alpha-u_\ell^\alpha\xi^\ell)+V^i\right]-\frac{d}{dx^i}\left(\frac{\partial L}{\partial u_i^\gamma}\eta^\gamma\right)=$$

$$=\frac{\partial L}{\partial u^\gamma}\eta^\gamma+\frac{\partial L}{\partial u_i^\gamma}\frac{d\eta^\gamma}{dx^i}-\frac{\partial L}{\partial u_i^\gamma}\frac{d\eta^\gamma}{dx^i}-\eta^\gamma\frac{d}{dx^i}\left(\frac{\partial L}{\partial u_i^\gamma}\right)=\eta^\gamma[L]_\gamma=0.$$

Now, we have to check that the functions $\xi_1^k, V_1^1, \ldots, V_1^n$ lead to the same conservation law as the functions $\xi^1,\ldots,\xi^n,\eta^1,\ldots,\eta^m,$ $V^1,\ldots,V^n$ in the right-hand side of the relations (3) - (5). We proceed as follows : in the general form

$$\frac{d}{dx^k}\left(L\xi^k-\frac{\partial L}{\partial u_k^\alpha}u_k^\alpha\xi^k+V^k\right)+\frac{d}{dx^\delta}\left(-\frac{\partial L}{\partial u_\delta^\alpha}u_k^\alpha\xi^k+V^\delta\right)=0$$

of the conservation laws based on $\xi^k, V^1, \ldots, V^n$, we replace these functions by the values $\xi_1^k, V_1^1, \ldots, V_1^n$ given by (3) - (5); it results just the relation (2). A similar calculation for the functions $\eta_1^\gamma, V_1^1, \ldots, V_1^n$ in (6) and (7) completes the proof.

The proposition shows that, instead of studying the equation (1) in its general form, it is sufficient to consider that the functions $\xi^1,\ldots,\xi^n,\eta^1,\ldots,\eta^m$ vanish, except one of them, chosen so that the problem become as simple as possible.

Let us suppose, for example, that we are concerned with the motion of ideal fluids and use the Eulerian description. It can be shown (see [4] , [5]) that the equations of motion follow from the variational principle :

$$\delta\int_{t_0}^{t_1}\int_D L\,dx\,dt=0,$$

where

$$L=\rho\left(W+\frac{\partial\phi}{\partial t}+S\frac{\partial\eta}{\partial t}+\alpha\frac{\partial\beta}{\partial t}+U+\frac{1}{2}\vec{v}^{\,2}\right) \tag{8}$$

and $D\subset R^3$ is an arbitrary domain. We have used the representation

$$v_i=\frac{\partial\phi}{\partial x_i}+S\frac{\partial\eta}{\partial x_i}+\alpha\frac{\partial\beta}{\partial x_i}\quad(i=1,2,3)\ ;$$

W is the specific internal energy, S is the entropy, U is the potential of the exterior forces (with changed sign), α is one of the Lagrangian coordinates and ϕ,η,β are multipliers (see [5]).

The independent variables are x_1, x_2, x_3, t and the dependent variables are $\rho,\phi,S,\eta,\alpha,\beta$.

If we replace L in the condition (1) by the expression (8), we remark that we obtain the most simple equation when all the functions ξ^i and η^α vanish, except η^ϕ (corresponding to ϕ). In order to avoid any misunderstanding, the function will be denoted by η_1^ϕ. We arrive at the relation :

$$\rho \frac{D\eta_1^\phi}{Dt} + \frac{dV_1^i}{dx^i} = 0 \qquad (9)$$

where $\frac{D}{Dt}$ is substantial derivative.

Now, the functions V^i are V_1, V_2, V_3, V_t (corresponding to the independent variables) and the conservation law following from (9) may be written in the form :

$$\int_{D(t)} \frac{d}{dt}(\rho\eta_1^\phi + V_t^1)\,dx = -\int_{\partial D(t)} (\rho\eta_1^\phi v_i + V_i^1)\,n_i\,ds, \qquad (10)$$

$\vec{n}$ denoting the exterior normal to the boundary of the domain D(t).

We shall show how the classical theorems of ideal fluid mechanics can be deduced starting from (9) and (10).

Having in view the values of the functions $\xi^1, \dots, \xi^n, \eta^1, \dots, \eta^m; V^1, \dots, V^n$ which usually lead to these theorems (i.e'. having the condition (9) automatically fulfiled), according to the proposition), the relations (6), (7) and (10) yield us the following results :

The energy theorem is obtained from :

$$\eta_1^\phi = 0$$

$$V_i^1 = V_i + p v_i a + \rho v_i \left(W + U + \tfrac{1}{2}\vec{v}^2\right) a$$

$$V_t^1 = V_t + \rho\left(W + U + \tfrac{1}{2}\vec{v}^2\right) a ,$$

where V_1, V_2, V_3, V_t are solutions for $\frac{dV_i}{dx_i} + \frac{dV_t}{dt} = -\rho\frac{\partial U}{\partial t} a$ (a is an arbitrary constant) and has the form :

$$\frac{D}{Dt}\int_{D(t)} \rho\left(W + U + \tfrac{1}{2}\vec{v}^2\right) dx = \int_{D(t)} \rho \frac{\partial U}{\partial t}\,dx - \int_{\partial D(t)} p\,\vec{v}\cdot\vec{n}\,ds .$$

The momentum theorem follows from :

$$\eta_1^\phi = 0$$

$$V_i^1 = V_i - p a_i - \rho v_i v_\ell a_\ell$$

$$V_t^1 = V_t - \rho v_\ell a_\ell ,$$

with $\frac{dV_i}{dx_i} + \frac{dV_t}{dt} = -\rho \frac{\partial U}{\partial x_i} a_i$ (a_i are arbitrary constants) and has the form :

$$\frac{D}{Dt}\int_{D(t)} \rho \vec{v}\, dx = \int_{D(t)} \rho \vec{F}\, dx - \int_{\partial D(t)} p \vec{n}\, ds .$$

Putting $(\vec{\alpha} \times \vec{x})_i$ instead of a_i ($i=1, 2, 3$), we get the angular momentum theorem :

$$\frac{D}{Dt}\int_{D(t)} \vec{x} \times \rho \vec{v}\, dx = \int_{D(t)} \vec{x} \times \rho \vec{F}\, dx - \int_{\partial D(t)} \vec{x} \times p \vec{n}\, ds ,$$

while for $a_i t$ instead of a_i we obtain the center-of-mass theorem :

$$\frac{D}{Dt}\int_{D(t)} \rho (\vec{x} - \vec{v} t)\, dx = -\int_{D(t)} \rho t \vec{F}\, dx + \int_{\partial D(t)} p t \vec{n}\, ds .$$

From

$$\eta_1^\phi = a$$
$$V_i^1 = 0$$
$$V_t^1 = 0$$

it follows the conservation-of-mass theorem :

$$\frac{D}{Dt}\int_{D(t)} \rho\, dx = 0 .$$

For

$$\eta_1^\phi = 0$$
$$V_i^1 = \rho \alpha v_i a$$
$$V_t^1 = \rho \alpha a ,$$

we are led to the conservation law : $\frac{D}{Dt}\int_{D(t)} \rho \alpha\, dx = 0.$

Finally, from

$$\eta_1^\phi = 0$$
$$V_i^1 = \rho S v_i a$$
$$V_t^1 = \rho S a$$

we obtain the entropy conservation law in the form :

$$\frac{D}{Dt}\int_{D(t)} \rho S\, dx = 0 .$$

In each case, one can verify directly that the chosen functions are solutions of the equation (9).

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