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THE FENCHEL-ROCKAFELLAR DUALITY THEORY FOR MATHEMATICAL PROGRAMMING IN ORDER COMPLETE VECTOR LATTICES AND APPLICATIONS

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Constantin ZĂLINESCU

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Constantin ZĂLINESCU*)

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*) *Faculty of Mathematics, University of Iași, 6600 Iași, Romania*

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0. Introduction

In the last few years one notes the attempt to generalize the duality theory for mathematical programming when the objective function takes values in an order complete vector lattice. Zowe [13] extended the Fenchel duality theorem for that case, and Rosinger [7] treated the same problem for the case when the objective function is not convex. In this paper we present the Fenchel-Rockafellar theory for the same case. Note that the objective function is not supposed to be convex, so that we reobtain the results in [7]. The approach, parallel to a certain extent to that of Rockafellar [6], is based on the notion of conjugate set in [10] and the subdifferentiability criterion of Zowe [14]. We apply the results to calculate conjugate operators and subdifferentials; so we reobtain the results of Kutateladze [3] concerning conjugate operators and subdifferentials, some of them in more general conditions. We also give a Kuhn-Tucker theorem which generalizes that one in [13]. Using a theorem of Ursescu [12], concerning multifunctions with closed convex graph, we establish a useful criterion for the continuous version, which represents a generalization of a similar result of Robinson [5].

1. Preliminaries

Throughout this paper X, Y, Z denote real vector spaces and Z is also an order complete vector lattice, i.e., Z is an order vector space (order symbol $\leq$), $\inf(u,v) = u \wedge v$ and $\sup(u,v) = u \vee v$ exist for all $u,v \in Z$, and for each nonempty $A \subset Z$ such that A is order bounded from below in Z, infA exists. The set $C = \{z : z \geqslant 0\}$ of pozitive elements of Z is called the <u>positive cone</u> of Z. By the definition of an order vector space, $C \cap -C = \{0\}$, $C + C = C$, $\lambda C \subset C$ for all $\lambda \in R_+$ ($R_+ = \{\lambda \in R : \lambda \geqslant 0\}$.

If $A \subset X$, ${}^iA$, A^i, <u>co</u>A denote the intrinsic core (relative algebraic interior), the core (the algebraic interior) and the convex hull of A, respectively. If X is a topological vector space, then <u>int</u>A denotes the topological interior (see [4]). A subset $A \subset X$ is said to be <u>lineally closed</u> if each line meets A in a closed subset of the line. One has (see [9]);

<u>Proposition</u> 1.1. The positive cone of an order complete vector lattice is lineally closed.

Let $f : D(f) \subset X \to Y$ be an operator, and $A \subset X$. Then $f(A) = \{f(x) : x \in A \cap D(f)\}$. In the sequel we shall need the following.

<u>Lemma</u> 1.1. Let X, Y be a linear vector spaces, $A, B \subset X$, $D \subset Y$ and $S : X \to Y$ a linear operator. Then

(i) $\underline{co}(A \times D) = (\underline{co}A) \times (\underline{co}D)$,

(ii) $S(\underline{co}A) = \underline{co}S(A)$,

(iii) $\underline{co}(A+B) = \underline{co}A + \underline{co}B$.

Proof. (i) It is clear that $A \times D \subset \underline{co}A \times \underline{co}D$. Let $x \in \underline{co}A$ and $y \in \underline{co}D$. Then $\exists \lambda_i \in (0,1]$, $x_i \in A$, $i=1,\dots,n$ such that $x = \sum_{i=1}^{n} \lambda_i x_i$ and $\sum_{i=1}^{n} \lambda_i = 1$, $\exists \mu_j \in (0,1]$, $y_j \in D$, $j=1,\dots,m$ such that $y = \sum_{j=1}^{m} \mu_j y_j$ and $\sum_{j=1}^{m} \mu_j = 1$. Then

$$x = \sum_{i=1}^{n} \lambda_i \Big(\sum_{j=1}^{m} \mu_j \Big) x_i = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j x_i ,$$

$$y = \sum_{j=1}^{m} \mu_j \Big(\sum_{i=1}^{n} \lambda_i \Big) y_j = \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i \mu_j y_j = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j y_j .$$

Hence

$$(x,y) = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\,\mu_j x_i,\ \sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\,\mu_j y_i\right) =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\,\mu_j (x_i,y_j).$$

But $\lambda_i \cdot \mu_j > 0$ and $\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\,\mu_j = 1$, $(x_i,y_j) \in A \times D$, so that $(x,y) \in \underline{co}(A \times D)$.

(ii) it is obvious.

(iii) Let $S: X \times X \to X$, $S(x,y) = x+y$, and $A,B \subset X$; then $S(A \times B) = A + B$.

Applying (i) and (ii) one obtains $\underline{co}(A + B) = \underline{co}\,A + \underline{co}B$.

Lemma 1.2 Let $A_1, A_2, \ldots, A_n$ be convex subsets of X and $M = \{(x - x_1, \ldots, x - x_n) : x \in X, x_k \in A_k,\ 1 \le k \le n\} \subset X^n$. Then $(0,0,\ldots,0) \in {}^{i}M$ if and only if

$$0 \in {}^{i}\left(\bigcap_{i=1}^{k-1} A_i - A_k\right) \text{ for all } k,\quad 2 \le k \le n. \qquad (1.1)$$

Proof. Note that if A is convex then $0 \in {}^{i}A$ if and only if $\forall x \in A\ \exists \lambda_x > 0: -\lambda_x x \in A$; if $(0,0,\ldots,0) \in M$ or (1.1) holds then $\exists \bar{x} \in \bigcap_{i=1}^{n} A_i$.

Suppose that $(0,\ldots,0) \in {}^{i}M$. Let $x \in \bigcap_{i=1}^{k-1} A_i, x_k \in A_k$; then $(-x,\ldots,-x,-x_k,-\bar{x},\ldots,-\bar{x}) \in M$. It follows that $\exists \lambda > 0$ such that $(\lambda x,\ldots,\lambda x,\lambda x_k,\lambda\bar{x},\ldots,\lambda\bar{x}) \in M$, i.e., $\exists y \in X, x_i' \in A_i$ such that

$$\lambda x = y - x_i' \quad \forall i,\ 1 \le i \le k-1,$$

$$\lambda x_k = y - x_k',$$

$$\lambda \bar{x} = y - x_i' \quad \forall i,\ k+1 \le i \le n.$$

It follows that $x_1' = x_2' = \ldots = x_{k-1}' = x' \in \bigcap_{i=1}^{k-1} A_i$, so that

$\lambda x = y - x'$, $\lambda x_k = y - x'_k$, which imply $-\lambda(x - x_k) =$

$= x' - x'_k \in \bigcap_{i=1}^{k-1} A_i - A_k$. Therefore, $0 \in {}^i(\bigcap_{i=1}^{k-1} A_i - A_k)$ for all k, $2 \leq k \leq n$.

Conversely, suppose that (1.1) holds. Note that it is sufficient to show that for $x_i \in A_i$, $1 \leq i \leq n$ (hence $(-x_1, \ldots, -x_n) \in M$) $\exists \lambda > 0$ such that $\lambda(x_1, \ldots, x_n) \in M$. We shall show, by induction, that $\forall k$, $2 \leq k \leq n$

$$(P_k) \ \exists \lambda > 0 \ \forall i, \ 1 \leq i \leq k \ \exists x'_i \in A_i : \ \lambda x_1 + x'_1 = \ldots = \lambda x_k + x'_k.$$

If (P_n) is true, denoting by x the common value, we have

$$\lambda(x_1, \ldots, x_n) = (x - x'_1, \ldots, x - x'_n) \in M.$$

so that $(0, \ldots, 0) \in {}^i M$.

Let $k = 2$. We have $x_2 - x_1 \in A_2 - A_1$. Since $0 \in {}^i(A_2 - A_1)$ it follows that $\exists \lambda > 0, x_1 \in A_1, x_2 \in A_2$ such that $-\lambda(x_2 - x_1) = x'_2 - x'_1$, so that $\lambda x_1 + x'_1 = \lambda x_2 + x_2$. Hence (P_2) is true. Suppose that (P_k) is true and show that (P_{k+1}) is also true. Let $\lambda > 0$, $x'_i \in A_i$, $1 \leq i \leq k$ such that

$$\lambda x_1 + x'_1 = \ldots = \lambda x_k + x'_k = x.$$

Then $\frac{1}{\lambda + 1} \cdot x \in \bigcap_{i=1}^{k} A_i$; hence $\frac{1}{\lambda + 1} x - x_{k+1} \in \bigcap_{i=1}^{k} A_i - A_{k+1}$.

Since $0 \in {}^i(\bigcap_{i=1}^{k} A_i - A_{k+1})$, it follows $\exists \lambda' > 0, x' \in \bigcap_{i=1}^{k} A_i$, $x'_{k+1} \in A_{k+1}$ such that $-\lambda'(\frac{1}{\lambda + 1} x - x_{k+1}) = x' - x'_{k+1}$. Hence

$$\lambda' x_{k+1} + x'_{k+1} = \frac{\lambda'}{\lambda + 1} x + x' = \frac{\lambda'}{\lambda + 1}(\lambda x_i + x'_i) + x' \text{ for } 1 \leq i \leq k,$$

so that

$$\frac{\lambda\lambda'}{\lambda + 1} x_i + \underbrace{\frac{\lambda'}{\lambda + 1} x'_i}_{+x'} = \frac{\lambda\lambda'}{\lambda + 1} x_{k+1} + \frac{\lambda'}{\lambda + 1} x_{k+1} + x'_{k+1}, \quad 1 \leq i \leq k.$$

Dividing by $1 + \frac{\lambda'}{\lambda + 1}$, taking into account that A_i are convex sets, we see, that (P_{k+1}) is true, so that (1.1) holds.

Definition 1.1. Let $A_i \subset X$, $1 \leq i \leq n$ be convex sets. We say that $A_1, \ldots, A_n$ are in general position if there exists a permutation $\{i_1, \ldots, i_n\}$ of the set $\{1, 2, \ldots, n\}$ such that, for the corresponding rearrangement, (1.1) holds.

Remark 1.1. The above definition for A_i convex cones can be found in [3] and [8].

Remark 1.2. From Lemma 1.2 we see that $A_1, \ldots, A_n$ are in general position if and only if (1.1) holds for every rearrangement of the sets $A_1, \ldots, A_n$.

If $A \subset X$, let $C(A, \bar{x}) = \bigcup_{\lambda \geqslant 0} \lambda(A - \bar{x})$ for $\bar{x} \in A$ and $H(A) = \{(x, t) : t \geqslant 0, x \in tA\}$.

Remark 1.3. Let $A_1, \ldots, A_n$ be convex sets. Then $A_1, \ldots, A_n$ are in general position if and only if $C(A_1, \bar{x}), \ldots, C(A_n, \bar{x})$ are in general position for some (every) $\bar{x} \in \bigcap_{i=1}^{n} A_i$. If $H(A_1), \ldots, H(A_n)$ are in general position then $A_1, \ldots, A_n$ are in general position.

Remark 1.4. If $A_1 \cap \bigcap_{k=2}^{n} A_k^i \neq \emptyset$, then $A_1, \ldots, A_n$ are in general position.

Lemma 1.3. Let Z be a complete vector lattice. If the set $\{\lambda z : \lambda \in R_+\}$ is upper (below) bounded then $z \leq 0$ $(z \geqslant 0)$.

Proof. Let $\lambda z \leq z_0$ for all $\lambda \in R_+$. Then $\forall \lambda \in R_+$, $z_0 - \lambda z \in C$. Let $\mu \in (0,1)$; $\mu z_0 - (1 - \mu) z \in C$. Since C is lineally closed it follows that for $\mu = 0$ one obtains an element of C, i.e., $-z \in C$.

We denote by P_X the operator $P_X : X \times Y \to X$, $P_X(x,y) = x$ and by $P_X D$ the set $P_X(D)$, where $D \subset X \times Y$.

As in the case $Z = R$ we have:

Lemma 1.4. (i) Let $f : D \subset X \times Y \to Z$ be an operator. For $x \in P_X D$ let $D_x = \{y : (x,y) \in D\}$ and for $y \in P_Y D$ let $D_y = \{x : (x,y) \in D\}$. Then

$$\inf_{x\in P_X D}\ \inf_{y\in D_x} f(x,y) = \inf_{y\in P_Y D}\ \inf_{x\in D_y} f(x,y) = \inf_{(x,y)\in D} f(x,y),$$

every time when one of them exists.

(ii) Let $A,B\subset Z$. Then

$$\inf(A+B) = \inf A + \inf B .$$

when $\inf A$ and $\inf B$, of $\inf(A+B)$ exist.

Let $f\colon D(f)\subset X\to Y$ be an operator and Y an order vector space with the positive cone P. We say that f is <u>convex</u> if $D(f)$ is a convex set and $f(\lambda x + (1-\lambda)y)\le \lambda f(x) + (1-\lambda)f(y)$ for all $x,y\in D(f)$ and $\lambda\in(0,1)$; if in addition $D(f)$ is a cone and $f(x+y)\le f(x)+f(y)$ for all $x, y\in D(f)$, f is <u>sublinear</u>. Let $A\subset X\times Z$; we say that A is a <u>set of epigraph type</u> if $(x,z)\in A, z\le z'$ $\Rightarrow (x,z')\in A$.

<u>Theorem</u> 1.1. (i) Let $f\colon D(f)\subset X\to Z$ be an operator. Then f is convex if and only if

$$\underline{\text{epi}}\ f = \{(x,z) : x\in D(f), z\in Z, f(x)\le z\}\subset X\times Z$$

is a convex set.

(ii) Let $A\subset X\times Z$ be a convex set of epigraph type. If $\forall x\in P_X A$ $\exists \inf\{z\colon (x,z)\in A\}$ then the operator

$$\varphi_A\colon P_X A\to Z,\ \varphi_A(x) = \inf\{z\colon (x,z)\in A\}$$

is convex. Furthermore, φ_A is the greatest convex operator φ with the property $A\subset \underline{\text{epi}}\ \varphi$.

Proof. (i) The proof is the same as in the case $Z = R$.

(ii) Since $A\subset X\times Z$ is a convex set it follows from Lemma 1.1 (ii) that $P_X A$ is convex. Let $x_1,x_2\in P_X A$, $\lambda\in(0,1)$ be fixed. Let $(x_1,z_1)\in A$ and $(x_2,z_2)\in A$; then $(\lambda x_1 + (1-\lambda)x_2,$ $\lambda z_1 + (1-\lambda)z_2)\in A$, so that

$$\varphi_A(\lambda x_1 + (1-\lambda)x_2)\le \lambda z_1 + (1-\lambda)z_2 .$$

Fix z_2 and take z_1 arbitrarily such that $(x_1, z_1) \in A$. Then

$$\forall z_1 \in Z, (x_1, z_1) \in A: (\varphi_A(\lambda x_1 + (1-\lambda)x_2) - (1-\lambda)z_2)/\lambda \leq z_1,$$

and consequently,

$$(\varphi_A(\lambda x_1 + (1-\lambda)x_2) - (1-\lambda)z_2)/\lambda \leq \varphi_A(x_1),$$

or equivalently

$$(\varphi_A(\lambda x_1 + (1-\lambda)x_2) - \lambda\,\varphi_A(x_1))/(1-\lambda) \leq z_2.$$

Taking now z_2 arbitrarily such that $(x_2, z_2) \in A$, it follows that

$$\varphi_A(\lambda x_1 + (1-\lambda)x_2) \leq \lambda \varphi_A(x_1) + (1-\lambda)\,\varphi_A(x_2).$$

Hence φ_A is a convex operator. Since $(x,z) \in A$ implies $\varphi_A(x) \leq z$, we have $A \subset \underline{\text{epi}}\ \varphi_A$. Let $g\colon\ P_X A \to Z$ be such that $A \subset \underline{\text{epi}}\ g$; then $(x,z) \in A \Rightarrow (x,z) \in \underline{\text{epi}}\ g \Rightarrow g(x) \leq z \Rightarrow g(x) \leq \inf\{z\colon (x,z) \in A\} = \varphi_A(x)$.

The proof is complete.

Theorem 1.2. Let $A \subset X \times Z$ be a convex set of epigraph type. Suppose that there exists $x_o \in {}^i(P_X A)$ such that $\inf\{z\colon (x_o, z) \in A\}$ exists. Then $\inf\{z\colon (x,z) \in A\}$ exists for all $x \in P_X A$.

Proof. Since A is convex it follows that $P_X A$ is also convex. Let $x \in P_X A$; since $x_o \in {}^i(P_X A)$ then $\exists t_o > 0\ \ \forall t \in [-t_o, 1]\colon$
$(1-t)x_o + tx \in P_X A$. Let $x_1 = (1+t_o)x_o - t_o x$; then $x_o = \frac{1}{1+t_o} x_1 + \frac{t_o}{1+t_o} \cdot x$. Since $x_1 \in P_X A$, there is $z_1 \in Z$ such that $(x_1, z_1) \in A$. Then for every $z \in Z$ such that $(x,z) \in A$, we have

$$\left(\frac{1}{1+t_o} x_1 + \frac{t_o}{1+t_o} x, \frac{1}{1+t_o} z_1 + \frac{t_o}{1+t_o} z\right) =$$

$$= \left(x_o, \frac{1}{1+t_o} z_1 + \frac{t_o}{1+t_o} z\right) \in A.$$

Hence

$$\frac{1}{1+t_o} z_1 + \frac{t_o}{1+t_o} \cdot z \geqslant \inf\{z : (x_o, z) \in A\},$$

so that

$$z \geqslant \frac{1 + t_o}{t_o} \inf \{ z: (x_o, z) \in A \} - \frac{1}{t_o} \cdot z_1 .$$

Consequently $\inf \{ z: (x,z) \in A \}$ exists. The proof is complete.

Corollary 1.1. Let $f: D(f) \subset X \rightarrow Z$ be an operator. If there exists $x_o \in {}^i(\mathrm{co}D(f))$ such that $\inf \{ z: (x_o,z) \in \mathrm{co}(\mathrm{epi}\ f) \}$ exists, then $\inf \{ z: (x,z) \in \mathrm{co}(\mathrm{epi}\ f) \}$ exists for every $x \in \mathrm{co}D(f)$.

Proof. Take $A = \mathrm{co}(\mathrm{epi}\ f)$; then $\mathrm{co}D(f) = P_X A$. Hence $x_o \in {}^i(P_X A)$, so that Theorem 1.2. applies.

When $\inf \{ z: (x,z) \in \mathrm{co}(\mathrm{epi}\ f) \}$ exists for every $x \in \mathrm{co}D(f)$, we call the operator $\varphi_{\mathrm{co}(\mathrm{epi}\ f)}$ the convex hull of f and we denote it by cof. Let X,Y be vector spaces; L(X,Y) denote the space of linear operators from X into Y.

Theorem 1.3. [14]. X, Z be vector spaces, Z be a complete vector lattice and $f: D(f) \subset X \rightarrow Z$ a convex operator. If $x_o \in {}^iD(f)$ then there exists $T \in L(X,Z)$ such that

$$Tx - Tx_o \leq f(x) - f(x_o) \text{ for all } x \in D(f). \qquad (1.2)$$

Definition 1.2. Let $f: D(f) \subset X \rightarrow Z$ be an operator and $x_o \in D(f)$. The set of all linear operators satisfying (1.2) is the subdifferential of f at x_o, denoted by $\partial f(x_o)$.

2. Duality Theory

Let $A \subset X \times Z$ be a set of epigraph type. According to Stoer and Witzgall [10, Def.4.65] we introduce the conjugate of the set A as follows:

$$A^c = \{ (T, z'): T \in L(X,Z),\ z' \in Z,\ \forall (x,z) \in A: z + z' \geqslant Tx \}.$$

Proposition 2.1.

(i) A^c is a convex set of epigraph type,

(ii) $A^c = (\mathrm{co}A)^c$,

(iii) If $A^c \neq \emptyset$ then $\inf \{ z: (x,z) \in \mathrm{co}A \}$ exists for all $x \in P_X(\mathrm{co}\ A)$ and

$$\inf\{z: (x,z)\in \underline{coA}\} \geqslant \sup\{Tx - z' : (T,z')\in A^c\}, \quad \forall x \in \underline{co}P_X A. \tag{2.1}$$

Proof. (i) and (ii) are obvious. (iii) Let $(T,z')\in A^c$; then $\forall (x,z)\in \underline{co}\, A$: $z + z' \geqslant Tx$. Fix $x \in \underline{co}P_X A$; then $\forall z\in Z, (x,z)\in \underline{coA} \Rightarrow z \geqslant Tx - z'$, so that $\inf\{z: (x,z)\in \underline{coA}\}$ exists and $\inf\{z: (x,z)\in \underline{coA}\} \geqslant Tx - z'$. Taking the supremum in the right hand side with $(T,z')\in A^c$ we obtain (2.1).

Theorem 2.1. Let $A\subset X\times Z$ be a set of epigraph type and $x_o\in {}^i(P_X(\underline{coA}))$. If $\inf\{z: (x_o,z)\in \underline{coA}\}$ exists, then

$$\inf\{z: (x_o,z)\in \underline{coA}\} = \max\{Tx_o - z' : (T,z')\in A^c\}. \tag{2.2}$$

Proof. Theorem 1.2 assures that there exists the operator $\varphi_{\underline{coA}}: P_X(\underline{coA})\to Z$, given by $\varphi_{\underline{coA}}(x) = \inf\{z: (x,z)\in \underline{coA}\}$. From Theorem 1.1 we have that $\varphi_{\underline{coA}}$ is a convex operator. Since $x_o\in {}^i(P_X(\underline{coA})) = {}^i(D(\varphi_{\underline{coA}}))$, we can apply Theorem 1.4 for the convex operator $\varphi_{\underline{coA}}$. Hence $\exists\, T_o\in L(X,Z)$ such that

$$T_o x - T_o x_o \leq \varphi_{\underline{coA}}(x) - \varphi_{\underline{coA}}(x_o) \quad \text{for all } x\in D(\varphi_{\underline{coA}}).$$

Hence

$$z + T_o x_o - \varphi_{\underline{coA}}(x_o) \geqslant T_o x \quad \text{for all } (x,z)\in \underline{epi}\,\varphi_{\underline{coA}} \supset A,$$

so that

$$z + T_o x_o - \varphi_{\underline{coA}}(x_o) \geqslant T_o x \text{ for all } (x,z)\in A.$$

It follows that $(T_o, T_o x_o - \varphi_{\underline{coA}}(x_o))\in A^c$, so that

$$\inf\{z: (x_o,z)\in \underline{coA}\} = \varphi_{\underline{coA}}(x_o) \geqslant \sup\{Tx_o - z' : (T,z')\in A^c\}$$

$$\geqslant T_o x_o - (T_o x_o - \varphi_{\underline{coA}}(x_o)) = \varphi_{\underline{coA}}(x_o).$$

Hence (2.2) holds, which completes the proof.

Definition 2.1. Let $f: D(f)\subset X\to Z$ be an operator. The conjugate operator of f is the operator f^c: $D(f^c) \subset L(X,Z) \to Z$,

$$f^c(T) = \sup \{ Tx - f(x) : x \in D(f) \} ,$$

where $D(f^c) = \{ T \in L(X,Z) : \sup \{ Tx - f(x) : x \in D(f) \} \text{ exists} \}$.

Theorem 2.2. Let f: $D(f) \subset X \to Z$ be an operator. Then $(\underline{epi}\ f)^c = \underline{epi}\ f^c$. Hence f^c is a convex operator.

Proof. We have

$$\begin{aligned}(T,z') \in (\underline{epi}\ f)^c &\Longleftrightarrow z + z' \geq Tx \quad \text{for all } (x,z) \in \underline{epi}\ f \\ &\Longleftrightarrow f(x) + z' \geq Tx \quad \text{for all } x \in D(f) \\ &\Longleftrightarrow z' \geq Tx - f(x) \quad \text{for all } x \in D(f) \\ &\Longleftrightarrow T \in D(f^c) \quad \text{and } z' \geq f^c(T) \\ &\Longleftrightarrow (T,z') \in \underline{epi}\ f^c .\end{aligned}$$

Corollary 2.1. Let f: $D(f) \subset X \to Z$ be an operator . If $(\underline{epi}\ f)^c \neq \emptyset$, then $\underline{co}f$ exists and $(\underline{co}f)^c = f^c$.

Proof. It is an immediate consequence of Proposition 2.1 (ii) and (iii) and the above theorem.

Corollary 2.2. Let f: $D(f) \subset X \to Z$ and $x_o \in D(f)$. Then $T \in \partial f(x_o)$ if and only if $T \in D(f^c)$ and

$$f(x_o) + f^c(T) = Tx_o .$$

Proof. It is immediate.

Let now $\phi : D(\phi) \subset X \times Y \to Z$ be an operator. We consider the following primal problem

$$(\mathcal{P}) \qquad \inf_{(x,0) \in D(\phi)} \phi(x,0) = \inf \{ \phi(x,0) : x \in P_X(D(\phi) \cap X \times \{0\}) \} .$$

When $\underline{co}\phi$ exists we can associate to $(\mathcal{P})$ the relaxed problem

$$(\bar{\mathcal{P}}) \qquad \inf_{(x,0) \in \underline{co}D(\phi)} \underline{co}\,\phi(x,0) = \inf \{ \underline{co}\,\phi(x,0) : x \in P_X(\underline{co}D(\phi) \cap X \times \{0\}) \} .$$

Let

$$A = \{ (y,z) : \exists\, x \in X \text{ such that } \phi(x,y) \leq z \} . \qquad (2.3)$$

We have the following relations:

$$A = P_{Y\times Z}(\underline{\text{epi}}\,\Phi), \qquad (2.4)$$

$$\underline{\text{co}}A = P_{Y\times Z}(\underline{\text{co}}(\underline{\text{epi}}\,\Phi)), \qquad (2.4')$$

$$P_Y A = P_Y D(\Phi) \qquad (2.5)$$

$$P_Y(\underline{\text{co}}A) = P_Y(\underline{\text{co}}D(\Phi)). \qquad (2.5')$$

Remark 2.1. A is a set of epigraph type; if Φ is a convex operator, then A is a convex set.

Proposition 2.2.

(i) $\inf\{\Phi(x,0):\ (x,0)\in D(\Phi)\} = \inf\{z:\ (0,z)\in A\}$. (2.6)

(ii) If $\underline{\text{co}}\Phi$ exists then

$\inf\{\underline{\text{co}}\,\Phi(x,0): (x,0)\in \underline{\text{co}}D(\Phi)\} = \inf\{z: (0,z)\in \underline{\text{co}}A\}$. (2.6')

Proof. Let $D = \{(x,z)\in X\times Z:\ (x,0,z)\in \underline{\text{epi}}\,\Phi\}$ and $f: D\to Z$, $f(x,z) = z$. Applying Lemma 1.4 (i) one obtains

$$\inf_{x\in P_Y D}\ \inf_{z\in D_x} z = \inf_{z\in P_Z D}\ \inf_{x\in D_z} z = \inf_{z\in P_Z D} z.$$

But $z\in P_Z D \Longleftrightarrow \exists x\in X$ such that $(x,0,z)\in \underline{\text{epi}}\,\Phi \Longleftrightarrow (0,z)\in A$; hence $\inf_{z\in P_Z D} z = \inf\{z:\ (0,z)\in A\}$. Let $x\in P_X D \Longleftrightarrow \exists z\in Z$ such that $(x,0,z)\in \underline{\text{epi}}\,\Phi \Longleftrightarrow (x,0)\in D(\Phi)$. Then $\inf_{z\in D_x} z = \inf\{z:\ (x,0,z)\in \underline{\text{epi}}\,\Phi\} = \Phi(x,0)$. Hence

$$\inf_{x\in P_X D}\ \inf_{z\in D_x} z = \inf\{\Phi(x,0):\ (x,0)\in D(\Phi)\}.$$

Therefore (2.6) holds. To obtain (2.6') we take $D = \underline{\text{co}}(\underline{\text{epi}}\,\Phi)$.

Let us determinate the conjugate of A given by (2.3).

$$\begin{aligned}(T,z')\in A^c &\Longleftrightarrow \forall (y,z)\in A:\ z + z' \geqslant Ty\\ &\Longleftrightarrow \forall (x,y)\in D(\Phi):\ \Phi(x,y) + z' \geqslant Ty\\ &\Longleftrightarrow \forall (x,y)\in D(\Phi):\ z' \geqslant 0x + Ty - \Phi(x,y)\\ &\Longleftrightarrow (0,T)\in D(\Phi^c)\ \text{and}\ z' \geqslant \Phi^c(0,T).\end{aligned}$$

Hence

$$A^c = \{(T,z'):\ (0,T)\in D(\Phi^c),\ z' \geqslant \Phi^c(0,T)\}. \qquad (2.7)$$

Taking into account (2.1) and (2.6) it is natural to consider as the dual problem of $(\mathcal{P})$ (or $(\overline{\mathcal{P}})$) the problem

$$(\mathscr{D}) \quad \sup \{ -\Phi^c(0,T) : (0,T) \in D(\Phi^c) \} .$$

If $A^c \neq \emptyset$ then, from the definition of $(\mathscr{P})$, $(\overline{\mathscr{P}})$ and (2.1) we have

$$\inf \mathscr{P} \geqslant \inf \overline{\mathscr{P}} \geqslant \sup \mathscr{D} . \tag{2.8}$$

Theorem 2.3. If $0 \in {}^i(\underline{co}\, P_Y D(\Phi))$ and $\inf \{ z : (x,0,z) \in \underline{co}(\underline{epi}\, \Phi) \}$ exists, then $\underline{co}\,\Phi$ exists and

$$\inf \{ \underline{co}\, \Phi(x,0) : (x,0) \in \underline{co}D(\Phi) \} =$$
$$= \max \{ -\Phi^c(0,T) : (0,T) \in D(\Phi^c) \} . \tag{2.9}$$

Moreover, x_o is a solution for $(\mathscr{P})$ and $\inf \mathscr{P} = \max \mathscr{D}$ if and only if there exist $T \in L(Y,Z)$ such that

$$\Phi(x_o,0) + \Phi^c(0,T) = 0,$$

or equivalently

$$\exists T \in L(Y,Z) \text{ such that } (0,T) \in \partial \Phi(x_o,0).$$

Proof. From the conditions of the theorem, passing through (2.5') and (2.6'), we have that $0 \in {}^i(\underline{co}P_Y A) = {}^i(\underline{co}P_Y D(\Phi))$ and $\inf \{ z : (0,z) \in \underline{co}A \} = \inf \{ z : (x,0,z) \in \underline{co}(\underline{epi}\, \Phi) \}$ exists. Thus we can apply Theorem 2.1 to obtain

$$\inf \{ z : (0,z) \in \underline{co}A \} = \max \{ -z' : (T,z') \in A^c \} .$$

Hence $A^c \neq \emptyset$, and, consequently, by (2.7), $\underline{epi}\, \Phi^c = (\underline{epi}\, \Phi)^c \neq \emptyset$, so that $\underline{co}\,\Phi$ exists and $\max \{ -z' : (T,z') \in A^c \} = \max \{ -\Phi^c(0,T) : (0,T) \in D(\Phi^c) \}$. The rest of the theorem is immediate.

In what follows we obtain some important cases particularizing Φ.

Theorem 2.4. Let $F : D(F) \subset X \times Y \to Z$ be an operator and $S \in L(X,Y)$. Suppose that $0 \in {}^i \{ Sx - y : (x,y) \in \underline{co}D(F) \}$ and $\inf \{ z : (x, Sx, z) \in \underline{co}(\underline{epi}\, F) \}$ exists. Then $\underline{co}\, F$ exists and

$$\inf \{ \underline{co}F(x,Sx) : (x,Sx) \in \underline{co}D(F) \} =$$
$$\max \{ -F^c(T \circ S, -T) : (T \circ S, -T) \in D(F^c) \} . \tag{2.10}$$

Moreover, x_o is a solution of the primal problem and $\inf \mathcal{P} = \max \mathcal{D}$ if and only if there exists $T \in L(Y,Z)$ such that

$$F(x_o, Sx_o) + F^c(T \circ S, -T) = 0,$$

or, equivalently

there exists $T \in L(Y,Z)$ such that $(T \circ S, -T) \in \partial F(x_o, Sx_o)$.

Proof. We take $\phi: D(\phi) \subset X \times Y \to Z$, $\phi(x,y) = F(x, Sx-y)$, where $D(\phi) = \{(x,y): (x, Sx-y) \in D(F)\} = \{(x, Sx-y): (x,y) \in D(F)\}$. Thus $\underline{co}\,(P_Y D(\phi)) = \underline{co}\,\{Sx-y: (x,y) \in D(F)\} = \{Sx-y: (x,y) \in \underline{co}D(F)\}$, by Lemma 1.1 (ii). The conditions of the theorem assure that $0 \in {}^i(\underline{co}P_Y D(\phi))$. We also have

$$\begin{aligned}\phi^c(T_1, T_2) &= \sup\{T_1 x + T_2 y - \phi(x,y): (x,y) \in D(\phi)\} \\ &= \sup\{T_1 x + T_2 y - F(x, Sx-y): (x, Sx-y) \in D(F)\} \\ &= \sup\{T_1 x + T_2(Sx-y) - F(x,y): (x,y) \in D(F)\} \\ &= \sup\{(T_1 + T_2 \circ S)x + (-T_2)y - F(x,y): (x,y) \in D(F)\} \\ &= F^c(T_1 + T_2 \circ S, -T_2).\end{aligned}$$

To apply Theorem 2.3 we must calculate $\underline{co}(\underline{epi}\,\phi)$. We have $\underline{epi}\,\phi = \{(x,y,z): F(x, Sx-y) \le z\} = \{(x, Sx-y, z): F(x,y) \le z\} = U(\underline{epi}\,F)$ where $U: X \times Y \times Z \to X \times Y \times Z$, $U(x,y,z) = (x, Sx-y, z)$. It is obvious that U is a linear operator. Hence by Lemma 1.1 (ii) we have

$$\underline{co}(\underline{epi}\,\phi) = \{(x, Sx-y, z): (x,y,z) \in \underline{co}(\underline{epi}\,F)\}.$$

Therefore $\underline{epi}\,\phi^c \neq \emptyset$, so that $\underline{epi}\,F^c \neq \emptyset$, hence $\underline{co}F$ exists and

$$\begin{aligned}\inf\{z: (x,0,z) \in \underline{co}(\underline{epi}\,\phi)\} &= \inf\{z: (x, Sx, z) \in \underline{co}(\underline{epi}\,F)\} \\ &= \inf\{\underline{co}F(x, Sx): (x, Sx) \in \underline{co}D(F)\}\end{aligned}$$

exists.

Now (2.10) follows from (2.9). The rest of the theorem is immediate.

Another important case is furnished taking $F(x,y) = f(x) + g(y)$.

<u>Theorem</u> 2.5. Let $f: D(f) \subset X \to Z$, $g: D(g) \subset Y \to Z$ be two operators and $S \in L(X,Y)$. If $0 \in {}^{i}(S(\underline{co}D(f)) - \underline{co}D(g))$ and $\inf\{z_1 + z_2 : (x,z_1) \in \underline{co}(\underline{epi}f),\ (Sx,z_2) \in \underline{co}(\underline{epi}\ g)\}$ exists, then $\underline{co}f$ and $\underline{co}g$ exist and

$$\inf\{\underline{co}f(x) + \underline{co}g(Sx) : x \in \underline{co}D(f) \cap S^{-1}(\underline{co}D(g))\} \qquad (2.11)$$
$$= \max\{-f^c(T \circ S) - g^c(-T) : T \circ S \in D(f^c), -T \in D(g^c)\}.$$

Moreover, x_o is a solution of the primal problem and $\inf \mathcal{P} = \max \mathcal{D}$ if and only if there exists $T \in -D(g^c)$ such that $T \circ S\ D(f^c)$ and

$$f(x_o) + f^c(T \circ S) = T \circ S(x_o), \qquad (2.12)$$
$$g(Sx_o) + g^c(-T) = -T \circ S(x_o),$$

or, equivalently

$$\exists T \in L(Y,Z) \text{ such that } T \circ S \in \partial f(x_o),\ -T \in \partial g(Sx_o).$$

Proof. Let us take $\phi: D(\phi) \to Z$, $\phi(x,y) = f(x) + g(Sx-y)$, where $D(\phi) = \{(x,y) : x \in D(f),\ Sx-y \in D(g)\} = \{(x,Sx-y) : x \in D(f),\ y \in D(g)\}$. Thus $P_Y D(\phi) = \{Sx-y : x \in D(f),\ y \in D(g)\} = S(D(f)) - D(g)$. Hence $0 \in {}^{i}(S(\underline{co}D(f)) - \underline{co}D(g)) = {}^{i}(\underline{co}P_Y D(\phi))$, where we have used Lemma 1.1(ii) - (iii). On the other hand we have

$$\underline{epi}\ \phi = \{(x,y,z) : f(x) + g(Sx-y) \le z\}$$
$$= \{(x,Sx-y,z) : f(x) + g(y) \le z\}$$
$$= \{(x,Sx-y,z_1+z_2) : (x,z_1) \in \underline{epi}\ f, (y,z_2) \in \underline{epi}\ g\}.$$

Let $U: X \times Z \times Y \times Z \to X \times Y \times Z$, $U(x,z_1,y,z_2) = (x,Sx-y,z_1+z_2)$. U is a linear operator and $U(\underline{epi}\ f \times \underline{epi}\ g) = \underline{epi}\,\phi$. By Lemma 1.1(i) we obtain

$$\underline{co}(\underline{epi}\,\phi) = U(\underline{co}(\underline{epi}\ f) \times \underline{co}(\underline{epi}\ g)) \qquad (2.13)$$
$$= \{(x,Sx-y,z_1+z_2) : (x,z_1) \in \underline{co}(\underline{epi}\ f), (y,z_2) \in \underline{co}(\underline{epi}\ g)\}.$$

Hence

$$\inf\{z : (x,0,z) \in \underline{co}(\underline{epi}\,\phi)\} = \inf\{z_1 + z_2 : (x,z_1) \in \underline{co}(\underline{epi}\ f),$$
$$(Sx,z_2) \in \underline{co}(\underline{epi}\ g)\}$$

exists. Thus the conditions of Theorem 2.3 are verified. So

$$\inf\{\underline{co}\,\phi(x,0):(x,0)\in \underline{co}D(\phi)\} = \max\{-\phi^c(0,T):(0,T)\in D(\phi^c)\}. \tag{2.14}$$

We have

$$\begin{aligned}\phi^c(T_1,T_2) &= \sup\{T_1x + T_2y - f(x) - g(Sx-y):\ x\in D(f), Sx-y\in D(g)\}\\ &= \sup\{T_1x + T_2(Sx-y) - f(x)-g(y):\ x\in D(f), y\in D(g)\}\\ &= \sup\{(T_1 + T_2\circ S)x + (-T_2)y-f(x)-g(y): x\in D(f), y\in D(g)\}\\ &= \sup\{(T_1 + T_2\circ S)x - f(x):\ x\in D(f)\} + \sup\{(-T_2)y-g(y):\ y\in D(g)\}\end{aligned}$$

Hence

$$\phi^c(T_1,T_2) = f^c(T_1+ T_2\circ S) + g^c(-T_2), \tag{2.15}$$

when $T_1 + T_2\circ S\in D(f^c)$ and $-T_2\in D(g^c)$. Since $D(\phi^c)\neq\emptyset$ it follows that $D(f^c)\neq\emptyset$ and $D(g^c)\neq\emptyset$, and, consequently, $\underline{co}f$ and $\underline{co}g$ exist. Take $(x,0)\in \text{co}D(\phi) = \{(x,Sx-y):\ x\in \underline{co}D(f),\ y\in\underline{co}D(g)\}$. From (2.13), applying Lemma 1.4(ii), we obtain

$$\begin{aligned}\underline{co}\,\phi(x,0) &= \inf\{z:\ (x,0,z)\in \underline{co}(\text{epi}\,\phi)\}\\ &= \inf\{z_1+ z_2:\ (x,z_1)\in \underline{co}(\underline{\text{epi}}\ f), (Sx,z_2)\in \underline{co}(\underline{\text{epi}}\ g)\}\\ &= \inf\{z_1:\ (x,z_1)\in \underline{co}(\underline{\text{epi}}\ f)\} + \inf\{z_2:\ (Sx,z_2)\in \underline{co}(\underline{\text{epi}}\ g)\}.\end{aligned}$$

Hence

$$\underline{co}\ \phi(x,0) = \underline{co}f(x) + \underline{co}g(Sx). \tag{2.16}$$

From (2.14), (2.15) and (2.16) one obtains (2.11).

Suppose that $\inf\mathcal{P} = \max\mathcal{D}$ and x_o is a solution of the primal problem. Then there exists $T\in L(Y,Z)$ such that $T\circ S\in D(f^c)$, $-T\in D(g^c)$ and

$$f(x_o) + g(Sx_o) = -f^c(T\circ S) - g^c(-T)\Leftrightarrow$$

$$f(x_o) + f^c(T\circ S) - T\circ S(x_o) + g(Sx_o) + g^c(-T) + T\circ S(x_o)=0$$

But

$$f(x_o) + f^c(T\circ S) - T\circ S(x_o)\geqslant 0,$$

$$g(Sx_o) + g^c(-T) + T\circ S(x_o)\geqslant 0.$$

Since $z_1, z_2 \geqslant 0$, $z_1 + z_2 = 0$ imply $\underbrace{z_1}_{=z_2} = 0$ (when $C \cap -C = \{0\}$) it follows that (2.12) takes place. The last equivalence of the theorem is obvious, and the proof is complete.

From Theorem 2.5 one obtains immediately:

Theorem 2.6. Let $g: D(g) \subset Y \to Z$ be an operator and $S \in L(X,Y)$. If $0 \in {}^i(S(X) - \underline{co}D(g))$ and $\inf \{ z: (Sx,z) \in \underline{co}(\underline{epi}\ g)\}$ exists, then $\underline{co}g$ exists and

$$\inf \{ \underline{co}g(Sx): Sx \in \underline{co}D(g)\} = \max \{ -g^c(T): T \in D(g^c), \\ T \circ S = 0\}.$$

Moreover, x_o is a solution of the primal problem and $\inf \varphi = \max \vartheta$ if and only if there exists $T \in D(g^c)$ such that

$$T \circ S = 0 \text{ and } g(Sx_o) + g^c(T) = 0,$$

or equivalently

$$\exists T \in \partial g(Sx_o) \text{ such that } T \circ S = 0.$$

Proof. Take in Theorem 2.5 $f: X \to Z, f(x) = 0$.

Ar a first glance it seems that Theorem 2.4 is more general than Theorem 2.6. In reality we can obtain Theorem 2.4 from Theorem 2.6 taking $g = F$ and replacing $S: X \to Y$ by $\bar{S}: X \to X \times Y$ $\bar{S}x = (x,Sx)$. It is easy to verify that $(0,0) \in {}^i(\bar{S}(X) - \underline{co}D(F))$ if and only if $0 \in {}^i\{Sx - y: (x,y) \in \underline{co}D(F)\}$. Thus we have the full equivalence between Theorems 2.4 and 2.6. The form of the function in Theorem 2.4 is more convenient for optimal control problems.

Theorem 2.7. Let $f_k: D(f_k) \subset X \to Z$, $1 \leqslant k \leqslant n$. If $\underline{co}D(f_1), \dots, \underline{co}D(f_n)$ are in general position and $\inf \{ z_1 + \dots + z_n: x \in X, (x,z_k) \in \underline{co}(\underline{epi}\ f_k), 1 \leqslant k \leqslant n\}$ exists, then $\underline{co}f_k, 1 \leqslant k \leqslant n$ exist and

$$\inf \Big\{ \sum_{k=1}^{n} \underline{co}f_k(x): x \in \bigcap_{k=1}^{n} \underline{co}D(f_k)\Big\} = \max \Big\{ -\sum_{k=1}^{n} f_k^c(T_k): T_k \in D(f_k^c),$$

$$\sum_{k=1}^{n} T_k = 0 \Big\} .$$

Moreover, x_o is an optimal solution for the primal problem and $\inf \mathcal{P} = \max \mathcal{D}$ if and only if $\exists\, T_k \in D(f_k^c)$, $1 \leq k \leq n$ such that $\sum_{k=1}^{n} T_k = 0$ and

$$f_k(x_o) + f_k^c(T_k) = T_k x_o, 1 \leq k \leq n \quad ,$$

or, equivalently

$$0 \in \partial f_1(x_o) + \partial f_2(x_o) + \ldots + \partial f_n(x_o).$$

Proof. Consider $g\colon D(g) \subset X^n \to Z$, $g(x_1, \ldots, x_n) = \sum_{k=1}^{n} f_k(x_k)$, $D(g) = \bigtimes_{k=1}^{n} D(f_k)$, and $S\colon X \to X^n$, $Sx = (x, \ldots, x)$. It is easy to see that $D(g^c) = \bigtimes_{k=1}^{n} D(f_k^c)$, $g^c(T_1, \ldots, T_n) = \sum_{k=1}^{n} f_k^c(T_k)$. Since $\underline{co}D(f_1), \ldots, \underline{co}D(f_n)$ are in general position, Lemma 1.2 shows that $(0, \ldots, 0) \in {}^i\{(x - x_1, \ldots, x - x_n)\colon x \in X,\ x_k \in \underline{co}D(f_k), 1 \leq k \leq n\} = {}^i(S(X) - \underline{co}D(g))$. Hence all the hypotheses of Theorem 2.6 hold, so that we can apply it. Therefore the conclusions of the theorem are true, using a similar argument to that of Theorem 2.4.

Let now Y be ordered by the convex cone Q. We say that $g\colon D(g) \subset Y \to Z$ is increasing if $D(g) - Q \subset D(g)$ and $x \leq y$ implies $g(x) \leq g(y)$. Denote by $L^+(Y,Z)$ the space of increasing (positive) linear operators from Y into Z.

<u>Theorem 2.8</u>. Let $f\colon D(f) \subset X \to Y$ be a convex operator and $g\colon D(g) \subset Y \to Z$ an increasing convex operator. Suppose that $0 \in {}^i(D(g) - f(X))$ and $\inf \{ g(f(x))\colon x \in D(f),\ f(x) \in D(g)\}$ exists. Then

$$\inf \{ g(f(x))\colon x \in D(f), f(x) \in D(g)\} =$$

$$= \max \{ -g^c(T) - (T \circ f)^c(0)\colon T \in D(g^c), 0 \in D((T \circ f)^c)\}$$

Moreover, x_o is a solution of the primal problem if and only if $\exists\, T \in D(g^c)$ such that $0 \in D((T \circ f)^c)$ and $g(f(x_o)) + g^c(T) =$

$$= Tf(x_o) = -(T \circ f)^c(0),$$

or, equivalently

$$\exists T \in \partial g(f(x_o)) \text{ such that } 0 \in \partial (T \circ f)(x_o).$$

Proof. Consider $\phi : D(\phi) \subset X \times Y \to Z$, $\phi(x,y) = g(f(x)+y)$, where $D(\phi) = \{(x,y): x \in D(f),\ f(x)+y \in D(g)\}$. It is easy to see that ϕ is a convex operator and $P_Y D(\phi) = D(g) - f(X)$. On the other hand

$$\begin{aligned}\phi^c(T_1,T_2) &= \sup\{T_1x + T_2y - \phi(x,y): (x,y) \in D(\phi)\}\\ &= \sup\{T_1x + T_2y - g(f(x)+y): x \in D(f), f(x)+y \in D(g)\}\\ &= \sup\{T_1x - T_2f(x) + T_2y - g(y): x \in D(f),\ y \in D(g)\}\\ &= \sup\{T_1x - (T_2 \circ f)(x): x \in D(f)\} + \sup\{T_2y - g(y): y \in D(g)\}\\ &= (T_2 \circ f)^c(T_1) + g^c(T_2),\end{aligned}$$

when $T_2 \in D(g^c)$ and $T_1 \in D((T_2 \circ f)^c)$. Applying Theorem 2.3 one obtains the assertion of the theorem.

Remark 2.1. If $g: D(g) \subset Y \to Z$ is an increasing operator, then $D(g^c) \subset L^+(Y,Z)$.

Indeed, let $T \in D(g^c)$ and $y_o \in D(g)$. Then for every $q \in Q$, $y_o - q \leq y_o$, so that $T(y_o - q) \leq g(y_o - q) + g^c(T) \leq g(y_o) + g^c(T)$; hence $Tq \geqslant Ty_o - g(y_o) - g^c(T)$. Applying Lemma 1.3 one obtains $Tq \geqslant 0$ for every $q \in Q$. Hence $T \in L^+(Y,Z)$.

Corollary 2.3. In Theorem 2.8 suppose that g is sublinear. Then

$$\inf\{g(f(x)): x \in D(f),\ f(x) \in D(g)\} = \max\{-(T \circ f)^c(0): T \in \partial g(0),\ 0 \in D((T \circ f)^c)\}.$$

Moreover, x_o is an optimal solution of the primal problem if and only if

$$\exists T \in \partial g(0) \text{ such that } g(f(x_o)) = Tf(x_o) \text{ and } 0 \in \partial (T \circ f)(x_o).$$

Proof. The corollary is an immediate consequence of Theorem 2.8 taking into account that $D(g^c) = \partial g(0), g^c(T) = 0$.

<u>Theorem</u> 2.9. Let $f_k : D(f_k) \subset X \to Y$, $1 \leq k \leq n$, be convex operators, Y a vector lattice and $T \in L^+(Y,Z)$. Suppose that $\inf \{ T(f_1(x) \vee \ldots \vee f_n(x)) : x \in \bigcap_{k=1}^{n} D(f_k) \}$ exists and $D(f_1), \ldots, D(f_n)$ are in general position. Then

$$\inf \{ T(f_1(x) \vee \ldots \vee f_n(x)) : x \in \bigcap_{k=1}^{n} D(f_k) \} =$$

$$= \max \{ -\sum_{k=1}^{n} (T_k \circ f_k)^c (S_k) : T_k \in L^+(Y,Z), \sum_{k=1}^{n} T_k = T, S_k \in D((T_k \circ f_k)^c), \sum_{k=1}^{n} S_k = 0 \}.$$

Moreover, x_o is an optimal solution of the primal problem if and only if

$$\exists T_k \in L^+(Y,Z),\ 1 \leq k \leq n, \text{ such that } \sum_{k=1}^{n} T_k = T, T(f_1(x_o) \vee \ldots \vee f_n(x_o)) = \sum_{k=1}^{n} T_k f_k(x_o) \text{ and } 0 \in \sum_{k=1}^{n} \partial (T_k \circ f_k)(x_o).$$

Proof. Let us take $f: \bigcap_{k=1}^{n} D(f_k) \subset X \to Y^n$, $f(x) = (f_1(x), \ldots, f_n(x))$ and $g: Y^n \to Z$, $g(y_1, \ldots, y_n) = T(y_1 \vee \ldots \vee y_n)$. It is easy to see that f is a convex operator and g is an increasing convex operator. After some calculations one obtains $\partial g(0) = \{ (T_1, \ldots, T_n) : T_k \in L^+(Y,Z), \sum_{k=1}^{n} = T \}$. Since $D(g) - f(X) = Y^n$, we can apply the preceding corollary. Thus

$$\inf \{ T(f_1(x) \vee \ldots \vee f_n(x)) : x \in \bigcap_{k=1}^{n} D(f_k) \} =$$

$$= \max \{ -(\sum_{k=1}^{n} T_k \circ f_k)^c (0) : T_k \in L^+(Y,Z), \sum_{k=1}^{n} T_k = T, 0 \in D((\sum_{k=1}^{n} T_k \circ f_k)^c) \}.$$

Applying Theorem 3.4 in the following section, taking into account that $D(f_1), \ldots, D(f_n)$ are in general position, the conclusion of the theorem follows.

As a consequence of Theorem 2.7 we have the following Sandwich theorem (see [14]).

Corollary 2.4. Let $f\colon D(f)\subset X\to Z$, $g\colon D(g)\subset X\to Z$ be such that f and $-g$ are convex operators. If $0\in {}^{i}(D(f)-D(g))$ and $f(x)\geqslant g(x)$ for all $x\in D(f)\cap D(g)$, then there exists $T\in L(X,Z)$ and $z\in Z$ such that

$$Tx - z\leqslant f(x) \quad \text{for all } x\in D(f), \tag{2.17}$$

$$Tx - z\geqslant g(x) \quad \text{for all } x\in D(g). \tag{2.18}$$

Proof. Since $f(x)\geqslant g(x)$ for all $x\in D(f)\cap D(g)$, it follows that there exists $\inf\{ f(x) - (-g)(x)\colon x\in D(f)\cap D(g)\}\geqslant 0$. Applying theorem 2.7 we obtain

$$0\leqslant \inf\{ f(x) - g(x)\colon x\in D(f)\cap D(g)\} =$$

$$= \max\{- f^{c}(T) - (-g)^{c}(-T)\colon T\in D(f^{c})\cap - D((-g)^{c})\}.$$

Hence, there is $T\in L(X,Z)$ such that $0\leqslant -f^{c}(T) - (-g)^{c}(-T)$. Take $z = f^{c}(T)$; then $(-g)^{c}(-T)\leqslant -z$, so that

$$Tx - f(x)\leqslant z \Longleftrightarrow Tx - z\leqslant f(x) \quad \text{for all } x\in D(f),$$

and

$$-Tx - (-g)(x)\leqslant - z \Longleftrightarrow Tx - z\geqslant g(x) \quad \text{for all } x\in D(g).$$

The proof is complete.

In the sequel we give a necessary and sufficient condition to exist an operator $T\in L(X,Z)$ with the properties (2.17)-(2.18). Compare with Sandwich theorem 4.3 of [14] .

Theorem 2.10. Let $f\colon D(f)\subset X\to Z$ and $g\colon D(g)\subset X\to Z$ be such that f and $-g$ are convex operator. Then the following assertions are equivalent:

(i) there exists a convex operator $F\colon X\to Z$ with $F(0)\leqslant 0$ such that

$$f(x_1) - g(x_2)\geqslant -F(x_1 - x_2) \quad \text{for all } x_1\in D(f),\ x_2\in D(g), \tag{2.19}$$

(ii) there exist $T \in L(X,Z)$ and $z \in Z$ satisfying (2.17) and (2.18).

Proof. Suppose (ii) takes place. Hence

$$Tx_1 - z \leq f(x_1) \text{ for all } x_1 \in D(f),$$
$$Tx_2 - z \geqslant g(x_2) \text{ for all } x_2 \in D(g),$$

so that

$$f(x_1) - g(x_2) \geqslant Tx_1 - z - Tx_2 + z = T(x_1 - x_2) \ \forall x_1 \in D(f),\ x_2 \in D(g).$$

Taking $F = T$ it is clear that (i) is verified.

Suppose now that (i) is verified. Let $G: D(f) \times D(g) \to Z$, $G(x_1,x_2) = f(x_1) - g(x_2)$. It is obvious that G is a convex operator. Let $\overline{F}: X \times X \to Z$ be defined by $\overline{F}(x_1,x_2) = F(x_1 - x_2)$. Since F is convex, so is $\overline{F}$. Clearly, we have $D(G) - D(\overline{F}) = X \times X$, so that $(0,0) \in {}^{i}(D(G) - D(\overline{F}))$. We also have $G(x_1,x_2)\ \ (-F)(x_1,x_2)$ for all (x_1,x_2) $D(G)\ \ D(F) = D(G)$. Consequently we can apply the preceding corollary. Hence there exist $T_1, T_2 \in L(X,Z)$ and $z_o \in Z$ such that:

$$G(x_1,x_2) \geqslant T_1x_1 + T_2x_2 - z_o \ \forall (x_1,x_2) \in D(G) \qquad (2.20)$$
$$T_1x_1 + T_2x_2 - z_o \geqslant -F(x_1 - x_2) \ \forall (x_1,x_2) \in X \times X. \qquad (2.21)$$

From (2.21) we have

$$T_1x + T_2x - z_o \geqslant -F(0) \ \forall x \in X \Leftrightarrow$$
$$(T_1 + T_2)x \geqslant z_o - F(0) \ \forall x \in X.$$

Taking $x = 0$ we obtain $z_o \leq F(0)$, and Lemma 1.3 shows that $T_1 + T_2 = 0$. Let $T = T_1 = -T_2$; from (2.20) we obtain

$$Tx_1 - Tx_2 - z_o \leq f(x_1) - g(x_2) \ \forall x_1 \in D(f),\ x_2 \in D(g),$$

or, equivalently,

$$f(x_1) - Tx_1 \geqslant g(x_2) - Tx_2 - z_o \geqslant g(x_2) - Tx_2$$
$$\forall x_1 \in D(f),\ x_2 \in D(g).$$

Taking $z = \inf\{f(x) - Tx: x \in D(f)\}$, which exists since $D(g) \neq \emptyset$, we obtain $f(x) \geqslant Tx - z \ \forall x \in D(f)$ and $g(x) \leq Tx - z \ \forall x \in D(g)$, which completes the proof.

We want to establish a condition to assure that $\inf \mathcal{P}_1 = \inf \overline{\mathcal{P}}_1$, where

$$(\mathcal{P}_1) \quad \inf \{ f(x) + g(x) : x \in D(f) \cap D(g) \},$$

and

$$(\overline{\mathcal{P}}_1) \quad \inf \{ \underline{co}f(x) + \underline{co}g(x) : x \in \underline{co}D(f) \cap \underline{co}D(g) \},$$

when $\underline{co}f$ and $\underline{co}g$ exist. In that case, if $v = \inf \{f(x) + g(x) : x \in D(f) \cap D(g)\}$, we have $\inf \mathcal{P}_1 = \inf \overline{\mathcal{P}}_1$ if and only if $\underline{co}f(x) + \underline{co}g(x) \geq v$ for all $x \in \underline{co}D(f) \cap \underline{co}D(g)$, or, equivalently

$$\begin{cases} \forall \lambda_i > 0, x_i \in D(f),\ 1 \leq i \leq n,\ \sum_{i=1}^{n} \lambda_i = 1,\quad \forall \mu_j > 0, y_j \in D(g),\ 1 \leq j \leq m,\ \sum_{j=1}^{m} \mu_j = 1: \\ \sum_{i=1}^{n} \lambda_i x_i = \sum_{j=1}^{m} \mu_j y_j \Rightarrow \sum_{i=1}^{n} \lambda_i f(x_i) + \sum_{j=1}^{m} \mu_j g(y_j) \geq v. \end{cases}$$

We shall say that $f: D(f) \subset X \to Z$ and $g: D(g) \subset X \to Z$ satisfy condition (H) if

$$(H) \begin{cases} \forall \lambda_i > 0, x_i \in D(f), 1 \leq i \leq n, \sum_{i=1}^{n} \lambda_i = 1;\ \forall \mu_j > 0, y_j \in D(g), 1 \leq j \leq m, \sum_{j=1}^{m} \mu_j = 1: \\ \sum_{i=1}^{n} \lambda_i x_i = \sum_{j=1}^{m} \mu_j y_j \Rightarrow \sum_{i=1}^{n} \lambda_i f(x_i) + \sum_{j=1}^{m} \mu_j g(y_j) \geq 0. \end{cases}$$

Note that if $0 \in {}^{i}(\underline{co}D(f) - \underline{co}D(g))$ condition (H) implies the existence of $\underline{co}f$ and $\underline{co}g$ (see Theorem 2.7) and $\underline{co}f(x) + \underline{co}g(x) \geq 0$ for all $x \in \underline{co}D(f) \cap \underline{co}D(g)$. So we have

<u>Proposition 2.3</u>. Let $0 \in {}^{i}(\underline{co}D(f) - \underline{co}D(g))$ and $v =$

$= \inf \left\{ f(x) + g(x) : x \in D(f) \cap D(g) \right\}$. Then f-v and g satisfy (H) if and only if $\inf \varphi_1 = \inf \overline{\varphi}_1$.

In [7] Rosinger has used the following condition:

$$\text{(CI)} \quad \begin{cases} \forall \lambda_1, \dots, \lambda_n \in R_+,\ x_1, \dots, x_n \in D(f),\ y_1, \dots, y_n \in D(g): \\ \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n} \lambda_i y_i \Rightarrow \sum_{i=1}^{n} \lambda_i f(x_i) \geqslant \sum_{i=1}^{n} \lambda_i g(y_i). \end{cases}$$

It is obvious that in condition (CI) we can take $\lambda_i > 0$, $\sum_{i=1}^{n} \lambda_i = 1$.

<u>Proposition</u> 2.4. f and g satisfy (H) if and only if f and -g satisfy (CI).

Proof. It is clear that if f and g satisfy (H) then f and -g satisfy (CI). Conversely, suppose f and -g satisfy (CI), i.e.,

$$\begin{cases} \forall \lambda_1, \dots, \lambda_n \in (0, \infty),\ x_1, x_2, \dots, x_n \in D(f),\ y_1, \dots, y_n \in D(g), \\ \qquad \sum_{i=1}^{n} \lambda_i = 1: \\ \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n} \lambda_i y_i \Rightarrow \sum_{i=1}^{n} \lambda_i f(x_i) + \sum_{i=1}^{n} \lambda_i g(y_i) \geqslant 0. \end{cases} \tag{2.22}$$

Take $\lambda_i > 0$, $x_i \in D(f)$, $1 \leq i \leq n$, $\sum_{i=1}^{n} \lambda_i = 1$, $\mu_j > 0$, $y_j \in D(g)$, $1 \leq j \leq m$, $\sum_{j=1}^{m} \mu_j = 1$ such that $\sum_{i=1}^{n} \lambda_i x_i = \sum_{j=1}^{m} \mu_j y_j = x$. Then

$$x = \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n} \lambda_i \Big(\sum_{j=1}^{m} \mu_j \Big) x_i = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j x_i,$$

$$x = \sum_{j=1}^{m} \mu_j y_j = \sum_{j=1}^{m} \mu_j \Big(\sum_{i=1}^{n} \lambda_i \Big) y_j = \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i \mu_j y_j = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j y_j .$$

From (2.22) it follows that

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\mu_j f(x_i)+\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\mu_j g(y_j)\geqslant 0,$$

or, equivalently,

$$\sum_{i=1}^{n}\lambda_i f(x_i)+\sum_{j=1}^{m}\mu_j g(y_j)\geqslant 0.$$

Thus (H) is satisfied.

Remark 2.2. From Propositions 2.3, 2.4 and Theorem 2.7 one obtains Theorems 1 and 2 and Lemma 3 of [7], taking into account that $(0,\infty)\cdot(D(f)-D(g))=X$ implies $0\in(\underline{co}D(f)-\underline{co}D(g))^i$.

Remark 2.3. From Theorem 2.7, taking into account the discussion in Section 5, one obtains Theorems 2.3 of Dragomirescu [2] and Theorems 2 and 3 of Zowe [13] concerning the Fenchel duality.

Remark 2.4. From Theorem 2.7 one obtains Theorem 6 of Bair [1] in the case of vector lattices. Note, if Z has only the least upper bound property, in Theorem 6 [1] one has rather equivalence than equality.

3. Applications

In this section we apply the results of Section 2 to calculate conjugate operators and subdifferentials.

Theorem 3.1. Let $F: D(F)\subset X\times Y\rightarrow Z$ a convex operator, $S\in L(X,Y)$ and $\varphi: D(\varphi)\rightarrow Z$, $\varphi(x)=F(x,Sx)$, where $D(\varphi)=\{x: (x,Sx)\in D(F)\}$. If $0\in {}^i\{Sx-y: (x,y)\in D(F)\}$, then

$$\begin{cases} D(\varphi^c)=\{U: U=T_1+T_2\circ S,\ (T_1,T_2)\in D(F^c)\}, \\ \varphi^c(U)=\min\{F^c(T_1,T_2): (T_1,T_2)\in D(F^c),\ U=T_1+T_2\circ S\}, \end{cases} \tag{3.1}$$

and

$$\partial\varphi(x) = \{T_1 + T_2 \circ S: (T_1,T_2) \in \partial F(x,Sx)\} \quad \text{for all } x \in D(\varphi), (3.2$$

Proof. Let $U \in D(\varphi^c)$; take $\bar{F}: D(\bar{F}) \to Z, \bar{F}(x,y) = F(x,y)-U(x)$, where $D(\bar{F}) = D(F)$. Hence $0 \in^i \{Sx-y: (x,y) \in D(\bar{F})\}$. Since $U \in D(\varphi^c)$, it follows that $\varphi^c(U) = \sup \{Ux-F(x,Sx): (x,Sx) \in D(F)\}$ exists. So, we can apply Theorem 2.4:

$$-\varphi^c(U) = \inf \{\bar{F}(x,Sx): (x,Sx) \in D(\bar{F})\} =$$

$$= \max \{ -\bar{F}^c(T \circ S, -T: (T \circ S, -T) \in D(\bar{F}^c)\},$$

or equivalently

$$\varphi^c(U) = \min \{ F^c(T \circ S, - T): (T \circ S, -T) \in D(\bar{F}^c)\}.$$

But

$$\bar{F}^c(T_1,T_2) = \sup \{ T_1x + T_2y - \bar{F}(x,y): (x,y) \in D(\bar{F})\}$$

$$= \sup\{T_1x + T_2y - F(x,y) + Ux: (x,y) \in D(F)\}$$

$$= F^c(T_1 + U, T_2),$$

with $D(\bar{F}^c) = \{(T_1 - U, T_2): (T_1,T_2) \in D(F^c)\}$. Consequently,

$$\varphi^c(U) = \min\{F^c(T \circ S + U,-T): (T \circ S + U,-T) \in D(F^c)\}$$

$$= \min \{F^c(T_1,T_2): (T_1,T_2) \in D(F^c), T_1 + T_2 \circ S = U\}.$$

Hence it is verified (3.1). It is easy to verify that $\{T_1 + T_2 \circ S: (T_1,T_2) \in \partial F(x,Sx)\} \subset \partial\varphi(x)$ for all $x \in D(\varphi)$. Let us show the converse inclusion. Let $U \in \partial\varphi(x_o)$; then $\varphi(x_o) + \varphi^c(U) = U(x_o)$. Therefore, there exists $(T_1,T_2) \in D(F^c)$ such that $U = T_1 + T_2 \circ S$ and

$$F(x_o,Sx_o) + F^c(T_1,T_2) = (T_1 + T_2 \circ S)x_o = T_1x_o + T_2(Sx_o),$$

so that $(T_1,T_2) \in \partial F(x_o,Sx_o)$. Hence $\partial\varphi(x_o) \subset \{T_1 + T_2 \circ S: (T_1,T_2) \in \partial F(x_o,Sx_o)\}$, and (3.2) is verified.

<u>Theorem</u> 3.2. Let $f: D(f) \subset X \to Z$ and $g: D(g) \subset Y \to Z$ be convex operators and $S \in L(X,Y)$. Let $\varphi: D(\varphi) \to Z$, $\varphi(x) = f(x) + g(Sx)$, where $D(\varphi) = D(f) \cap S^{-1}(D(g))$. If $0 \in^i (S(D(f) -$

$-D(g))$ then

$$\begin{cases} D(\varphi^c) = \{T_1 + T_2 \circ S: (T_1,T_2) \in D(f^c) \times D(g^c)\} , \\ \varphi^c(U) = \min \{ f^c(T_1) + g^c(T_2): T_1 \in D(f^c), T_2 \in D(g^c), U = T_1 + T_2 \circ S \} , \end{cases}$$

and

$$\partial\varphi(x) = \{T_1 + T_2 \circ S: T_1 \in \partial f(x), T_2 \in \partial g(Sx)\} \text{ for all } x \in D(\varphi).$$

Proof. Take $F(x,y) = f(x) + g(y)$, $D(F) = D(f) \times D(g)$ in the preceding theorem. Then $F^c(T_1,T_2) = f^c(T_1) + g^c(T_2)$ and $(T_1,T_2) \in \partial F(x,y)$ if and only if $(T_1,T_2) \in D(F^c)$ and $f(x) + g(y) + f^c(T_1) + g^c(T_2) = T_1x + T_2y$, which is equivalent by the same argument as in Theorem 2.5, to $T_1 \in \partial f(x)$ and $T_2 \in \partial g(y)$. The proof is complete, applying Theorem 3.1.

Theorem 3.3. Let $g: D(g) \subset Y \to Z$ be a convex operator and $S \in L(X,Y)$ Let $\varphi: D(\varphi) \to Z$, $\varphi(x) = g(Sx)$, where $D(\varphi) = S^{-1}(D(g))$. If $0 \in {}^i(S(X) - D(g))$, then

$$\begin{cases} D(\varphi^c) = \{T \circ S: T \in D(g^c)\} , \\ \varphi^c(U) = \min \{ g^c(T) : T \in D(g^c), T \circ S = U\} , \end{cases}$$

and

$$\partial\varphi(x) = \{T \circ S: T \in \partial g(Sx)\} \text{ for all } x \in D(\varphi).$$

Proof. We take in Theorem 3.2 $f: X \to Z$, $f(x) = 0$.

From Theorems 2.7, 2.8 and 2.9 one obtains, respectively:

Theorem 3.4. Let $f_k: D(f_k) \subset X \to Z, 1 \le k \le n$, be convex operators. Suppose that $D(f_1),\ldots,D(f_n)$ are in general position. Let $\varphi: \bigcap_{k=1}^{n} D(f_k) \to Z$, $\varphi(x) = \sum_{k=1}^{n} f_k(x)$. Then

$$\begin{cases} D(\varphi^c) = \left\{ \sum_{k=1}^{n} T_k: T_k \in D(f_k^c), 1 \le k \le n \right\} , \\ \varphi^c(U) = \min \left\{ \sum_{k=1}^{n} f_k^c(T_k): \sum_{k=1}^{n} T_k = U \right\}, \end{cases}$$

and

$$\partial\varphi(x) = \sum_{k=1}^{n} \partial f_k(x) \quad \text{for all} \quad x \in D(\varphi).$$

Theorem 3.5. Let $f\colon D(f) \subset X \to Y$ be a convex operator and $g\colon D(g) \subset Y \to Z$ an increasing convex operator, where Y is ordered. Let $\varphi\colon D(\varphi) \to Z$, $\varphi(x) = g(f(x))$, where $D(\varphi) = \{x\colon x \in D(f),\ f(x) \in D(g)\}$. Suppose that $0 \in {}^{i}(D(g) - f(X))$; then

$$\begin{cases} D(\varphi^c) = \bigcup\{D((T\circ f)^c)\colon T \in D(g^c)\}, \\ \varphi^c(U) = \min\{g^c(T) + (T\circ f)^c(U)\colon T \in D(g^c), \\ \qquad\qquad U \in D((T\circ f)^c)\}, \end{cases}$$

and

$$\partial\varphi(x) = \bigcup\{\partial(T\circ f)(x)\colon T \in \partial g(f(x))\} \quad \text{for all } x \in D(\varphi).$$

Theorem 3.6. Let $f_k\colon D(f_k) \subset X \to Y$ be convex operators, $1 \le k \le n$, Y a vector lattice and $T \in L^+(Y,Z)$. Let $\varphi\colon \bigcap_{k=1}^{n} D(f_k) \to Z$, $\varphi(x) = T(f_1(x) \vee \ldots \vee f_n(x))$. Suppose that $D(f_1), \ldots, D(f_n)$ are in general position; then

$$D(\varphi^c) = \left\{\sum_{k=1}^{n} S_k\colon T_k \in L^+(Y,Z), \sum_{k=1}^{n} T_k = T,\ S_k \in D((T_k\circ f_k)^c)\right\},$$

$${}^{c}(U) = \min\Big\{\sum_{k=1}^{n} (T_k\circ f_k)^c(S_k)\colon T_k \in L^+(Y,Z),$$

$$\sum_{k=1}^{n} T_k = T,\ S_k \in D((T_k\circ f_k)^c),\ \sum_{k=1}^{n} S_k = U\Big\},$$

and

$$\partial\varphi(x) = \bigcup\Big\{\sum_{k=1}^{n} \partial(T_k\circ f_k)(x)\colon T_k \in L^+(Y,Z), \sum_{k=1}^{n} T_k = T,\ T(f_1(x)\vee\ldots$$

$$\ldots\vee f_n(x)) = \sum_{k=1}^{n} T_k f_k(x)\Big\} \quad \text{for all, } x \in D(\varphi).$$

Remark 3.1. The most part of the formulae for calculating conjugate operators and subdifferentials in this section are given also by Kutateladze [3]. But Theorems 3.3, 3.5 and 3.6 are stated in more general conditions. Thus, the formulae in

Theorem 3.1 and 3.2 do not follow from those of Kutateladze.

<u>Theorem</u> 3.7. Let $F\colon D(F)\subset X\to Z$ be a sublinear operator, $P\subset X$, $Q\subset Y$ be convex cones, $S\in L(X,Y)$ and $y_o\in Y$. Suppose that $D(F)-P$ is a linear subspace of X and $y_o\in {}^{i}(S(D(F)\cap P)-Q$. Then

$$x\geqslant 0, Sx\geqslant y_o \Rightarrow F(x)\geqslant z_o, \tag{3.3}$$

if and only if

$$\begin{cases} T_1\in L^+(X,Z),\ T_2\in L^+(Y,Z) \quad \text{such that} \\ T_1+T_2\circ S\in\partial F(0),\ T_2y_o\geqslant z_o. \end{cases} \tag{3.4}$$

Proof. Let $T_1\in L^+(X,Z)$ and $T_2\in L^+(Y,Z)$ satisfy (3.4) and $x\geqslant 0$, $Sx\geqslant y_o$. Then

$$z_o\leq T_2y_o\leq T_2\circ Sx\leq T_1x+T_2\circ Sx\leq F(x),$$

so that (3.3) holds.

Suppose now that (3.3) holds. To show that (3.4) is verified, let us consider $\phi\colon D(\phi)\subset X\times Y\to Z$, $\phi(x,y)=F(x)$, where $D(\phi)=\{(x,y)\colon x\in D(F)\cap P,\ Sx\in y_o+y+Q\}$. It is clear that ϕ is convex. We have

$$P_YD(\phi)=\{y\colon\ x\in D(F)\cap P, Sx\in y_o+y+Q\}=$$
$$=S(D(F)\quad P)-Q-y_o,$$

so that $0\in{}^{i}(P_YD(\phi))$. We also have $z_o\leq\inf\{\phi(x,0)\colon (x,0)\in D(\phi)\}=\inf\{F(x)\colon x\geqslant 0, Sx\geqslant y_o\}$. Hence we can apply Theorem 2.3, so that

$$\inf\{\phi(x,0)\colon (x,0)\in D(\phi)\}=\max\{-\phi^c(0,T_2)\colon (0,T_2)\in D(\phi^c)\}.$$

Let $(0,T_2)\in D(\phi^c)$. Then

$$\begin{aligned}\phi^c(0,T_2)&=\sup\{T_2y-\phi(x,y)\colon (x,y)\in D(\phi)\}\\ &=\sup\{T_2(Sx-y_o-y)+F(x)\colon x\in D(F)\cap P, y\in Q\}\\ &=\sup\{T_2\circ Sx-F(x)\colon x\in D(F)\cap P\}+\sup\{(-T_2)y\colon\\ &\qquad y\in Q\}-T_2y_o.\end{aligned}$$

Since $D(F)\cap P$ is a cone and $T_2\circ S - F$ is positive homogeneous, it follows that $T_2\circ Sx - F(x)\leq 0$ $\forall x \in D(F)\cap P$ and $\sup\{T_2\circ Sx - F(x) : x \in D(F)\cap P\} = 0$. Analogously, we have $T_2\geqslant 0$ and $\sup\{-T_2 y : y\in Q\} = 0$. Therefore $\phi^c(0,T_2) = -T_2 y_o, T_2\geqslant 0$ and $T_2\circ Sx\leq F(x)$ $\forall x \in D(F)\cap P$. Let $I_p : P\to Z$, $I_p(x) = 0$. The last assertion is equivalent to $T_2\circ S \in \partial(F + I_p)(0)$. Since $D(F) - P$ is a linear subspace of X we can apply Theorem 3.4 to obtain $\partial(F + I_p)(0) = \partial F(0) + \partial I_p(0)$. It is obvious that $\partial I_p(0) = -L^+(X,Z)$. Hence

$$\inf\{F(x): x \geqslant 0, Sx\geqslant y_o\} = \max\{T_2 y_o: T_1 \in L^+(X,Z), T_2\in L^+(Y,Z),\ T_1 + T_2\circ S\in\partial F(0)\},$$

which shows that (3.4) holds.

Remark 3.2. Theorem 3.7 represents an analogous generalisation of the Farkas lemma to that one in Zălinescu [15], but under different conditions.

4. Applications to the Kuhn-Tucker Theorem

Suppose that Y is ordered by the convex cone P. Let $f: D(f)\subset X\to Z$ and $g: D(g)\subset X\to Y$ be convex operator. Consider the following problems:

$$(\mathcal{P})\quad \inf\{f(x):\ g(x)\leq 0\}.$$

Theorem 4.1. Suppose that $0\in {}^i(g(D(f)) + P)$. Then x_o is an optimal solution for $(\mathcal{P})$ if and only if x_o is admisible and there exists $T\geqslant 0$ such that

$$Tg(x_o) = 0 \text{and } 0 \in \partial(f + T\circ g)(x_o). \tag{4.1}$$

Proof. Suppose x_o is an optimal solution for $(\mathcal{P})$. Let $\phi: D(\phi)\subset X\times Y\to Z$, $\phi(x,y) = f(x)$, where $D(\phi) = \{(x,y): x\in D(f)\cap D(g),\ g(x)\leq y\}$. We have

$$P_Y D(\phi) = \{y:\ x\in D(f)\cap D(g),\ g(x)\leq y\} = g(D(f)) + P.$$

$$P_Y D(\phi) = \{y: \exists x \in D(f)\cap D(g),\ g(x)\leq y\} = g(D(f)) + P.$$

Hence $D \in {}^{i}P_Y D(\phi)$. Since x_o is an optimal solution for $(\mathcal{P})$, it follows that $\inf\{\phi(x,0): (x,0)\in D(\phi)\}$ exists. Thus we can apply Theorem 2.3. If follows there exists $T \in L(Y,Z)$ such that $(0,T)\in \partial\phi(x_o,0)$, i.e.,

$$Ty \leq \phi(x,y) - \phi(x_o,0) \quad \forall (x,y)\in D(\phi) \Leftrightarrow$$

$$Ty \leq f(x) - f(x_o) \quad \forall x \in D(f)\cap D(g), \quad g(x)\leq y \Leftrightarrow$$

$$Tg(x) + Ty \leq f(x) - f(x_o) \quad \forall x \in D(f)\cap D(g),\ y\in P \Rightarrow$$

$$Ty \leq 0 \text{ for all } y\in P.$$

Hence $T \leq 0$. Thus there exists $T \geqslant 0$ such that

$$f(x_o) \leq f(x) + Tg(x) \quad \forall x \in D(f)\cap D(g).$$

Taking $x = x_o$ we obtain $0 \leq Tg(x_o) \leq T0 = 0$, so that $Tg(x_o) = 0$ and

$$f(x_o) + Tg(x_o) \leq (f + T\circ g)(x) \quad \forall x \in D(f)\cap D(g) \Leftrightarrow$$

$$0 \in \partial(f + T\circ g)(x_o).$$

Conversely, suppose that x_o is admissible and there exists $T \geqslant 0$ such that $Tg(x_o) = 0$ and $0\in \partial(f + T\circ g)(x_o)$. Then for all $x\in D(f)\cap D(g)$ we have

$$f(x_o) + Tg(x_o) \leq f(x) + Tg(x),$$

so that for all $x\in D(f)\cap D(g)$ such that $g(x)\leq 0$ we have $f(x_o)\leq f(x)$. Therefore x_o is an optimal solution.

<u>Corollary</u> 4.1. If $0\in {}^{i}(g(D(f)) + P)$ and $0\in {}^{i}(D(f) - D(g))$, then x_o is an optimal solution for $(\mathcal{P})$ if and only if x_o is an admissible solution and

$$\exists T \geqslant 0 \text{ such that } Tg(x_o) = 0 \text{ and } \partial f(x_o)\cap(-\partial(T\circ g)(x_o)) \neq \emptyset.$$

Proof. Since $0\in {}^{i}(D(f) - D(g)) = {}^{i}(D(f) - D(T\circ g))$. We can apply Theorem 3.4 to obtain $\partial(f + T\circ g)(x) = \partial f(x) + \partial(T\circ g)(x)$. Now apply Theorem 4.1.

<u>Corollary</u> 4.2. Let $g: D(g)\subset X\to Y$ be a convex operator and $\tilde{g}: D(\tilde{g})\to Z$, $\tilde{g}(x) = 0$, where $D(\tilde{g}) = \{x: g(x)\leq 0\}$. If $0\in$

${}^{i}(g(X) + P)$ then

$$\partial \tilde{g}(x) = \bigcup \{\partial(T \circ g)(x) : T \geq 0, Tg(x) = 0\} \text{ for all } x \in D(\tilde{g}).$$

Proof. Let $x_o \in D(\tilde{g})$ and $T \geq 0$ such that $Tg(x_o) = 0$ and $U \in \partial(T \circ g)(x_o)$ Then

$$Ux - Ux_o \leq T \circ g(x) - T \circ g(x_o) \text{ for all } x \in D(g).$$

Thus for all $x \in D(\tilde{g})$ we have $Ux - Ux_o \leq 0 = \tilde{g}(x) - \tilde{g}(x_o)$, so that $U \in \partial \tilde{g}(x_o)$. Conversely, let $U \in \partial \tilde{g}(x_o)$. In the preceding corollary take $f = -U$; it is clear that x_o is an optimal solution for problem $(\mathcal{P})$. Consequently, there exists $T \geq 0$ such that $Tg(x_o) = 0$ and $U \in \partial(T \circ g)(x_o)$. The proof is complete

Consider now $f: D(f) \subset X \to Z$ and $g_k: D(g_k) \subset X \to Y_k$, $1 \leq k \leq n$, be convex operators, where Y_k is ordered by the convex cone P_k. Let $G_k = \{x : g_k(x) \leq 0\}$. Consider the following problem:

$$(\mathcal{P}_1) \quad \inf\{f(x) : g_k(x) \leq 0, 1 \leq k \leq n\}.$$

<u>Theorem</u> 4.2. Suppose that $D(f)$, $G_1, \ldots, G_n$ are in general position and $0 \in {}^{i}(g_k(X) + P_k)$ for all $k, 1 \leq k \leq n$. Then x_o is an optimal solution for $(\mathcal{P}_1)$ if and only if x_o is admissible and there exists $T_k \in L^+(Y_k, Z)$, $T_k g_k(x_o) = 0$, $1 \leq k \leq n$, and

$$0 \in \partial f(x_o) + \sum_{k=1}^{n} \partial (T_k \circ g_k)(x_o) \qquad (4.2)$$

Proof. Note that x_o is an optimal solution for $(\mathcal{P}_1)$ if and only if

$$0 \in \partial(f + \sum_{k=1}^{n} \tilde{g}_k)(x_o),$$

where $\tilde{g}_k : G_k \to Z$, $\tilde{g}_k(x) = 0$. For the operators f and $\tilde{g}_k$, $1 \leq k \leq n$ we can apply Theorem 3.4 so that

$$\partial(f + \sum_{k=1}^{n} \tilde{g}_k)(x) = \partial f(x) + \sum_{k=1}^{n} \partial \tilde{g}_k(x).$$

Since $0 \in {}^{i}(g_k(X) + P_k)$, we can apply Corollary 4.2 to get (4.2).

5. The continuous case

Throughout this section all vector spaces are topological vector spaces and the cone $C \subset Z$ <u>is normal</u>, i.e., there exists a base $\mathcal{W}$ of neighborhoods of the origin in Z such that

$$W = (W + C) \cap (W - C) \quad \text{for all } W \in \mathcal{W}.$$

The system of balanced neighborhoods of the origin in X is denoted by $\mathcal{V}(X)$. The operator $f: D(f) \subset X \to Y$ is said to be <u>continuous</u> at x_o if $x_o \in \text{int}D(f)$ and f is continuous in the usual sense at x_o.

<u>Theorem</u> 5.1. Let X,Z be topological vector spaces, $C \subset Z$ be a normal cone and $f: D(f) \subset X \to Z$ a convex operator. Then f is continuous at x_o if and only if

$$\exists z \in Z \ \forall W \in \mathcal{W} \ \exists V \in \mathcal{V}(X) \ \forall x \in x_o + V: f(x) \in z + W - C. \qquad (5.1)$$

Proof. Suppose that f is a continuous at $x_o \in \text{int}D(f)$. Then

$$\forall W \in \mathcal{W} \ \exists V \in \mathcal{V}(X) \ \forall x \in x_o + V: f(x) \in f(x_o) + W \Longrightarrow$$
$$\exists z = f(x_o) \ \forall W \in \mathcal{W} \ \exists V \in \mathcal{V}(X) \ \forall x \in x_o + V: f(x) \in z + W - C.$$

Let us show now the sufficiency of (5.1). Without loss of generality we can suppose that $x_o = 0$ and $f(0) = 0$ (otherwise take $\tilde{f}(x) = f(x_o + x) - f(x_o)$). Let $W \in \mathcal{W}$; since $\mathcal{W}$ is a base of neighborhoods of the origin, it follows there exists $W_1 \in \mathcal{W}$ such that $[0,1] \cdot W_1 + [0,1] \cdot W_1 \subset W \cap (-W)$. Using (5.1) $\exists V \in \mathcal{V}(X) \ \forall x \in V$: $f(x) \in z + W_1 - C$. Since $W_1 \in \mathcal{V}(Z)$ $\exists \lambda_o \in (0,1] \ \forall \lambda \in [0, \lambda_o]: \lambda z \in W_1$. Let $\lambda \in [0, \lambda_o]$ and $x \in V$. We have

$$f(\lambda x) = f(\lambda x + (1 - \lambda)0) \leq \lambda f(x) + (1 - \lambda) f(0) = \lambda f(x),$$

so that

$$f(\lambda x) \in \lambda f(x) - C \subset \lambda z + \lambda W_1 - \lambda C \subset W_1 + [0,1] \cdot W_1 - C$$
$$\subset W \cap (-W) - C \subset W - C.$$

On the other hand

$$0 = f(0) = f\left(\tfrac{1}{2}\lambda x + \tfrac{1}{2}(-\lambda x)\right) \leq \tfrac{1}{2} f(\lambda x) + \tfrac{1}{2} f(-\lambda x),$$

so that

$$f(\lambda x) \in -f(-\lambda x) + C \subset -(W \cap (-W) - C) + C = W \cap (-W) + C +$$
$$+ C \subset W + C,$$

since $-x \in V$. Consequently [illegible] $\lambda \in [0, \lambda_0]$ $\forall x \in V$: $f(\lambda x) \in (W-C) \cap (W+C) = W$, Hence f is continuous at x_o.

Corollary 5.1. Let $f: D(f) \subset X \to Z$ be a convex operator. If f is continuous at x_o then f is continuous on int $D(f)$.

Proof. Let $x \in$ int$D(f)$. Since the map $\rho \to x_o + \rho(x - x_o)$ is continuous, it follows that there is $\rho > 1$ such that $u = x_o + \rho(x - x_o) \in$ int $D(f)$. Let $W \in \mathcal{W}$; $\exists W_1 \in \mathcal{W}$ such that $[0,1] \cdot W_1 \subset W$. Since f is continuous at $x_o \in$ int$D(f)$, $\exists V_1 \in \mathcal{V}(X)$ $\forall x \in x_o + V_1$: $f(x) \in f(x_o) + W_1$. Let $V_2 = (1 - \frac{1}{\rho})(x_o + V_1) + \frac{1}{\rho}(x_o + \rho(x - x_o)) =$ $= (1 - \frac{1}{\rho})V_1 + \frac{\rho - 1}{\rho} x_o + \frac{1}{\rho} x_o + x - x_o = x + (1 - \frac{1}{\rho})V_1$. Hence V_2 is a neighborhood of x. Let $x' \in V_2$; hence $x' = (1 - \frac{1}{\rho})v + \frac{1}{\rho} u$ for some $v \in x_o + V_1$. We have $f(x') \leq (1 - \frac{1}{\rho})f(v) + \frac{1}{\rho} f(u)$, so that

$$f(x') \in (1 - \tfrac{1}{\rho})f(v) + \tfrac{1}{\rho}f(u) - C \subset (1 - \tfrac{1}{\rho})(f(x_o) + W_1) + \tfrac{1}{\rho}f(u) - C$$
$$= (1 - \tfrac{1}{\rho})f(x_o) + \tfrac{1}{\rho}f(u) + (1 - \tfrac{1}{\rho})W_1 - C \subset (1 - \tfrac{1}{\rho})f(x_o) + \tfrac{1}{\rho}f(u) + W - C.$$

Consequently, $\exists z = (1 - \frac{1}{\rho})f(x_o) + \frac{1}{\rho}f(u)$ $\forall W \in \mathcal{W}$ $\exists V = (1 - \frac{1}{\rho})V_1$ $\forall x' \in x + V$: $f(x') \in z + W - C$, which completes the proof.

Corollary 5.2. (i) Let $f: D(f) \subset X \to Z$ be a convex operator Suppose there are some $z \in Z$ and $V \in \mathcal{V}(X)$ such that $f(x) \leq z$ for all $x \in x_o + V$. Then f is continuous at x_o, and consequently, on int$D(f)$.

(ii) If int$C \neq \emptyset$ the above condition is also necessary, i.e.

if f is continuous at $x_o \in \underline{\mathrm{int}}D(f)$, then there are some $z \in Z$ and $V \in \mathcal{V}(X)$ such that $f(x) \leq z$ for all $x \in x_o + V$.

Proof. (i) suppose that $\exists z_o \in Z, V_o \in \mathcal{V}(X)\ \forall x \in x_o + V$: $f(x) \leq z_o$, or equivalently $\exists z_o \in Z, V_o \in \mathcal{V}(X)\ \forall x \in x_o + V_o$: $f(x) \in z_o - C$. Then $\exists z = z_o\ \forall W \in \mathcal{W}\ \exists V = V_o\ \forall x \in x_o + V$: $f(x) \in z + W - C$, so that, by Theorem 5.1, f is continuous at x_o, and by Corollary 5.1, f is continuous on $\underline{\mathrm{int}}D(f)$.

(ii) Let $z_1 \in \underline{\mathrm{int}}C$; then $z_1 - C \in \mathcal{V}(Z)$. Since f is continuous at x_o, $\exists V_o \in \mathcal{V}(X)\ \forall x \in x_o + V_o$: $f(x) \in f(x_o) + z_1 - C$. Therefore, $\exists z_o = z_1 + f(x_o)\ \exists V_o \in \mathcal{V}(X)\ \forall x \in x_o + V_o$: $f(x) \leq z_o$.

Corollary 5.3. Let $f: D(f) \subset X \to Z$ be an operator, continuous at $x_o \in \underline{\mathrm{int}}D(f)$ and A convex set of epigraph type, such that $A \supset \underline{\mathrm{epi}}\ f$. If $\inf\{z: (x_o, z) \in A\}$ exists, then $\varphi_A: P_X A \to Z$, $\varphi_A(x) = \inf\{z: (x,z) \in A\}$ is continuous at x_o, and consequently on $\underline{\mathrm{int}}(P_X A) \supset \underline{\mathrm{int}}\ (\underline{\mathrm{co}}D(f))$.

Proof. We have

$$\underline{\mathrm{co}}D(f) = \underline{\mathrm{co}}P_X(\underline{\mathrm{epi}}\ f) = P_X(\underline{\mathrm{co}}(\underline{\mathrm{epi}}\ f)) \subset P_X A.$$

Since $x_o \in \underline{\mathrm{int}}D(f)$, it follows that $x_o \in \underline{\mathrm{int}}\ P_X A$. By hypothesis, $\inf\{z: (x_o, z) \in A\}$ exists, so that, according to Theorem 1.2, $\varphi_A(x)$ exists for all $x \in P_X A$, and by Theorem 1.1 (ii), φ_A is convex. But f is continuous at x_o, so that $\forall W \in \mathcal{W}\ \exists V \in \mathcal{V}(X)\ \forall x \in x_o + V$: $f(x) \in f(x_o) + W$. Since $\underline{\mathrm{epi}}\ f \subset A$, we have $\varphi_A(x) \leq f(x)$ for $x \in D(f)$, so that $\forall x \in x_o + V$: $\varphi_A(x) \in f(x_o) + W - C$. Applying Theorem 5.1, φ_A is continuous at x_o, and consequently on $\underline{\mathrm{int}}(P_X A)$.

Denote by B(X,Y) the space of continuous linear operator between X and Y.

Remark 5.1. Let $f: D(f) \subset X \to Z$. If f is continuous at some $x_o \in \underline{\mathrm{int}}D(f)$, then $D(f^c) \subset B(X,Z)$, and consequently $\partial f(x) \subset B(X, Z)$ for every $x \in D(f)$.

Indeed, if $T \in D(f^c)$ then $f^c(T) + f(x) \geq Tx$ for every $x \in D(f)$. Corollary 5.3 implies that T is continuous at x_o, and consequently on X.

To obtain the continuous version of Theorem 2.3 it is sufficient to stengthen the condition $0 \in {}^i(P_Y(\underline{co}D(\phi)))$ in such a way to obtain that φ_{coA} be continuous at 0, where $A = P_{Y \times Z}(\underline{epi}\,\phi)$. Taking into account Theorem 5.1, such that a condition is the following:

$$\exists z \in Z \quad \forall W \in \mathcal{W} \quad \exists V \in \mathcal{V}(Y) \quad \forall y \in V \quad \exists x \in X: \quad \phi(y,x) \in z + W + C. \tag{5.2}$$

To hold (5.2) it is sufficient to have

$$\exists z \in Z \quad \exists V \in \mathcal{V}(Y) \quad \forall y \in V \quad \exists x \in X: \ \phi(x,y) \leq z, \tag{5.3}$$

or,

$$\exists x_o \in X \ \text{ such that } \ \phi(x_o, \cdot) \text{ is continuous at } 0. \tag{5.4}$$

To obtain the continuous version of the other results in Section 2 we must take $S \in B(X,Y)$ and rewrite conditions (5.2)-(5-4) for the corresponding operator ϕ.

In Section 3 to assure that φ_{coA} is continuous at 0 for every $\tilde{\phi}$, $\tilde{\phi}(x,y) = \phi(x,y) - Sx$, where $S \in B(X,Z)$, we must use a condition of the following type:

$$\begin{cases} \exists z \in Z \ \forall W \in \mathcal{W} \quad \forall U \in \mathcal{V}(X) \quad \exists V \in \mathcal{V}(Y) \quad \exists x \in X \ \forall y \in V \\ \exists x' \in x + U: \ \phi(x',y) \in z + W - C. \end{cases} \tag{5.2'}$$

It is obvious that a necessary condition for having φ_{coA} continuous at 0 is $0 \in (P_Y(\underline{co}D(\phi)))^i$. The following result shows that this condition is sufficient for (5.1) to hold in ruther general cases.

<u>Theorem</u> 5.2. Let X, Z be Fréchet spaces and Y a barrelled space, and $\phi: D(\phi) \subset X \times Y \to Z$ a convex operator with closed

epigraph. If $0 \in (P_Y D(\phi))^i$ and $\inf \{\phi(x,0):(x,0) \in D(\phi)\}$ exists, then $\varphi(y) = \inf \{\phi(x,y):(x,y) \in D(\phi)\}$ exists for every $y \in P_Y D(\phi)$ and φ is continuous at 0. Moreover (5.2') holds.

Note that in our condition, by Theorem 3.3, $\varphi(y)$ exists for every $y \in P_Y D(\phi)$. Thus we must only show that (5.2') holds. To do this, we shall use the following theorem of Ursescu stated in somewhat more general setting in [12] .

<u>Theorem</u> . Let X be a Fréchet space and Y be a barrelled space. Let $F: X \to Y$ be a closed convex multifunction (i.e. <u>graph</u> F $= \{(x,y): y \in F(x)\}$ is a closed convex set. If $(\underline{\text{Range}}\ F)^i \neq \emptyset$ then

$$F(x) \cap (\underline{\text{Range}}\ F)^i \subset \underline{\text{int}}\ F(x+U),$$

$$F(x) \subset \underline{\text{lin}}\ \underline{\text{int}}\ F(x+U)$$

for all $x \in D(F)$ and $U \in \vartheta(X)$, where <u>lin</u> A denote the algebraic closure of A.

Proof. of Theorem 5.2. Consider the multifunction $F: X \times Z \to Y$ defined by $F(x,z) = \{y: (x,y,z) \in \underline{\text{epi}}\,\phi\} = \{y: \phi(x,y) \le z\}$. Hence <u>graph</u> $F = \{(x,z,y): (x,y,z) \in \underline{\text{epi}}\,\phi\}$. This constitutes a reorientation of <u>epi</u> ϕ, so that F is a closed convex multifuncti on. <u>Range</u> $F = P_Y(\underline{\text{graph}}\ F) = P_Y(\underline{\text{epi}}\,\phi) = P_Y(D(\phi))$. Hence $0 \in (\underline{\text{Range}}\ F)^i$. Let $(x_o, z_o) \in X \times Z$ such that $0 \in F(x_o, z_o) \Leftrightarrow \phi(x_o, 0) \le z_o$. The above theorem shows that $\forall W \in \vartheta\ \forall U \in \vartheta(X)\ \exists V \in \vartheta(Y)$

such that $V \subset F((x_o, z_o) + U \times W)$, or equivalently

$$\forall W \in \vartheta, U \in \vartheta(X)\ \exists V \in \vartheta(Y)\ \forall y \in V\ \exists x \in x_o + U, z \in z_o + W: \phi(x,y) \le z \Leftrightarrow$$

$$\forall W \in \vartheta, U \in \vartheta(X)\ \exists V \in \vartheta(Y)\ \forall y \in V\ \exists x \in x_o + U: \phi(x,y) \in z_o + W - C.$$

Hence (5.2') holds, and consequently φ is continuous at 0. The proof is complete.

Theorem 5.2 represents a generalization of Corollary 2.1 in [5] .

We want to remark that formula (4.3) holds with $T \in B(Y,Z)$ if $0 \in \underline{\text{int}}\,(g(X)+P)$. If $\text{int}P \neq \emptyset$ then the above condition is equivalent to

$$\exists\, x_o \in X \text{ such that } g(x_o) \in -\underline{\text{int}}P. \tag{5.5}$$

Indeed, let $y_o \in \underline{\text{int}}P$ and suppose that $0 \in \underline{\text{int}}\,(g(X)+P)$, then $\exists \lambda_o > 0$ such that $-\lambda_o y_o \in g(X) + P$, so that $\exists x_o \in X, p \in P$ such that $-\lambda_o y_o = g(x_o) + p$. Hence $-g(x_o) = \lambda_o y_o + p \in \underline{\text{int}}P$, since $P + \underline{\text{int}}P = \underline{\text{int}}P$.

The following theorem represents the continuous version of Theorem 4.2 and a generalization of Theorem 6 in [13]. In this case X, Y_k, Z, are topological vector spaces, the cone $P_k \subset Y_k$ is closed and convex, $\underline{\text{int}}\,P_k \neq \emptyset$, $1 \leq k \leq n$ and $C \subset Z$ is closed.

<u>Theorem</u> 5.3. Let $f: D(f) \subset X \to Z$, $g_k: D(g_k) \subset X \to Y_k$ be convex operators such that g_k is continuous on $\underline{\text{int}D}(g_k) \neq \emptyset$. Suppose there is some $x_o \in D(f)$ such that $g_k(x_o) \in -\underline{\text{int}}P_k$, $1 \leq k \leq n$. Then $\bar{x}$ is an optimal solution of the problem

$$\inf \{ f(x):\ g_k(x) \leq 0,\ 1 \leq k \leq n \},$$

if and only if $\bar{x}$ is admissible and $\exists T_k \in B^+(Y_k, Z)$, $T_k g_k(x) = 0$ $1 \leq k \leq n$ such that

$$0 \in \partial f(\bar{x}) + \sum_{k=1}^{n} \partial (T_k \circ g_k)(\bar{x}) \quad .$$

Moreover, if g_k are Gateaux differentiable at $\bar{x}$, then $\bar{x}$ is an optimal solution for the above problem if and only if $\bar{x}$ is admissible and $\exists T_k \in B^+(Y_k, Z)$, $T_k g_k(\bar{x}) = 0$, $1 \leq k \leq n$ such that

$$\sum_{k=1}^{n} T_k \circ g_k' \ (\bar{x}) \in -\partial f(\bar{x}) .$$

Proof. We must only show that if $g: D(g) \subset X \to Y$ is comvex and Gateaux differentiable at $\bar{x}$ ($\in \underline{\text{int}} D(g)$), then

$$\partial g(\bar{x}) = \{g'(\bar{x}) + S:\ S \in L(X,Y),\ \underline{\text{Range}}\ S \subset P \cap (-P)\}. \quad (5.6)$$

Since g is Gateaux differentiable at $\bar{x}$ it follows that $\bar{x} \in \underline{\text{int}} D(g)$. Let $x \in X$ be such that $\bar{x} + x \in D(g)$; there is some $t_0 > 0$ such that $0 < t \leq t_0$ implies $\bar{x} - tx \in D(g)$. We have for $t \in (0, t_0]$

$$g(\bar{x}) = g\left(\frac{1}{1+t}(\bar{x} + x) + \frac{1}{1+t}(\bar{x} - tx)\right)$$

$$\leq \frac{t}{1+t}\, g(\bar{x} + x) + \frac{1}{1+t}\, g(\bar{x} - tx)$$

$$\Longleftrightarrow g(\bar{x}) + tg(\bar{x}) \leq tg(\bar{x} + x) + g(\bar{x} - tx)$$

$$\Longleftrightarrow g(\bar{x}) - g(\bar{x} - tx) \leq t(g(\bar{x} + x) - g(\bar{x}))$$

$$\Longleftrightarrow \frac{g(\bar{x}) - g(\bar{x} - tx)}{t} \leq g(\bar{x} + x) - g(\bar{x})$$

$$\Longleftrightarrow \frac{g(\bar{x} - tx) - g(\bar{x})}{-t} \leq g(\bar{x} + x) - g(\bar{x}).$$

Letting $t \downarrow 0$ one gets

$$g'(\bar{x})(x) \leq g(\bar{x} + x) - g(\bar{x}) \text{ for all } x \in X \text{ such that } \bar{x} + x \in$$

$D(g)$ or, equivalently

$$g'(\bar{x})(x - \bar{x}) \leq g(x) - g(\bar{x}) \quad \text{for all } x \in D(g).$$

Hence $g'(\bar{x}) \in \partial g(\bar{x})$. It easily follows that $\{g'(\bar{x}) + S:\ S \in L(X,Y),\ \underline{\text{Range}}\ S \subset P \cap (-P)\} \subset \partial g(\bar{x})$. Let now $S \in L(X,Y)$, $S \in \partial g(\bar{x})$; we have

$$Sx \leq g(\bar{x} + x) - g(\bar{x}) \quad \text{for all } x \in D(g) - \bar{x}.$$

Since $\bar{x} \in \underline{\text{int}}\ D(g)$, there is some $t_x > 0$ such that $\bar{x} + tx \in D(g)$ for $t \in (0, t_x)$. For such t we have

$$S(tx) \leq g(\bar{x} + tx) - g(\bar{x}) \Longleftrightarrow Sx \leq (g(\bar{x} + tx) - g(\bar{x}))/t.$$

Letting $t \downarrow 0$, taking into account that P is closed, we obtain

$$Sx \leq g'(\bar{x})(x) \quad \text{for all } x \in X,$$

so that $(S - g'(\bar{x}))(x) \in P \cap (-P)$ for all $x \in X$. Therefore $S \in \{g'(\bar{x}) + U:\ U \in L(X,Y),\ \underline{\text{Range}}\ U \subset P \cap (-P)\}$. Now, if $T \in B^+(Y,Z)$, then $T \circ g$ is convex and $(T \circ g)'(\bar{x}) = T \circ g'(\bar{x})$. Since $C \cap (-C) = \{0\}$, it follows from (5.6) that $\partial (T \circ g)(\bar{x}) = \{T \circ g'(\bar{x})\}$. The proof is complete.

After this paper was elaborated we taked knowledge of the Thera's paper [11] . Some of our formulae to calculate subdifferentials result from that of Thera. We also remark that the Thera's formulae to calculate ε- subdifferentials ($\varepsilon \geq 0$) follow from our results. Let us show that, in the conditions of Theorem 3.3, we have

$$\partial_{\varepsilon}\varphi(x) = \{T \circ S: T \in \partial_{\varepsilon} g(Sx)\} \text{ for all } x \in D(\varphi).$$

It is clear that $\{T \circ S: T \in \partial_{\varepsilon} g(Sx)\} \subset \partial_{\varepsilon}\varphi(x)$. Let show the converse inclusion. Let $U \in L(X,Z)$, $U \in \partial_{\varepsilon}\varphi(x)$; this means that

$$Ux - Ux_o \leq \varphi(x) - \varphi(x_o) + \varepsilon \quad \forall x \in D(\varphi) \Longleftrightarrow$$

$$\varphi(x_o) - Ux_{1o} - \varepsilon \leq \varphi(x) - Ux \quad \forall x \in D(\varphi) \Longleftrightarrow$$

$$\varphi(x_o) - Ux_o - \varepsilon \leq \inf \{\varphi(x) - Ux: x \in D(\varphi)\}$$

$$= \inf \{g(Sx) - Ux: Sx \in D(g)\} .$$

Taking in Theorem 2.5 $f = -U$, it follows that $\exists T \in D(g^c)$ such that $T \circ S = U$ and

$$\varphi(x_o) - Ux_o - \varepsilon \leq -g^c(T) \Longleftrightarrow$$

$$g(Sx_o) + g^c(T) - T \circ S(x_o) \leq \varepsilon \Longleftrightarrow$$

$$Ty - g(y) + g(Sx_o) - T \circ Sx_o \leq \varepsilon \quad \forall y \in D(g) \Longleftrightarrow$$

$$Ty - T \circ Sx_o \leq g(y) - g(Sx_o) + \varepsilon \quad \forall y \in D(g) \Longleftrightarrow$$

$$T \in \partial_{\varepsilon} g(Sx_o).$$

In the paper of Thera is also stated Theorem 5.2 (without proof).

R e f e r e n c e s

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