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A C^*-ALGEBRA APPROACH TO THE COWEN-DOUGLAS THEORY

C.Apostol and M.Martin

Let H be a separable infinite-dimensional Hilbert space over the complex field C and let $L(H)$ denote the algebra of all bounded linear operators on H.

For any open connected subset Ω of C and for any positive integer n, let $B_n(\Omega)$ denote the operators S in $L(H)$ which satisfy:

(i) $(\omega-S)(H)=H, \quad \omega\varepsilon\Omega$

(ii) $\bigvee_{\omega\varepsilon\Omega} \ker(\omega-S)=H$

(iii) $\dim\ker(\omega-S)=n, \quad \omega\varepsilon\Omega.$

Let S be a subset in $L(H)$ containing the identity operator I and an operator $T\varepsilon B_n(\Omega)$ and let $\varphi:S\to L(H)$ be a map such that $\varphi(I)=I$ and $\varphi(T)\varepsilon B_n(\Omega)$.

M.J.Cowen and R.G.Douglas [2] initiated a systematic study of the unitary orbit associated with an element of $B_n(\Omega)$ by means of complex Hermitian geometry techniques. To be more specific, they proved that $\varphi(T)$ is unitarily equivalent with T if and only if $\varphi(T)|\ker(\omega-\varphi(T))^{n+1}$ is unitarily equivalent with $T|\ker(\omega-T)^{n+1}$ for any $\omega\varepsilon\Omega$ (the corresponding unitary operators depend on ω).

Suppose S is included in $\{T\}'$, the commutant of T and $\varphi(S)\subset\{(T)\}'$; in Theorem C below we show that φ is the restriction to S of an inner automorphism in $L(H)$ if and only if $\varphi(X)|\ker(\omega-\varphi(T))^{n+1}$ is unitarily equivalent with $X|\ker(\omega-T)^{n+1}$ for any $X\varepsilon S$, $\omega\varepsilon\Omega$ (the corresponding unitary operators depend on ω only). If $S=\{T,I\}$ we recapture the result of Cowen and Douglas.

In fact we shall give a local description of the restrictions to S of inner automorphisms in $L(H)$, without the assumption $S\subset\{T\}'$ (see Theorem B).

The above results are consequences of our main Theorem A on some C^{∞}-fields of finite-dimensional C^*-algebras.

Throughout the paper S will denote a subset in $L(H)$ containing I and $T\varepsilon B_n(\Omega)$, where Ω is an open subset in C.

For any $\omega\varepsilon\Omega$, the operators R_ω, P_ω will be defined by the equations:

$$R_\omega=(\omega-T)^*[(\omega-T)(\omega-T)^*]^{-1}$$
$$P_\omega=I-R_\omega(\omega-T).$$

It is plain that P_ω is the orthogonal projection of H onto $\ker(\omega-T)$.

For each $\omega\varepsilon\Omega$ and each non-negative integer k put

$$A^k_\omega=\{P_\omega R^{*p}_\omega Y^*XR^q_\omega P_\omega : 0\leq p,q\leq k,\quad X,Y\varepsilon S\}$$

$$B^k_\omega= P_\omega R^{*p}_\omega Y^*XR^q_\omega P_\omega : \max(p,q)=k+1, \min(p,q)\leq k,\quad X,Y\varepsilon S\}$$

and denote by C^k_ω, D^k_ω the C^*-algebras generated in $L(H)$ by A^k_ω, resp. $A^k_\omega\cup B^k_\omega$.

The union $\bigcup_{k\geq 0} C^k_\omega$ is obviously a C^*-algebra which we shall denote by C^{∞}_ω.

Let $C^{\infty}(\Omega,L(H))$ denote the *-algebra of all $L(H)$-valued infinitely differentiable functions defined in Ω, with the involution defined by the equation

$$A^*(\omega)=A(\omega)^*,\quad A\varepsilon C^{\infty}(\Omega,L(H))$$

and let $C^{\infty}(\Omega)$ denote all C-valued infinitely differentiable functions defined in Ω.

We shall denote by $\Gamma(\Omega,C^k)$, (Ω,D^k), $\Gamma(\Omega,C^{\infty})$ the *-subalgebras in $C^{\infty}(\Omega,L(H))$ determined by the conditions:

$$\Gamma(\Omega,C^k)=\{A\varepsilon C^{\infty}(\Omega,L(H)) : A(\omega)\varepsilon C^k_\omega\}$$
$$\Gamma(\Omega,D^k)=\{A\varepsilon C^{\infty}(\Omega,L(H)) : A(\omega)\varepsilon D^k_\omega\}$$
$$\Gamma(\Omega,C^{\infty})=\{A\varepsilon C^{\infty}(\Omega,L(H)) : A(\omega)\varepsilon C^{\infty}_\omega\}.$$

We have $P\varepsilon\Gamma(\Omega,C^0)$, $R\varepsilon C^{\infty}(\Omega,L(H))$ where P and R are defined by the equations

$$P(\omega)=P_\omega,\quad R(\omega)=R_\omega.$$

Finally observe that the usual $\frac{\partial}{\partial\omega}$ and $\frac{\partial}{\partial\bar{\omega}}$ derivatives determine two linear maps in $C^\infty(\Omega, L(H))$. We shall denote this maps by D resp. $\bar{D}$. It is plain that we have

$$(DA)^* = \bar{D}A^*, \qquad A \in C^\infty(\Omega, L(H)).$$

THEOREM A. *There exist an open nonempty subset* $\Omega_0 \subset \Omega$ *and* $1 \le k \le n$ *with the properties:*

(i) $\Gamma(\Omega_0, C^k) = \Gamma(\Omega_0, C^\infty)$

(ii) *if* $\psi : \Gamma(\Omega_0, C^\infty) \to C^\infty(\Omega_0, L(H))$ *is an algebraic homomorphism such that*

$$\psi(P(D^p\bar{D}^qA)P) = \psi(P)(D^p\bar{D}^q\psi(A))\psi(P), \quad 0 \le p,q \le 1, \quad A \in \Gamma(\Omega_0, C^{k-1})$$

then

$$\psi(P(D^p\bar{D}^qA)P) = \psi(P)(D^p\bar{D}^q\psi(A))\psi(P), \quad 0 \le p,q, \quad A \in \Gamma(\Omega_0, C^\infty).$$

The proof of this theorem will be given after some preliminary lemmas.

1. LEMMA. *For any* ω *in* Ω *we have:*

(i) $(\omega - T)R_\omega = I$ *and* $P_\omega R_\omega = 0$

(ii) $\ker(\omega - T)^{k+1} = \bigvee_{j=0}^{k} R_\omega^j P_\omega(H)$ *for each* $0 \le k$

(iii) $H = \bigvee_{j \ge 0} R_\omega^j P_\omega(H)$.

PROOF. The relations (i) are obvious. Clearly, (ii) will easily follow if we prove that

$$\ker(\omega - T)^{k+1} = P_\omega(H) \oplus R_\omega(\ker(\omega - T)^k).$$

Since $(\omega - T)(\ker(\omega - T)^{k+1}) \subset \ker(\omega - T)^k$ and $R_\omega(\omega - T) = I - P_\omega$ we have $\ker(\omega - T)^{k+1} \ominus P_\omega(H) \subset R_\omega(\ker(\omega - T)^k)$ hence

$$\ker(\omega - T)^{k+1} \subset P_\omega(H) \oplus R_\omega(\ker(\omega - T)^k)$$

and the reverse inclusion is obvious.
Using [1], Lemma 1.7 we know that we have

$$H = \bigvee_{\lambda \in \Omega} \ker(\lambda - T) = \bigvee_{k \ge 0} \ker(\omega - T)^k$$

thus (iii) becomes a consequence of (ii).

2. LEMMA. *The following relations hold:*

$$DR=-R^2,\ DR^*=R^*RP \ \text{ and } \ DP=-RP.$$

The proof is obvious, therefore we omit it.

As easy corollary of Lemma 2 is the following

3. LEMMA. *If* $P D\Gamma(\Omega_o, C^k) \subset \Gamma(\Omega_o, C^k)$ *for some open nonempty subset* $\Omega_o \subset \Omega$ *and* $1 \leqslant k$, *then*

$$\Gamma(\Omega_o, C^k) = \Gamma(\Omega_o, C^\infty).$$

4. LEMMA. *Let* $V, W \in C^\infty(\Omega, L(H))$ *be given such that* $VWV=V$. *Then we have:*

$$DV = V(DF) + (DE)V - V(DW)V$$
$$\bar{D}V = V(\bar{D}F) + (\bar{D}E)V - V(\bar{D}W)V$$

where $F=WV$, $E=VW$.

PROOF. Since $VF=EV=V$ it follows that

$$V(DF)+(DE)V = V(DW)V + VW(DV) + (DE)V =$$
$$= V(DW)V + E(DV) + (DE)V = V(DW)V + D(EV) = V(DW)V + DV.$$

The rest of the proof is similar.

5. LEMMA. *Let* $E \in \Gamma(\Omega, C^\infty)$ *be a selfadjoint projection such that* $P(DE)=0$ *and* $E\Gamma(\Omega, C^1)(P-E) = \{0\}$. *Then we have* $E\Gamma(\Omega, C^\infty)(P-E) = \{0\}$.

PROOF. Let $A \in \Gamma(\Omega, C^\infty)$ be such that $EA(P-E)=0$. Since we have $0 = D(EA(P-E)) = (DE)A(P-E) + E(DA)(P-E) + EA(D(P-E))$ and by our assumption and Lemma 2 $E(DE) = P(D(P-E)) = 0$, it follows $E(DA)(P-E)=0$ and analogously $E(\bar{D}A)(P-E)=0$. Because $E(\omega)A^1_\omega(P_\omega - E(\omega)) = \{0\}$ applying again Lemma 2 we derive easily $E\Gamma(\Omega, C^\infty)(P-E) = \{0\}$.

Our next lemma is a restatement of [2] , Lemma 3.4.

6. LEMMA. *Let* $A \in C^\infty(\Omega, L(H))$ *be such that* $A=A^*=PA$. *Then there exist an open nonempty subset* $\Omega_o \subset \Omega$, *and two collections* $\{P_\alpha : 1 \le \alpha \le m\} \subset C^\infty(\Omega_o, L(H))$, $\{\mu_\alpha : 1 \le \alpha \le m\} \subset C^\infty(\Omega_o)$ *with the properties:*

(i) $\{P_\alpha(\omega) : 1 \le \alpha \le m\}$ *are selfadjoint pairwise orthogonal projections in the* C^**-algebra generated in* $L(H)$ *by* $\{P_\omega, A(\omega)\}$.

(ii) $P = \sum_\alpha P_\alpha$ *and* $A = \sum_\alpha \mu_\alpha P_\alpha$ *in* $C^\infty(\Omega_o, L(H))$.

THE PROOF OF THEOREM A. By a repeated use of Lemma 6 we can find an open connected nonempty subset $\Omega_o \subset \Omega$, an integer $\min(2,n) \le k \le n$ and a pairwise orthogonal decomposition of P:

$$\{P_\alpha : 1\le\alpha\le m\}\subset\Gamma(\Omega_o, \mathcal{C}^{k-1})$$

where $P_\alpha(\omega)$ is a selfadjoint projection which is minimal in $\mathcal{C}^k_\omega$, $\omega\in\Omega_o$. Moreover, arguing as in [4], Ch.I, §11, we may assume that

$$P_\alpha\Gamma(\Omega_o,\mathcal{D}^{k-1})P_\beta = C^\infty(\Omega_o)U_{\alpha\beta}$$

where $U_{\alpha\beta}$ enjoys the properties:

$$U_{\alpha\alpha}=P_\alpha \quad, U^*_{\alpha\beta}=U_{\beta\alpha}=P_\beta U_{\beta\alpha} \quad, (U_{\alpha\beta}U^*_{\alpha\beta})^2=U_{\alpha\beta}U^*_{\alpha\beta}.$$

It is clear that either $U_{\alpha\beta}=0$ or $U_{\alpha\beta}U^*_{\alpha\beta}=P_\alpha$, $U^*_{\alpha\beta}U_{\alpha\beta}=P_\beta$.

Let $1\le\alpha,\beta\le m$ be given and suppose $U_{\alpha\beta}=0$. Denote by E the sum of all P_γ such that $U_{\alpha\gamma}\ne 0$. Since obviously E is a central projection in $\Gamma(\Omega_o, \mathcal{D}^{k-1})$ and under our assumption we have $k\ge 2$, therefore $\Gamma(\Omega_o,\mathcal{D}^{k-1})\supset(\Omega_o,\mathcal{C}^1)$ and consequently

$$E\Gamma(\Omega_o,\mathcal{C}^1)(P-E)=\{0\}.$$

But we also have

$$E(DE)=E(DE)E=0, \quad (P-E)(D(P-E))=(P-E)(D(P-E))(P-E)=0$$

whence it follows $P(DE)=0$. Now applying Lemma 5 we derive $P_\alpha\Gamma(\Omega_o,\mathcal{C}^\infty)P_\beta=\{0\}$ and in particular

$$P_\alpha\Gamma(\Omega_o,\mathcal{C}^k)P_\beta=C^\infty(\Omega_o)U_{\alpha\beta}.$$

Remark that if $U_{\alpha\beta}\ne 0$, the last relation holds valid in view of the minimality of $P_\alpha(\omega)$ and $P_\beta(\omega)$ in $\mathcal{C}^k_\omega$, $\omega\in\Omega_o$. Because we have $\Gamma(\Omega_o,\mathcal{C}^k)=\sum_{\alpha,\beta=1}^{m} P_\alpha\Gamma(\Omega_o,\mathcal{C}^k)P_\beta$ we deduce

(*) $$\Gamma(\Omega_o,\mathcal{C}^k)=\sum_{\alpha,\beta=1}^{m} C^\infty(\Omega_o)U_{\alpha\beta}.$$

To conclude the proof it sufficies to prove that we have

(1) $P(DU_{\alpha\beta})\in\Gamma(\Omega_o,\mathcal{C}^k)$

$\psi(P(DU_{\alpha\beta}))=\psi(P)(D(\psi(U_{\alpha\beta})))$

$\psi((\bar{D}U_{\alpha\beta})P)=(\bar{D}(\psi(U_{\alpha\beta})))\psi(P)$, $\quad 1\le\alpha,\beta\le m$.

Indeed (i) of our theorem will follow by Lemma 3 from (1) and (*), whence (ii) will be a consequence of (2) and (*).

Let us put $G=\{P_\alpha A(D^pD^qB)CP_\beta : 0\le p,q\le 1, A,B,C\in\Gamma(\Omega_o,\mathcal{C}^{k-1})\}$. We leave to the reader as an exercise to show that, eventually decreasing Ω_o, we may suppose that any $U_{\alpha\beta}\ne 0$ is a finite product of $U_{\alpha',\beta'}$'s belonging to G. A hint for this exercise is that if

$\omega_o \varepsilon \Omega_o$ is given then $U_{\alpha\beta}(\omega_o)$ is a finite product of elements belonging to G, valuated at ω_o. Thus we are allowed to prove (1) and (2) assuming $U_{\alpha\beta} \varepsilon G$.

Let $U_{\alpha\beta} = P_\alpha A(DB)CP_\beta$ be given. We derive easily that

$$(\bar{D}U_{\alpha\beta})P,\ P(DU^*_{\alpha\beta}) \varepsilon \Gamma(\Omega_o, C^k)$$

and

$$\psi((\bar{D}U_{\alpha\beta})P) = (\bar{D}(\psi(U_{\alpha\beta})))\psi(P)$$
$$\psi(P(DU^*_{\alpha\beta})) = \psi(P)(D(\psi(U^*_{\alpha\beta}))).$$

Putting in Lemma 4 $V=U_{\alpha\beta}$, $W=U^*_{\alpha\beta}$ and $E=P_\alpha$, $F=P_\beta$ we have

$$P(DU_{\alpha\beta}) = U_{\alpha\beta}(DP_\beta) + P(DP_\alpha)U_{\alpha\beta} - U_{\alpha\beta}(DU^*_{\alpha\beta})U_{\alpha\beta} \varepsilon \Gamma(\Omega_o, C^k).$$

If we put in Lemma 4 $V=\psi(U_{\alpha\beta})$, $W=\psi(U^*_{\alpha\beta})$ and $E=\psi(P_\alpha)$, $F=\psi(P_\beta)$ then, under our assumptions, we obtain

$$\psi(P)(D(\psi(U_{\alpha\beta}))) = \psi(U_{\alpha\beta})\psi(DP_\beta) + \psi(P)\psi(DP_\beta)\psi(U_{\alpha\beta}) - \psi(U_{\alpha\beta})\psi(DU^*_{\alpha\beta})\psi(U_{\alpha\beta}) = \psi(P(DU_{\alpha\beta}))$$

and (1) and (2) are proved. We proceed analogously if $U_{\alpha\beta} = P_\alpha A(\bar{D}B)CP_\beta$.

THEOREM B. *Let* $\varphi : S \to L(H)$ *be a map such that* $\varphi(I)=I$, $\tilde{T} = \varphi(T) \varepsilon B_n(\Omega)$. *The following conditions are equivalent:*

(i) φ *is the restriction to* S *of an inner automorphism in* $L(H)$.

(ii) *there exists a partial isometry* U_ω *such that*

$$U^*_\omega U_\omega = P_\omega\ ,\ U_\omega U^*_\omega = \tilde{P}_\omega$$

and

$$\tilde{P}_\omega \tilde{R}^{*p}_\omega \varphi(Y)^* \varphi(X) \tilde{R}^q_\omega \tilde{P}_\omega = U_\omega R^{*p}_\omega Y^* X R^q_\omega U^*_\omega$$

for any $\omega \varepsilon \Omega$, $0 \le p,q \le n$, $X,Y \varepsilon S$, *where* $\sim$*-symbols are associated with* $\tilde{T}$.

PROOF. It is clear that (i) implies (ii). Let $\Omega_o \subset \Omega$, $1 \le k \le n$ be produced by Theorem A. If we define ψ by the equation

$$\psi(A)(\omega) = U_\omega A(\omega) U^*_\omega,\ \omega \varepsilon \Omega_o,\ A \varepsilon \Gamma(\Omega_o, C^\infty)$$

under our assumption (ii) we have $\psi(A) \varepsilon \Gamma(\Omega_o, \tilde{C}^\infty)$ whenever $A = PR^{*p} Y^* X R^q P$, $0 \le p,q \le n$, $X,Y \varepsilon S$. Applying Theorem A and Lemma 2 we deduce $\psi(A) \varepsilon \Gamma(\Omega_o, \tilde{C}^\infty)$ for any $A \varepsilon \Gamma(\Omega_o, C^\infty)$ and $A \varepsilon \Gamma(\Omega_o, C^\infty)$ and

$$(*) \qquad P_\omega R^{*p}_\omega \varphi(Y)^* \varphi(X) R^q_\omega P_\omega = U_\omega R^{*p}_\omega Y^* X R^q_\omega U^*_\omega$$

for any $0 \leq p,q$, $X,Y \in S$.

Let $\omega_o \in \Omega_o$ be given. Since $I \in S$, Lemma 1 implies

$$H = \bigvee_{X \in S} \bigvee_{j \geq 0} XR^j_{\omega_o} P_{\omega_o} H = \bigvee_{j \geq 0} R^j_{\omega_o} P_{\omega_o} H.$$

Let $U \in L(H)$ be defined by the equation

$$U(XR^j_{\omega_o} P_{\omega_o} x) = \varphi(X) \tilde{R}^j_{\omega_o} \tilde{P}_{\omega_o} U_{\omega_o} x$$

for any $X \in S$, $0 \leq j$, $x \in H$. Using (*) we derive that U is a well defined unitary operator and $UX = \varphi(X)U$, $X \in S$.

THEOREM C. *Suppose* $S \subset \{T\}'$, $\varphi(S) \subset \{\tilde{T}\}'$. *The following conditions are equivalent:*

(i) φ *is the restriction to* S *of an inner automorphism in* $L(H)$

(ii) *there exists a unitary operator* $V_\omega : \ker(\omega - T)^{n+1} \to \ker(\omega - \tilde{T})^{n+1}$ *such that*

$$\varphi(X) | \ker(\omega - \tilde{T})^{n+1} = V_\omega X V^*_\omega | \ker(\omega - \tilde{T})^{n+1}$$

for any $\omega \in \Omega$, $X \in S$.

PROOF. It is sufficient to remark that under our assumptions, the present condition (ii) is equivalent with (ii) in Theorem B, where $U_\omega = V_\omega P_\omega$.

REFERENCES

1. Apostol, C.: The correction by compact perturbation of the singular behavior of operators, *Rev. Roumaine Math. Pures Appl.* 21(1976), 155-175.

2. Cowen, M.J.; Douglas, R.G.: Complex geometry and operator theory, *Acta Math.* 141(1978), 187-261.

3. Dixmier, J.: *Les C*-algebras et leurs representations*, Gauthier-Villars, Paris, 1969.

4. Takesaki, M.: *Theory of Operator Algebras I*, Springer-Verlag, New York, 1979.

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