ON AMPLE DIVISORS: II

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This paper is a continuation of our previous one [1], from which we shall borrow in general the terminology and notations. Using the same kind of techniques as in [1] we prove some other results.

§1. Let k be an algebraically closed field and Y a minimal model of rational surfaces over k. First of all we shall determine all smooth projective threefolds X containing Y as an ample divisor. By a well known theorem of Nagata, Y is isomorphic to one of the following surfaces: P^2, $P^1 \times P^1 = F_o$, or $F_e = P(O_{P^1} \oplus O_{P^1}(-e))$, where $e \geqslant 2$ and where P(E) denotes the projective bundle associated to any locally free sheaf E.

The problem we want to study is well known for $Y = P^2$ (and in this case X is isomorphic to P^3), while the case $Y = P^1 \times P^1$ is treated in [1], theorem 5. Thus we may assume $Y = F_e$ with $e \geqslant 2$. Denote by $p: Y = F_e \longrightarrow P^1$ the canonical projection. Then a base for Pic(Y) is the following: $O_Y(1) = O_{P(O \oplus O(-e))}(1)$ and $p^* O(1)$, where $O = O_{P^1}$ and $O(1) = O_{P^1}(1)$.

Theorem 1. Assume that char(k) = o and that $Y = F_e$, $e \geqslant 2$, is contained in the smooth projective threefold X as an ample divisor. Then there exists an exact sequence of O_{P^1}-modules of the form

$$o \longrightarrow O_{P^1} \longrightarrow E = O(a) \oplus O(b) \oplus O(c) \xrightarrow{\varphi} F = O(s) \oplus O(s-e) \longrightarrow o,$$

where $a > o$, $b > o$, $c > o$ are positive integers, such that X is isomorphic to P(E) and $Y \cong P(F)$ is contained in X via surjection φ.

<u>Proof</u>. Since $O_Y(1)$ and $p^*O(1)$ form a basis for Pic(Y), there are two integers α and β such that $O_X(Y)\otimes O_Y = O_Y(\beta)\otimes p^*O(\alpha)$, and since Y is an ample divisor on X, $\beta > o$ and $\alpha > \beta\cdot e$ (see for instance [3], page 380, 2.18). By proposition 1 in [1] the canonical map $\varepsilon : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y)$ is injective and Coker(ε) is torsion-free. (Here one uses essentially the hypothesis about char(k)!) Recalling that Pic(Y) is a free group of rank two, we have only one of the following cases:

1) <u>Pic(X) is a free group of rank one</u>. Choose then an ample generator L of Pic(X) and write $O_X(Y) = L^{\otimes r}$, $\omega_X = L^{\otimes t}$ and $L\otimes O_Y = O_Y(\gamma)\otimes p^*O(\delta)$, where $r,t,\gamma,\delta \in \mathbb{Z}$ and ω_X is the canonical sheaf of X. Since L is ample, $\gamma > o$ and $\delta > \gamma\cdot e$. The adjunction formula immediately yields

$$(r+t)\cdot\delta = -e-2 \qquad \text{and} \qquad (r+t)\cdot\gamma = -2.$$

If $\gamma = 1$ then $r+t = -2$ and thus $\delta = 1+\frac{e}{2}$, contradicting the inequalities $\delta > \gamma\cdot e$ and $e \geqslant 2$. If $\gamma = 2$ then $r+t = -1$ and thus $\delta = e+2$, contradicting again the inequalities $\delta > \gamma\cdot e$ and $e \geqslant 2$.

Therefore case 1) is impossible.

2) <u>ε is an isomorphism</u>. Then there are $L, M \in \mathrm{Pic}(X)$ such that $L\otimes O_Y \cong p^*O(1)$ and $M\otimes O_Y \cong O_Y(1)$. Since $O_X(Y)\otimes O_Y \cong O_Y(t)\otimes p^*O(s)$, with $s > te > o$, the injectivity of ε yields $O_X(Y) \cong L^{\otimes s}\otimes M^{\otimes t}$.

Let $\sigma \in \Gamma(X, O_X(Y)) \cong \Gamma(L^{\otimes s}\otimes M^{\otimes t})$ be such that $\mathrm{div}_X(\sigma) = Y$, i.e. a global equation for Y. The exact sequence ($u,m \in \mathbb{Z}$)

$$(1) \qquad o \longrightarrow L^{\otimes u}\otimes O_X((m-1)Y) \xrightarrow{\ \sigma\ } L^{\otimes u}\otimes O_X(mY) \longrightarrow O_Y(mt)\otimes p^*O(ms+u) \longrightarrow o$$

(where the first map is multiplication by σ) yields the exact sequence

$$(2) \qquad H^1(L^{\otimes u}\otimes O_X((m-1)Y)) \longrightarrow H^1(L^{\otimes u}\otimes O_X(mY)) \longrightarrow H^1(O_Y(mt)\otimes p^*O(ms+u)).$$

Since Y is ample on X and X is a smooth projective threefold we have

(3) $H^1(L^{\otimes u}\otimes O_X(mY)) = o$ for every $m \ll o$ and $u = o,1$.

I claim that

(4) $H^1(O_Y(mt)\otimes p^*O(ms+u)) = o$ for every $m \in \mathbb{Z}$ and $u = o,1$.

Proof of (4). Consider the Leray spectral sequence

$$E_2^{ij} = H^i(P^1, O(ms+u)\otimes R^jp_*O_Y(mt)) \Longrightarrow H^{i+j}(O_Y(mt)\otimes p^*O(ms+u)).$$

If $m \gg o$, $R^jp_*O_Y(mt) = o$ for every $j > o$ and hence this spectral sequence degenerates. In particular

$$H^1(O_Y(mt)\otimes p^*O(ms+u)) \cong H^1(P^1, O(ms+u)\otimes p_*O_Y(mt)) \cong$$
$$\cong H^1(P^1, O(ms+u)\otimes S^{mt}(O\oplus O(-e))) = \bigoplus_{i=0}^{mt} H^1(P^1, O(ms+u-ie)),$$

where $S^i(G)$ stands for the i^{th} symmetric power of the O_{P^1}-module G. But for $o \leq u \leq 1$ and $o \leq i \leq mt$ we have $ms+u-ie > o$ (recall that $s > te > o$), and therefore (4) follows in case $m \gg o$ from the explicit computation of the cohomology of P^1.

Assume now $m < o$. Consider the exact sequence

$$o \longrightarrow E_2^{1,o} \longrightarrow H^1 \longrightarrow E_2^{o,1}.$$

For proving (4) in this case it will be sufficient to show that $E_2^{1,o} = E_2^{o,1} = o$. We have

$$E_2^{1,o} = H^1(P^1, O(ms+u)\otimes p_*O_Y(mt)) = o \text{ since for } m < o,\ p_*O_Y(mt) = o\ ;$$
$$E_2^{o,1} = H^o(P^1, O(ms+u)\otimes R^1p_*O_Y(mt)).$$

The relative duality for the morphism p (see [4]) together with the equality $\omega_{Y/P^1} = O_Y(-2)\otimes p^*O(-e)$ give

$$R^1p_*O_Y(mt) \cong \mathcal{H}om_{O_{P^1}}(p_*[O_Y(-mt-2)\otimes p^*O(-e)], O_{P^1}) =$$
$$= \mathcal{H}om_{O_{P^1}}(O(-e)\otimes p_*O_Y(-mt-2), O_{P^1}).$$

Since $m < o$ and $t > o$, $-mt-2 \geq -1$. If $-mt-2 = -1$ (i.e. $t = 1$ and $m = -1$) then $p_*O_Y(-mt-2) = p_*O_Y(-1) = o$, or else $R^1p_*O(mt) = o$, and (4) is proved if $-mt-2 = -1$.

Assume therefore $-mt-2 \geqslant o$. Then $p_*O_Y(-mt-2) \cong S^{-mt-2}(O \oplus O(-e)) =$
$\cong \bigoplus_{i=o}^{-mt-2} O(-ie)$, and therefore $R^1 p_* O_Y(mt) \cong \bigoplus_{i=o}^{-mt-2} \mathcal{H}om_{O_{P^1}}(O(-e-ie), O_{P^1}) =$
$= \bigoplus_{i=o}^{-mt-2} O((i+1)e)$. Thus

$$E_2^{o,1} \cong H^o(P^1, \bigoplus_{i=o}^{-mt-2} O(ms+u+(i+1)e)) = o,$$

because for $m < o$, $-mt-2 \geqslant o$ and $o \leqslant u \leqslant 1$ we have $ms+u+(i+1)e \leqslant ms+1+(-mt-1)e =$
$= m(s-te) + (1-e) < o$.

Thus (4) is proved in all cases.

Now from (2), (3) and (4) we get by induction on m that $H^1(L^{\otimes u} \otimes O_X(mY)) =$
$= o$ for every $m \in \mathbb{Z}$ and $u = o,1$. In particular, (1) induces the exact sequence

$$(5_{u,m}) \qquad o \longrightarrow \Gamma(L^{\otimes u} \otimes O_X((m-1)Y)) \longrightarrow \Gamma(L^{\otimes u} \otimes O_X(mY)) \longrightarrow$$
$$\longrightarrow \Gamma(Y, O_Y(mt) \otimes p^* O(ms+u)) \longrightarrow o,$$

for every $m \in \mathbb{Z}$ and $u = o,1$.

Then $(5_{1,o})$ shows that the restriction map $\Gamma(L) \longrightarrow \Gamma(Y, p^*O(1))$ is an isomorphism (indeed, we have $\Gamma(L \otimes O_X(mY)) = o$ for $m \ll o$ and one applies an easy induction on $m < o$ using $(5_{1,m})$ in order to get that $\Gamma(L \otimes O_X(-Y)) = o$). Since $|p^*O(1)| = P^1$, there are two distinct divisors $\Delta, \Delta' \in |L|$ such that $\Delta \cap \Delta' \cap Y = \phi$. Since Y is ample on X, $\dim(\Delta \cap \Delta') \leqslant o$, and in fact we cannot have $\Delta \cap \Delta' \neq \phi$ because otherwise

$$3 = \operatorname{codim}_X(\Delta \cap \Delta') \leqslant \operatorname{codim}_X(\Delta) + \operatorname{codim}_X(\Delta) = 2.$$

Therefore $\Delta \cap \Delta' = \phi$. In particular, the linear system $|L|$ has no base points and hence the corresponding rational map $q = \varphi_L : X \longrightarrow |L| = P^1$ is a surjective morphism. This implies that for every $L' \in \mathrm{Pic}(X)$, $(L^{\cdot 2}.L') =$
$= o$ (see [5] for the intersection theory of line bundles). But the equalities

$$1 = (p^*O(1).O_Y(1))_Y = (L.M.Y)_X = s(L^{\cdot 2}.M) + t(L.M^{\cdot 2}) = t(L.M^{\cdot 2})$$

show that $t = 1$ and $(L.M^{\cdot 2}) = 1$. Therefore $O_X(Y) = L^{\otimes s}\otimes M$ with $s > e$. Let $\Delta \in |L|$ be an arbitrary member and set $T = O_X(Y)\otimes O_\Delta \cong M\otimes O_\Delta$. Then T is an ample invertible O_Δ-module, $(T^{\cdot 2})_\Delta = (L.M^{\cdot 2})_X = 1$, and moreover

(6) $\dim\Gamma(\Delta, T) \geqslant 3$.

<u>Proof of (6)</u>. The exact sequence $(5_{o,1})$ together with the fact that $t = 1$ give:

$$\dim\Gamma(L^{\otimes s}\otimes M) = 1 + \dim\Gamma(Y, O_Y(1)\otimes p^*O(s)) = 1 + \dim\Gamma(P^1, O(s)\oplus O(s-e)) =$$
$$= 1 + (s+1) + (s-e+1) = 2s - e + 3.$$

The exact sequence $(5_{1,1})$ gives

$$\dim\Gamma(L^{\otimes(s+1)}\otimes M) = \dim\Gamma(L) + \dim\Gamma(Y, O_Y(1)\otimes p^*O(s+1)) = 2 +$$
$$+ \dim\Gamma(P^1, O(s+1)\oplus O(s+1-e)) = 2 + (s+2) + (s+2-e) = 2s - s + 6.$$

Finally, the exact sequence

$$o \longrightarrow L^{\otimes s}\otimes M \longrightarrow L^{\otimes(s+1)}\otimes M \longrightarrow T \longrightarrow o$$

(recall that $\Delta \in |L|$ and $T = M\otimes O_\Delta$) yields the exact sequence

$$o \longrightarrow \Gamma(L^{\otimes s}\otimes M) \longrightarrow \Gamma(L^{\otimes(s+1)}\otimes M) \longrightarrow \Gamma(T),$$

and therefore $\dim\Gamma(T) \geqslant \dim\Gamma(L^{\otimes(s+1)}\otimes M) - \dim\Gamma(L^{\otimes s}\otimes M) = (2s-e+6) - (2s-e+3) = 3$, and (6) is proved.

Now a theorem of Kobayashi and Ochiai (see [6], or [1], theorem 3) and the facts that T is ample, $(T^{\cdot 2})_\Delta = 1$ and $\dim\Gamma(\Delta, T) \geqslant 3$ imply that $\Delta \cong P^2$ and $T \cong O_{P^2}(1)$. This happens for every $\Delta \in |L|$. Hence $O_X(Y)$ induces the tautological invertible sheaf on every $\Delta \cong P^2$. In these circumstances Hironaka has proved that $E = q_* O_X(Y)$ is a locally free O_{P^1}-module of rank 3, that $X \cong P(E)$ and that $O_{P(E)}(1) \cong O_X(Y)$ (see [4], theorem 1.8). Then the exact sequence $(5_{o,1})$ (with $t = 1$) yields the exact sequence of cohomology

$$o \longrightarrow q_*O_X \cong O_{P^1} \longrightarrow E \longrightarrow p_*[O_Y(1) \otimes p^*O(s)] \cong O(s) \oplus O(s-e) \longrightarrow R^1 q_* O_X = o.$$

By a theorem of Grothendieck E is of the form $E = O(a) \oplus O(b) \oplus O(c)$, where $a,b,c \in \mathbb{Z}$. Since Y is ample on X, $E = q_* O_X(Y)$ is ample on P^1, and therefore $a > o$, $b > o$ and $c > o$. Taking degrees we also get $a+b+c = 2s-e$. Q.E.D.

Corollary. Assume that the smooth projective threefold X over the algebraically closed field k of char. zero contains a minimal model of rational surfaces as an ample divisor. Then X is a rational threefold.

The corollary is a consequence of theorem of Nagata quoted above, theorem 1 above and [1], theorem 5 and proposition 2.

Remark. This corollary is no longer true if one drops the assumption about minimality of Y, as it is easily seen by taking for X any smooth cubic hypersurface in P^4, which is known to be not rational (see [2]). On the other hand, a generic hyperplane section of X is a smooth cubic surface in P^3, which is a rational surface, but not a minimal model.

A completely similar reasoning as that from theorem 1 proves the following:

Theorem 2. Assume that $Y = P(O_{P^1}(d_1) \oplus \dots \oplus O_{P^1}(d_n))$, with $d_1 \geq d_2 \geq \dots \geq d_n > o$ and $n \geq 3$, is an ample divisor in the smooth projective $(n+1)$-dimensional variety X over the algebraically closed field k of arbitrary char. Then there are: $n+1$ positive integers $a_1 > o, \dots, a_{n+1} > o$, $s \in \mathbb{Z}$ such that $s+d_n > o$ and the exact sequence of O_{P^1}-modules

$$o \longrightarrow O_{P^1} \longrightarrow E = O(a_1) \oplus \dots \oplus O(a_{n+1}) \xrightarrow{\varphi} F = O(s+d_1) \oplus \dots \oplus O(s+d_n) \longrightarrow o,$$

such that X is isomorphic to P(E) and $Y \cong P(F)$ is embedded in X via surjection φ.

Remarks. 1) Because theorem 2 holds in arbitrary char., one can ask whether theorem 1 is valid in arbitrary char. as well. The trouble lies in the fact that if char(k) = $p > o$, one knows only that the restriction map $\varepsilon : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y)$ is injective and Coker(ε) has no e-torsion with e prime to p. (In theorem 2 one apply the well known Lefschetz's theorem and deduce that ε is an isomorphism without the restriction about char(k).) Probably one can prove that Coker(ε) has no p-torsion by using crystaline cohomology. Another way (see [1]) would be to verify that X has a lifting to char. zero and then apply lemma 1 in [1]. But in general this is not so easy. It would be sufficient to show that $H^2(X, T_X) = o$, where T_X is the tangent bundle of X. In [1] we verified this condition if Y is P^2 or $P^1 \times P^1$, and using the same method one can also verify it for $Y = F_2$ (in other words, theorem 1 is valid in arbitrary char. if $Y = F_2$).

2) In the situation of theorems 1 or 2 one can easily see that the divisor Y is in fact very ample on X and that the associated embedding $\varphi_Y : X \hookrightarrow |Y| = P$ has the property that $\varphi_Y(X)$ is a rational scroll, i.e. $q^{-1}(b)$ is a line for every closed point $b \in P^1$ (where $q : X \longrightarrow P^1$ is the canonical projection). Moreover $\deg \varphi_Y(X) = \mathrm{codim}_P(Y) + 1$.

3) Exact sequences as in theorems 1 or 2 do exist (see [1]). In particular, theorem 2 is applicable to $Y = P^1 \times P^t$ with $t \geqslant 2$; one gets that for every $s \geqslant 2$ there exists an exact sequence of the form

$$(7) \quad o \longrightarrow O_{P^1} \longrightarrow E = O(1) \oplus O(s-1) \oplus H \xrightarrow{\varphi} O(s) \oplus H = F \longrightarrow o,$$

where $H = \underbrace{O(s) \oplus \ldots \oplus O(s)}_{t \text{ times}}$, and $\varphi = \varphi' \oplus \mathrm{id}_H$, with $\varphi' : O(1) \oplus O(s-1) \longrightarrow O(s)$ the surjection given by $\varphi'(u,v) = x_o^{s-1} u + x_1 v$ (x_o and x_1 being homogeneous coordinates on P^1). In other words, there is a smooth projective (t+2)-dimensional variety X (namely $X = P(E)$ with E from the exact sequence (7)) supporting $P^1 \times P^t$, $t \geqslant 2$, as an ample divisor. This shows that theorem 2 in [1] fails for $Y = P^s \times P^t$ with $s = 1$ and $t \geqslant 2$, or with $s \geqslant 2$ and $t = 1$. However, theorem 4 below shows in particular that one can find the same conclusion as in theorem 2 in [1] if one assumes moreover that the normal bundle $N_{Y,X}$ ($Y = P^1 \times P^t$, $t \geqslant 2$) is of the form $O(a,b)$, where either $a = b = 1$, or else $a \geqslant 1$ and $b \geqslant 2$.

§2. In this section k will be also an algebraically closed field. Let $Y \subset P^m$ be a smooth subvariety of P^m of dimension $d \geqslant 2$ satisfying the following two properties:

a) Y is arithmetically Cohen-Macaulay in P^m.

b) $\mathrm{Pic}(Y) = \mathbb{Z} \cdot O_Y(1)$, where $O_Y(1) = O_{P^m}(1) \otimes O_Y$.

Let $f_1, \ldots, f_n$ be a system of homogeneous generators of the ideal $I(Y) \subset k[T_o, \ldots, T_m]$ of Y in P^m. For every $s > o$ let us denote by $v_s : Y \hookrightarrow P^{m(s)}$, with $m(s) = \binom{m+s}{m} - 1$, the composition of the s-fold Veronese embedding $P^m \hookrightarrow P^{m(s)}$ with the inclusion $X \subset P^m$, and by $C(Y,s) \subset \subset P^{m(s)+1}$ the projective cone over $v_s(Y)$.

Theorem 3. *Let $Y \subset P^m$ be a smooth subvariety of dimension $d \geqslant 2$ of P^m satisfying properties a) and b) above. If $d = 2$ one assumes moreover that $\mathrm{char}(k) = o$. Assume furthermore that Y is contained in the normal projective variety X as an ample Cartier divisor. Let* $N_{Y,X} \cong O_Y(s)$ *be the normal sheaf of Y in X (necessarily $s > o$) and assume moreover that*

c) $s \geqslant \max_{j=1}^{n} \{\deg(f_j) + 1\}$.

In these conditions X is isomorphic to the cone C(Y,s) and Y is contained in X as the intersection of C(Y,s) with the hyperplane at infinity of $P^{m(s)+1}$.

Remarks. 1) Theorem 3 extends theorem 1 in [1].

2) Condition c) in theorem 3 is in general the best possible. Indeed,

take for Y any smooth hypersurface in P^m ($m \geqslant 4$) of degree $a \geqslant 2$ (resp. any generic surface in P^3 of degree $a \geqslant 4$). Then conditions a) and b) are in this case fulfilled (namely, condition b) holds by Lefschetz's theorem if $m \geqslant 4$ and by Noether's theorem if $m = 3$). In this situation condition c) reads " $s > a$ ". For $s = a$ there are at least two normal projective varieties X_1 and X_2 supporting Y as an ample Cartier divisor and with normal bundles both isomorphic to $O_Y(a)$, namely: $X_1 = P^m$ and $X_2 = C(Y,a)$. And since X_1 is smooth while X_2 is not, these two varieties cannot be isomorphic.

<u>Proof of theorem 3</u>. We shall analyze carefully the proof of theorem 1 in [1]. According to that proof, if U is the smooth locus of X, then there is an $L \in \mathrm{Pic}(U)$ such that $L \otimes O_Y \cong O_Y(1)$; then $O_X(Y)/U \cong L^{\otimes s}$. If $j: U \hookrightarrow X$ is the canonical inclusion, for every $a \in \mathbb{Z}$ put $F^{(a)} = j_*(L^{\otimes a})$. Then $F^{(as)} \cong O_X(aY)$ for every $a \in \mathbb{Z}$. Let $\sigma \in \Gamma(X, F^{(s)}) \cong \Gamma(X, O_X(Y))$ be a global equation for Y. Exactly as in [1] one can see that for every $a \in \mathbb{Z}$ there is the exact sequence

$$(8) \qquad o \longrightarrow \Gamma(X, F^{(a-s)}) \xrightarrow{\sigma} \Gamma(X, F^{(a)}) \longrightarrow \Gamma(Y, O_Y(a)) \longrightarrow o.$$

Set $S = \bigoplus_{a=0}^{\infty} \Gamma(X, F^{(a)}) = \bigoplus_{a=0}^{\infty} \Gamma(U, L^{\otimes a})$. Then S is a graded k-algebra whose homogeneous part of degree a is $S_a = \Gamma(X, F^{(a)})$, so that $\sigma \in S_s$. Then (8) shows that

$$S/\sigma S = \bigoplus_{a=0}^{\infty} \Gamma(Y, O_Y(a)) \overset{\text{by a)}}{\cong} k[T_o, \ldots, T_m]/I(Y).$$

Denoting by $t_i = T_i \bmod I(Y)$, let $t'_i \in S_1$ be such that $t'_i \bmod \sigma S = t_i$. Then $t'_o, \ldots, t'_m$ satisfy the equations:

$$(9) \qquad f_j(t'_o, \ldots, t'_m) = o \text{ for every } j = 1, \ldots, n.$$

Indeed, if $f_j(t'_o, \ldots, t'_m) \neq o$ then $f_j(t'_1, \ldots, t'_m)$ would be (by condition c)) a homogeneous element of S of degree $< s$, and hence $f_j(t'_o, \ldots, t'_m) \notin \sigma S$ because the degree of σ is s. But this is absurd because

$$f_j(t'_o,\dots,t'_m) \bmod \sigma S = f_j(t_o,\dots,t_m) = o.$$

Set $S' = S^{(s)} = \bigoplus_{a=o}^{\infty} \Gamma(X,F^{(as)}) = \bigoplus_{a=o}^{\infty} \Gamma(X,O_X(aY))$. Since Y is ample on X, $X \cong \mathrm{Proj}(S')$. Moreover, $\sigma \in S'_1 = S_s$ and $S'/\sigma S' = S'' = (k[T_o,\dots,T_m]/I(Y))^{(s)}$.

For every $(i_o,\dots,i_m)$ such that $i_h \geqslant o$ and $i_o+\dots+i_m = s$ denote by $\sigma_{i_o,\dots,i_m} = t'^{i_o}_o \dots t'^{i_m}_m$. Then $\sigma_{i_o,\dots,i_m} \bmod \sigma S' = t_o^{i_o}\dots t_m^{i_m} \in S''_1$. Because S'' is generated by the monomials $t_o^{i_o}\dots t_m^{i_m}$ (which are homogeneous elements of degree one in S''), then S' is generated by σ and $\{\sigma_{i_o,\dots,i_m}\}$, where $(i_o,\dots,i_m)$ runs over the set of multi-indexex with the above properties.

Then construct the following homomorphism of graded k-algebras

$$\psi : S''[T] \longrightarrow S' \quad \text{(with T an indeterminate over } S''\text{)},$$

by $\psi(T) = \sigma$ and $\psi(t_o^{i_o}\dots t_m^{i_m}) = \sigma_{i_o,\dots,i_m}$. Since $\{\sigma_{i_o,\dots,i_m}\}$ satisfy (by their definition) the well known Veronese equations and since $t'_o,\dots,t'_m$ satisfy equations (9), the definition of ψ is correct. Now is is obvious that ψ is an isomorphism, whence the conclusion. Q.E.D.

Corollary. Let $Y \subset P^m$ be a subvariety as in theorem 3, and assume that Y is an effective Cartier divisor on the normal complete variety X such that $N_{Y,X} \cong O_Y(s)$, with s as in condition c) of theorem 3. Then there is a morphism $f: X \longrightarrow C(Y,s)$ such that $f(Y)$ is the intersection of $C(Y,s)$ with the hyperplane at infinity of $P^{m(s)+1}$ and f is an isomorphism in a neighbourhood of Y.

The proof of this corollary is completely analogous to the proof of corollary 1 of theorem 1 in [1].

Before passing to the last theorem let me remind a notation from [1]: $X^{s,t}_{a,b}$ denotes the cone in P^N, $N = \binom{s+a}{s}\binom{t+b}{t}$, over $i_{a,b}(P^s \times P^t)$, where $i_{a,b}: P^s \times P^t \hookrightarrow P^{N-1}$ is the Segre-Veronese embedding given by forms of bidegree (a,b).

Theorem 4. a) Assume that $Y = P^1 \times P^t$, $t \geq 2$, is an ample Cartier divisor on the normal projective variety X. If the normal sheaf $N_{Y,X}$ is isomorphic to $O(a,b)$ (necessarily $a > o$ and $b > o$) and if one has one of the following situations:

i) $a = b = 1$, or

ii) $a \geq 1$ and $b \geq 2$,

then X is isomorphic to the cone $X^{1,t}_{a,b}$.

b) Assume that $Y = P^1 \times P^1$ is an ample Cartier divisor on the normal projective variety X over k, with char(k) = o. If $N_{Y,X} \cong O(a,b)$ with $a \neq b$, $a \geq 2$ and $b \geq 2$, then X is isomorphic to the cone $X^{1,1}_{a,b}$.

Proof. According to the proof of theorem 1 in [1] let U be the smooth locus of X. In case a) Lefschetz's theorem implies that the restriction map $\varepsilon : \mathrm{Pic}(U) \longrightarrow \mathrm{Pic}(Y)$ is an isomorphism.

In case b) Lefschetz's theory implies that either ε is an isomorphism, or that $\mathrm{Pic}(U) \cong \mathbb{Z}$ and $\mathrm{Coker}(\varepsilon)$ is torsion-free. We show that in case b) this last possibility does not occur. Indeed, if L were an ample generator of Pic(U), put $L \otimes O_Y = O(\alpha, \beta)$, where $\alpha > o$ and $\beta > o$. Moreover, since $\mathrm{Coker}(\varepsilon)$ is torsion-free, α and β are relatively prime each other. Write

$$O_X(Y)/U \cong L^{\otimes r},\ r > o, \text{ and } \omega_U \cong L^{\otimes d},\ d \in \mathbb{Z}.$$

The adjunction formula $\omega_U \otimes (O_X(Y)/U) \otimes O_Y \cong \omega_Y = O(-2,-2)$ yields

$$(r+d)\cdot\alpha = -2, \quad (r+d)\cdot\beta = -2,$$

which implies $\alpha = \beta$; recalling that α and β are relatively prime positive integers we get $\alpha = \beta = 1$. This implies $a = \alpha \cdot r = \beta \cdot r = b$, a contradiction.

Therefore in both cases the map ε is an isomorphism. In other words, there are $L_1, L_2 \in \mathrm{Pic}(U)$ such that $L_1 \otimes O_Y \cong O(1,o)$ and $L_2 \otimes O_Y \cong O(o,1)$.

For every integers m,n set $F^{(m,n)} = j_*(L_1^{\otimes m} \otimes L_2^{\otimes n})$, where $j: U \hookrightarrow X$ is the canonical inclusion. Then exactly as in the proof of theorem 1 in [1] one proves:

α) $F^{(m,n)}$ is coherent and $\mathrm{depth}_{O_x}((F^{(m,n)})_x) \geq 2$ for every closed point $x \in X$.

β) $F^{(ma,mb)} = O_X(mY)$ for every integer m.

γ) For every coherent O_X-module G such that $\mathrm{depth}_{O_x}(G_x) \geq 2$ for every closed point $x \in X$, $H^1(G \otimes O_X(mY)) = o$ for every $m \ll o$.

δ) For every $m,n \in \mathbb{Z}$ there is the exact sequence

$$(10_{m,n}) \quad o \longrightarrow F^{(m-a,n-b)} \xrightarrow{\ \sigma\ } F^{(m,n)} \longrightarrow O(m,n) \longrightarrow o,$$

where $\sigma \in \Gamma(X, F^{(a,b)}) \cong \Gamma(X, O_X(Y))$ is such that $\mathrm{div}_X(\sigma) = Y$.

From δ) one gets the exact sequence

$$(11_{m,n}) \qquad H^1(F^{(m-a,n-b)}) \longrightarrow H^1(F^{(m,n)}) \longrightarrow H^1(O(m,n)).$$

Then the exact sequences $(11_{1-ma,-mb})$ and $(11_{-ma,1-mb})$ with $m \geq o$, together with γ) and the fact that in our assumptions (of cases a) or b)) $H^1(O(1-ma,-mb)) = H^1(O(-ma,1-mb)) = o$, imply (by induction on $m \leq o$) that $H^1(F^{(1,o)}) = H^1(F^{(o,1)}) = o$. In particular, we have proved that the restriction maps $\Gamma(F^{(1,o)}) \longrightarrow \Gamma(O(1,o))$ and $\Gamma(F^{(o,1)}) \longrightarrow \Gamma(O(o,1))$ are both surjective. Therefore, if T_o and T_1 (resp. $U_o, \dots, U_t$) are homogeneous coordinates on P^1 (resp. on P^t), there exist $\alpha_o, \alpha_1 \in \Gamma(F^{(1,o)})$ and $\beta_o, \dots, \beta_t \in \Gamma(F^{(o,1)})$ such that $\alpha_i/Y = T_i$ and $\beta_j/Y = U_j$, $i = o,1$ and $j = o,1,\dots,t$.

Now $\bigoplus_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} \Gamma(X, F^{(m,n)}) = \bigoplus_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} \Gamma(U, L_1^{\otimes m} \otimes L_2^{\otimes n})$ has a natural structure of a two-fold graded k-algebra. Therefore it makes sense to consider the elements $a_{i,j} = a_{i_o,i_1;j_o,\dots,j_t} = \alpha_o^{i_o}\alpha_1^{i_1}\beta_o^{j_o}\cdots\beta_t^{j_t}\in\Gamma(F^{(a,b)})\simeq\Gamma(O_X(Y))$,

where $i_h\geqslant o$, $j_e\geqslant o$, $i_o+i_1 = a$ and $j_o+\dots+j_t = b$. Then by construction the elements $\{a_{i,j}\}$ satisfy the well known Segre-Veronese equations.

Now the exact sequence $(11_{ma,mb})$ together with γ) and the fact that $H^1(O(ma,mb)) = o$ for every $m\in\mathbb{Z}$ imply that $H^1(F^{(ma,mb)}) = o$ for every $m\in\mathbb{Z}$. In particular, for every $m\geqslant o$ one gets the exact sequence

$$o\longrightarrow\Gamma(O_X((m-1)Y))\longrightarrow\Gamma(O_X(mY))\longrightarrow\Gamma(O(ma,mb))\longrightarrow o.$$

Thus, denoting by $S = \bigoplus_{m=0}^{\infty}\Gamma(O_X(mY))$ and by $S' = \bigoplus_{m=0}^{\infty}\Gamma(O(ma,mb))$ we get: $X = \mathrm{Proj}(S)$ (Y is an ample divisor on X), $\sigma\in S_1$ and $S/\sigma S = S'$. Since we have also that $a_{i,j}\in S_1$ and that $\{a_{i,j} \bmod \sigma S\}$ generate S' as a graded k-algebra, σ and $\{a_{i,j}\}$ generate S as a graded k-algebra.

Now construct the homomorphism of graded k-algebras $\psi : S'[T]\longrightarrow S$, with T an indeterminate over S', by setting $\psi(T_o^{i_o}T_1^{i_1}U_o^{j_o}\dots U_t^{j_t}) = a_{i,j}$ and $\psi(T) = \sigma$. Since $a_{i,j}$ satisfy the Segre-Veronese equations, the definition of ψ is correct. The fact that ψ is an isomorphism is now obvious, completing the proof of theorem 4. Q.E.D.

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