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by

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1. Introduction

The problem of synthetizing open-loop optimal controls into an optimal feedback (or closed-loop) optimal control was formulated from the very beginning of the modern control theory ([1],[2],[17] , [26] , etc.) but the number of papers dedicated to this subject is comparatively small and the results obtained up to date seem far less satisfactory than those concerning open-loop optimal controls.

In what follows we try to review and comment some of the recent results and new trends in the study of the time-optimal feedback control.

There seem to be two main aspects of the theory of time-optimal feedback control that are considered in the works dedicated to this subject. The first consists in the study of its properties and existence; the second aspect concerns its stability to perturbations, strongly connected with generalized solutions of discontinuous differential equations. As will be seen in the sequel, both aspects generate very interesting and very difficult mathematical problems that are now only partially solved.

Since, as it is convincingly proved in [16] and [29] , the time-optimal feedback control defines a discontinuous differential equation whose Carathéodory solutions must be the optimal trajectories, some very special hypotheses should be made in order to make a "mathematical object" out of the time-optimal feedback control.

Such hypotheses were firstly made by V.G.Boltyanskii ([1], [2]); essentially, Boltyanskii's "regular synthesis" means a "stratification" of the phase space into differentiable manifolds called cells (very similar to the stratification considered in Differential Topology and Global Analysis, see [22]) such that the corresponding discontinuous vector field is everywhere tangent to some cells and everywhere transversal to the other cells, each of his (Carathéodory) integral curves passing successively through a finite number of cells until reaches the final point.

The results of P.Brunovsky ([6]) and H.Sussman ([29]) concerning the existence of such an object are briefly described and some suggestions for future research are made in section 2.

The next two sections are devoted to the survey of some results concerning stability to perturbations of discontinuous differential equations defined by the time-optimal feedback control of linear systems.

In the last section three problems motivating the need for stability results on the time-optimal feedback control are prezented: a linear system with slowly varying coefficients, a singularly perturbed linear system and the construction of a bang-bang state estimator for an input-output linear control system.

2. Properties and existence of the time-optimal feedback control

Let $U \subset R^p$ be a nonempty set called <u>the control space</u>, let $f: R \times R^n \times U \longrightarrow R^n$ be a continuous mapping that defines the parametrized differential equation:

(2.1) $\frac{dx}{dt} = f(t,x,u)$

and let $x_1 \in R^n$ be a given point called the <u>target</u> or the <u>final point</u>. For any $t_o \in R$, $x_o \in R^n$ we define the set $\mathcal{U}(t_o,x_o)$ of admissible controls with respect to (t_o,x_o) to be the set of all measurable bounded mappings $u(.):[t_o,t_1] \longrightarrow U$ such that the solution $\varphi(.;t_o,x_o;u(.)):[t_o,t_1] \longrightarrow R^n$ of the initial value problem:

(2.2) $\frac{dx}{dt} = f(t,x,u(t))$, $x(t_o) = x_o$

has the following properties:

(2.3) $\varphi(t_1;t_o,x_o;u(.)) = x_1$ and $\varphi(t;t_o,x_o;u(.)) \neq x_1$ for any $t \in [t_o,t_1)$.

For every admissible control $u(.) \in \mathcal{U}(t_o,x_o)$ we define the duration of steering x_o to x_1 as follows:

(2.4) $T(t_o,x_o,u(.)) = t_1 - t_o$

and we say that $\tilde{u}(.) \in \mathcal{U}(t_o,x_o)$ is a <u>time-optimal control</u> with respect to (t_o,x_o) if:

(2.5) $T(t_o,x_o,\tilde{u}(.)) \leqslant T(t_o,x_o,u(.))$ for any $u(.) \in \mathcal{U}(t_o,x_o)$.

The corresponding solution $\varphi(.;t_o,x_o;\tilde{u}(.))$ of (2.2) will be called <u>the optimal trajectory</u> of the optimal control $\tilde{u}(.)$

If $D \subset R \times R^n$ denotes the set of all points (t_o,x_o) for which a time-optimal control exists then $T(.,.):D \longrightarrow R_+$ defined by:

(2.6) $T(t_o,x_o) = T(t_o,x_o,\tilde{u}(.))$ if $\tilde{u}(.) \in \mathcal{U}(t_o,x_o)$ is optimal, is called <u>the minimal-time function</u>.

Obviously, the set D is contained into the controllability set, $\mathcal{C}(x_1)$, of all the points $(t_o,x_o) \in R \times R^n$ for which the set $\mathcal{U}(t_o,x_o)$ of admissible controls is not empty.

The problem of finding the time-optimal control may be considered only for a determined initial point $(t_o,x_o) \in R \times R^n$

but this is not allways the case. As it is suggested by some practical applications, the above time-optimal control problem may be interpreted as follows: an evolution process described by the equation (2.1) has $x_1 \in R^n$ as the equilibruium state and may be controlled using the parameter $u \in U$. Some perturbations deviate the system from the desired state x_1 and the control parameter $u \in U$ should be used to bring the system back to the state x_1 as quickly as possible. From this point of view a "regulator" that chooses a control $v(t,x) \in U$ at each point $(t,x) \in RxR^n$ need be devised such that the initial value problem:

(2.7) $$\frac{dx}{dt} = \overline{f}(t,x) \;,\; x(t_o) = x_o \;,\; \overline{f}(t,x) = f(t,x,v(t,x))$$

has a (Carathéodory) solution $\varphi(.;t_o,x_o)$ satisfying (2.3) and such that $t \longmapsto v(t,\varphi(t;t_o,x_o))$ is a time-optimal control with respect to (t_o,x_o).

We say that the mapping $v(.,.):D \longrightarrow U$ having these properties is a <u>time-optimal feedback control</u> for the system (2.1)

If for any point $(t,x) \in D$ there exists a unique open-loop optimal control $\tilde{u}(.;t,x) \in \mathcal{U}(t,x)$ which is piecewise continuous (as it happens for instance in the case of normal linear systems) then the time-optimal feedback control is uniquely defined by:

(2.8) $$v(t,x) = \tilde{u}(t;t,x)$$

but it is not yet proved that any Carathéodory solution of the equation (2.7) is an optimal trajectory unless some very special hypotheses about the "global picture" of the optimal trajectories are made.

If the open-loop optimal controls are not unique then the problem of the optimal feedback (closed-loop) control seems still more difficult.

As already mentioned in the Introduction, the special hypotheses that make a mathematical object out of the time-optimal feedback control are contained in the definition of the so called "regular synthesis" introduced by V.G.Boltyanskii ([1],[2]) or some of its modifications ([6], [11] , [20] , [27] , [29]).

For the sake of simplicity we assume in what follows that the vector field in (2.1) is autonomous so (2.1) takes the form:

(2.9) $$\frac{dx}{dt} = f(x,u)$$

(we may always write (2.1) in the autonomous form: $x'=f(x^o,x,u)$, $x^{o\prime}=1$; besides, the non-autonomous case is treated in [11] , [20] and [27]).

It is easy to see that in this case both, the time-optimal feedback control and the minimal-time function are also autonomous: we assume that the set $G\subset R^n$ of all the points $x\in R^n$ for which there exists a time-optimal (open-loop) control is an open neighbourhood of the target x_1.

In this case, instead of the system (2.7), we have to consider the following **autonomous** discontinuous differential equation:

(2.10) $$\frac{dx}{dt} = \overline{f}(x) \quad \text{where} \quad \overline{f}(x) = f(x,v(x)),\ x\in G\subset R^n.$$

We recall the definition of Boltyanskii's regular synthesis trying to distinguish four main characteristics:

Definition 2.1 ([1],[2])

The mapping $v(.):G\subset R^n \longrightarrow U\subset R^p$ is said to define a regular synthesis for the time-optimal control problem to the target $x_1\in Int(G)$ of the system (2.9) if the properties A) - D) listed bellow hold:

A) There exists a subset $\mathcal{N}\subset G$ and a partition of the set $G\setminus\mathcal{N}$ into a family $\mathcal{G}$ of connected differentiable manifolds called cells satisfying the following properties:

A.1. The family $\mathcal{S}$ is locally finite and admits the following partitions: $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ and if $\mathcal{S}^{(k)}$ denotes the set of k-dimensional cells, k = 0,1,... n, then $\mathcal{S}_1^{(k)} = \mathcal{S}_1^{(k)} \cup \mathcal{S}_2^{(k)}$, $\mathcal{S}_1^{(o)} = \mathcal{S}_2^{(n)} = \emptyset$ and $\{x_1\} \in \mathcal{S}^{(o)} = \mathcal{S}_2^{(o)}$.

A.2. For any cell $S \in \mathcal{S}$, the restriction mapp, $v_S(.)$, of v(.) to S is differentiable and may be extended to a differentiable mapping on a neighbourhood of S in G;

A.3. There exists a mapping $\Pi(.) : \mathcal{S}_1 \to \mathcal{S}$ such that if $S \in \mathcal{S}_1^{(k)}$ then $\Pi(S) \in \mathcal{S}^{(k-1)}$, the vector field $\bar{f}(.)$ defined by (2.10) is everywhere tangent to any cell $S \in \mathcal{S}_1$ and from each point $x \in S$ there starts a unique Carathéodory solution $\varphi(.;x)$ of (2.10) that leaves S after a finite time and reaches $\Pi(S)$ transversally.

A.4. There exists a mapping $\Sigma(.) : \mathcal{S}_2 \to \mathcal{S}_1$ such that if $S \in \mathcal{S}_2^{(k)}$ then $\Sigma(S) \in \mathcal{S}_1^{(k+1)}$, the restriction mapp of v(.) to $S \cup \Sigma(S)$ is differentiable and if $S \neq \{x_1\}$ then from every point $x \in S$ there starts a unique Carathéodory solution, $\varphi(.;x)$ of (2.10) for which there exists $\varepsilon > 0$ such that $\varphi(t;x) \in \Sigma(S)$ for $t \in (0, \varepsilon)$.

A.5. From any point in the "indifference" set $\mathcal{N}$ there may start several Carathédory solutions of (2.10) enterring cells from $\mathcal{S}_1$ and intersecting $\mathcal{N}$ only at the starting point.

A.6. Any solution of (2.10) starting at a point $x \in G$ reaches x_1 in a finite time, T(x), intersecting only a finite number of cells.

B) For any $x \in \mathcal{N}$, the time of reaching x_1 is the same along any trajectory of (2.10) starting at x , so the function $T(.) : G \to R_+$ representing the duration of reaching the target x_1 is well defined. The time-function $T(.)$ is assumed to be continuous.

C) If for any $k = 0,1,..\ n-1$ we denote:

(2.11) $\quad P^{(k)} = \bigcup\{S\ ;\ S \in \mathcal{S}^{(o)} \cup \mathcal{S}^{(1)} \ldots \cup \mathcal{S}^{(k)}\},\ P^{(n)} = G,$

(2.12) $\quad M = P^{(n-1)} \cup \mathcal{N}$

then the set M has the property in Boltyanskii's fundamental lemma: if $u(.) \in \mathcal{U}(x_o)$, $u(.):[0,t_1] \longrightarrow U$ is an admissible control then in any neighbourhood of x_o there exists a point y such that the solution $\varphi(.;y,u(.))$ of the problem:

(2.13 $\quad \frac{dx}{dt} = f(x,u(t))\ ,\ x(0) = y$

is defined on $[0,t_1]$ and intersects M for at most finitely many values of $t \in [0,t_1]$.

D) Any Carathéodory solution, $\varphi(.;x)$, of (2.10) satisfies Pontryagin's Maximum Principle with the admissible control $\tilde{u}(.) \in \mathcal{U}(x)$ defined by:

(2.14) $\quad \tilde{u}(t) = v(\varphi(t;x))$:

if we define $\mathcal{H}(.):R^n x R^n x U \longrightarrow R$ and $H:R^n x R^n \longrightarrow R$ by:

(2.15) $\quad \mathcal{H}(p,x,u) = \langle p,f(x,u)\rangle - 1$

(2,16) $\quad H(p,x) = \sup\{\mathcal{H}(p,x,u)\ ;\ u \in U\}$

then there exists an absolutely continuous mapping $p(.):[0,T(x)] \longrightarrow R^n$ such that:

(2.17) $\quad \frac{dp}{dt}(t) = -\frac{\partial\mathcal{H}}{\partial x}(p(t),\varphi(t;x),u(t))$ a.e. on $[0,T(x)]$

(2.18) $\quad H(p(t),\varphi(t;x)) = \mathcal{H}(p(t),\varphi(t;x),u(t))$ a.e. on $[0,T(x)]$

We recall that V.G.Boltyanskii ([1], [2]) proved that property C) in the above definition is satisfied if the set M defined by (2.11), (2.12) is a "piecewise smooth" set of dimension less than n and that the Carathéodory solutions of (2.10) ("marked trajectories") are time-optimal trajectories.

In [6] , P.Brunovsky proved that property C) in Definition 2.1 holds if the closure of the set M admits an analytic Whitney stratification; P.Brunovsky proved also the first existence theorem for the regular synthesis: <u>if</u> $U \subset R^p$ <u>is a compact convex polyhedron with the vertices</u>: $w_1, w_2, \ldots w_m \in R^p$:

(2.19) $\quad U = co\{w_1, w_2, \ldots w_m\}$

<u>if the system is linear</u>, i.e. (2.7) <u>is of the form</u>:

(2.20) $\quad \frac{dx}{dt} = Ax + Bu$

<u>and satisfies the following normality condition</u>: <u>the vectors</u> $B(w_i - w_j), AB(w_i - w_j), \ldots A^{n-1}B(w_i - w_j)$ <u>are linearly independent for any</u> $i \neq j$ <u>then the time-optimal feedback control with the target</u> $x_1 = 0 \in R^n$ <u>for the system</u> (2.20) <u>defines a regular synthesis with some additional properties</u>.

H.Sussman proved in [29] a very general theorem stating the existence of a non-optimal discontinuous feedback control for an analytic system of the form (2.7) on a real analytic manifold; as in [6], in [29] some results concerning stratifications by subanalytic sets are essentially used.

In an attempt to prove more general existence theorems for regular synthesis and possibly give algorithmical methods for its construction in particular cases several aspects should be considered:

1. It seems reasonable to try to construct the "stratification" $\mathcal{S}$ in Definition 2.1 from trajectories satisfying Pontryagin's Maximum Principle (2.15) - (2.18);

2. It may not be easy to prove the continuity of the time-function $T(.): G \to R_+$ in property B) but it seems possible to use the results in [28] to define a more general synthesis that does not require the continuity of the function $T(.)$;

3. It could be possible to find easier verifiable conditions for property C) using either Brunovsky's condition in [6] that the closure of the set M admits a Whitney stratification or Boltyanskii's condition assuming that M is a piecewise smooth set of dimension less than n.

It turns out that the main problem is to use Pontryagin's Maximum Principle to construct a stratification $\mathcal{S}$ of the phase space that satisfies the axioms A.1 - A.6 in Definition 21

On the other hand, as Boltyanskii himself pointed out in [3], Pontryagin's Maximum Principle (2.15) - (2.18) is difficult to use because it involves two different but simultaneous operations: the maximization in (2.18) and the "integration" of the differential equations (2.13) and (2.17). In [3], the separation of the two operations is suggested under the very restrictive hypothesis that for every $p \in R^n$ the function $H(p,.)$ defined by (2.16) is differentiable.

The introduction in [8] of the generalized gradient for locally-lipschitzian functions makes Boltyanskii's suggestion workable under hypotheses similar to those that insure the existence of the optimal controls.

Let us assume that $U \subset R^p$ is compact and let $f: R^n x U \to R^n$ that defines (2.9) be of class C^1 with respect to the x-variable i.e. $(x,u) \longmapsto \frac{\partial f}{\partial x}(x,u)$ is continuous.

From Theorem 2.1 in [8] it follows that under these hypotheses the "true Hamiltonian" ([9]), $H(.,.)$, defined by (2.16) is a locally-lipschitzian function and its generalized gradient is given by:

$$(2.21) \qquad H(p,x) = \overline{co} \left\{ (f(x,u), \frac{\partial \mathcal{H}}{\partial x}(p,x,u)) \ ; \ u \in V(p,x) \right\}$$

where the (upper- smicontinuous) multifunction $V(.,.): R^n x R^n \to \mathcal{P}(U)$ is defined by:

(2.22) $V(p,x) = \{u \in U \; ; \; H(p,x) = \mathcal{H}(p,x,u)\}$

Definition 2.2 ([3], [25])

The admissible control $u(.) \in \mathcal{U}(x_0)$, $u(.):[0,t_1] \to U$ is said to satsify the Support Principle if there exists an absolutely continuous mapping $p(.): [0,t_1] \to R^n$ such that the mapping $t \mapsto (\varphi(t;x_0,u(.)),p(t))$ is a solution of the "Hamiltonian inclusion" ([9]) :

(2.23) $(\dot{x},-\dot{p}) \in \partial H(p,x)$

and satisfies the condition:

(2.24) $u(t) \in V(p(t),\varphi(t;x_0,u(.)))$ a.e. on $[0,t_1]$

It is easy to prove ([25]) that under the above hypotheses any admissible control satisfying Pontryagin's Maximum Principle (2.15) - (2.18) satisfies also the Support Principle (2.23)-(2.24). The converse statement is true if the following convexity property, similar to that ensuring the existence of the optimal open-loop controls hold:

(2.25) $H(p,x) = \{ (f(x,u), \frac{\partial \mathcal{H}}{\partial x}(p,x,u)) \; ; \; u \in V(p,x)\}$

If the condition (2.25) is satisfied then for any absolutely continuous solution $(\varphi(.),p(.))$ of the Hamiltonian inclusion (2.23) there exists, via Filippov's lemma, a measurable mapping $u(.)$ satisfying (2.24), 2.13), (2.17) and therefore the Maximum Principle.

It seems likely that if (2.25) is not verified then for any solution of (2.23) there exists a corresponding "relaxed" open-loop optimal control.

In the case (2.25) is satisfied it seems reasonable to expect that the "backward" integration of (2.23), taking into account some properties of the "adjoint variable", p, ([11], [20], [21], [27]) may lead to the stratification in the Definition 2.1 so one may obtain more general and constructive existence theorems for the regular synthesis.

3. Generalized solutions for discontinuous differential equations

Let us suppose that there exists a time-optimal feedback control $v(.):G\subset R^n \longrightarrow U$ for the system (2.9) and that it defines a regular synthesis (Definition 2.1).

As it is remarked in [16], in practice, the value $v(x)$ of the control corresponding to the state x is determined after making a measurement of the state and if this measurement is in error, say $x+\xi(t)$ is measured rather than x, the governing equation of motion will have the form:

$$\frac{dx}{dt} = \bar{f}(x + \xi(t)) \tag{3.1}$$

rather than (2.10).

Likewise, as it is shown by the problems treated bellow as applications, in many cases the equation of motion may have the form:

$$\frac{dx}{dt} = \bar{f}(x) + \xi(t) \tag{3.2}$$

where $\xi(.):[0,\infty) \longrightarrow R^n$ is measurable and "small" in some norm (usually L_∞ or L_1 - norms).

It is intuitively obvious that the inner perturbation $\xi(.)$ in (3.1) as well as the "outer" one in (3.2) destroys the stratification structure in Definition 2.1. Moreover, as the following example shows, the systems (3.1) and (3.2) may not have any Carathéodory solution from some points in the phase space.

Example 3.1 ([4], [23])

Let us consider the time-optimal control problem to the origin $(x_1,y_1) = (0,0)\in R^2$ for the system:

$$\begin{cases} \dot{x} = -x + u^1 \\ \dot{y} = u^2 \end{cases} \tag{3.3}$$

where $u = (u^1, u^2) \in U = \mathrm{co}\{u_{(1)}, u_{(2)}, u_{(3)}\}$, $u_{(1)} = (1,0)$, $u_{(2)} = (-1/2, 1)$, $u_{(3)} = (-1,-1/2)$.

Applying Pontyagin's Maximum Principle (2.15) - (2.18) equivalent, in this case, to the Support Principle (2.23)-(2.24) one may prove that in the half-plane $\{(x,y) \; ; \; x < 0\}$ the time-optimal feedback control is given by:

$$(3.4) \qquad v(x) = \begin{cases} u_{(2)} & \text{if } x < 0,\ y < 0 \\ u_{(1)} & \text{if } x < 0,\ y = 0 \\ u_{(3)} & \text{if } x < 0,\ y > 0 \end{cases}$$

One may prove also that this time-optimal feedback control defines a regular synthesis in the sense of Definition 2.1

Let us assume now that a small outer perturbation of the form $\xi(t) = (\varepsilon \sin t, \varepsilon \cos t)$ where $\varepsilon \in (0,1/2)$ occurs in the system (2.10) corresponding to (3.3).

It is easy to see now that the perturbed system of the form (3.2) does not have a Carathédory solution starting from a point $(x_o, 0)$ if $x_o < 0$: the y-component of the state variable (x,y) for $x < 0$ satisfies the (un-perturbed) differential equation (corresponding to (2.10)):

$$(3.5) \qquad \dot{y} = \begin{cases} 1 & \text{if } y < 0 \\ 0 & \text{if } y = 0 \\ -1/2 & \text{if } y > 0 \end{cases}$$

(for which $y(t) \equiv 0$ is a Carathéodory solution) but the corresponding perturbed equation satisfied by the same component is the following:

$$(3.6) \qquad y = \begin{cases} 1 + \varepsilon \cos t & \text{if } y < 0 \\ \varepsilon \cos t & \text{if } y = 0 \\ -1/2 + \varepsilon \cos t & \text{if } y > 0 \end{cases}$$

which does not have any Carathéodory solution starting at the point $y(0) = 0$ (see also the Example 2.5 in [13]).

It is well known that Carathéodory solutions do not make sense for discontinuous differential equations unless some rather strong hypotheses, similar to those in Definition 2.1, are made. There are already several definitions for generalized solutions for discontinuous differential equations (see [10] and [13]) but it seems that there are many reasons (including the minimality property in [31]) to consider Filippov's generalized solutions as the most suitable at least for the problems in Control Theory.

The results on the regular synthesis discussed in the preceding section seemed to show that Carathéodory solutions are still suitable for optimal feedback control but the possibility of perturbations as in (3.1) or (3.2) and the example 3.1 compels us to consider some sort of generalized solutions.

In what follows we will consider only Filippov solutions (called F-solutions) though in [13] it is shown that Krassovskii or Hermes solutions may be as useful.

We recall first the definition of Filippov solutions:

Definition 3.2 ([10])

An absolutely continuous mapping $\varphi(.):[t_o,t_1] \to R^n$ will be called an F-solution of the equation (2.7) if it satisfies a.e. on $[t_o,t_1]$ the differential inclusion:

(3.7) $\dot{x} \in F_{\bar{f}}(t,x)$

where the set valued mapping $F_{\bar{f}}(.,.):RxR^n \to \mathcal{P}(R^n)$ is defined by:

(3.8) $F_{\bar{f}}(t,x) = \bigcap_{\varepsilon>0} \bigcap_{\mu(A)=0} \overline{co}\, \bar{f}(t,B_\varepsilon(x)\setminus A)$.

where $B_\varepsilon(x)$ denotes the ball of radius $\varepsilon > 0$ centered at x, $\mu(.)$ denotes the Lebesgue measure and $\overline{co}\, M$ denotes the closed convex hull of the set M.

The problem to be studied now is whether the F-solutions of systems of the form (3.1) or (3.2) are close to the optimal trajectories of (2.9) (which are the Carathéodory solutions of the equation (2.10)) if the perturbations $\xi(.)$ are sufficiently small in some sense.

A strong reason to consider the F-solutions more suitable than other generalized solutions is the existence in [10] of very good theorems concerning the existence, uniqueness, continuous dependence on initial data and the right-hand side, etc.

For the problem we are concerned with, the main tool is the following very strong "closure theorem" in [10]:

Theorem 3.3 ([10], Theorem 3)

Let $f, f_1, g_1, f_2, g_2, \ldots : R\times R^n \longrightarrow R^n$ be measurable mappings that define the differential equations:

(3.9) $\dot{x} = f(t,x)$

(3.10) $\dot{x} = f_k(t,x) + g_k(t,x)$, $k=1,2,\ldots$

and have the following properties:

(i) for any compact subset $D \subset R\times R^n$ there exist the measurable bounded real-valued functions $\eta(.), \eta_k(.), \psi_k(.)$ such that:

(3.11) $\|f(t,x)\| \leqslant \eta(t)$ a.e. on D

(3.12) $\|f_k(t,x)\| \leqslant \eta_k(t)$ a.e. on D, $k=1,2,\ldots$

(3.13) $\|g_k(t,x)\| \leqslant \psi_k(t)$ a.e. on D, $k=1,2,\ldots$

(3.14) $\int_a^b \psi_k(t)dt \longrightarrow 0$ as $k \longrightarrow \infty$, $D = [a,b] \times G \subset R\times R^n$,

(ii) there exists a sequence $r_k \longrightarrow 0$ as $k \longrightarrow \infty, r_k > 0$ such that:

(3.15) $f_k(t,x) \in \bigcap_{\mu(A)=0} \overline{co}\, f(t, B_{r_k}(x) \setminus A)$ a.e. on $D, k=1,2,\ldots$

Then if $\varphi_k(.) : [t_0, t_1] \subset [a,b] \longrightarrow R^n$, $k=1,2,\ldots$ is an F-solution of (3.10) remaining in D then there exists a subsequence $\{\varphi_{k_m}\}$ of $\{\varphi_k\}$ that converges uniformly to an F-solution of (3.9); in particular, if $\{\varphi_k(.)\}$ converges uniformly to $\varphi(.)$ then $\varphi(.)$ is an F-solution of (3.9).

Using this theorem one may easily derive results stating the stability or instability on a finite time-interval of the time-optimal feedback control to outer perturbations of the form (3.2).; for control problems it seems reasonable to require stability on the interval $[0,\infty)$ but this kind of results do not follow directly from the Theorem 3.3 above. The same is true for results stating the stability to inner perturbations of the form (3.1).

There are several kinds of stability to perturbations of the time-optimal feedback control that are studied in the literature:

Definition 3.4

The time-optimal feedback control $v(.):G\subset R^n \to U$ of the system (2.9) is said to be $L_1(L_\infty)$ - stable to outer (respectively to inner) perturbations if for any compact neighbourhood $G_0\subset G$ of the target x_1 and for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x\in G_0$ and any measurable bounded function $\xi(.):[0,T(x)]\to R^n$ satisfying:

(3.16) $\int_0^{T(x)}\|\xi(t)\|dt \leq \delta$ (respectively, $\|\xi(t)\|\leq\delta$ a.e.on $[0,T(x)]$)

any solution $\varphi(.;x;\xi(.))$ of (3.2) (respectively, of (3.1) for outer perturbations) satisfies:

(3.17) $\|\varphi(t;x) - \varphi(t;x;\xi(.))\|\leq\varepsilon$ fot any $t\in[0,T(x)]$

where $\varphi(.;x)$ is the Carathéodory solution through $(0,x)$ of (2.10) (the time-optimal trajectory through x) and $T(.):G\to R_+$ is the minimal-time function.

Definition 3.5

The time-optimal feedback control v() of the system (2.9) is said to be L_1^{τ} ($L\infty$) - strongly stable to outer (respectively to inner) perturbations if for any compact neighbourhood $G_0 \subset G$ of the target x_1 and for any τ, $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable bounded mapping $\xi(.) : [0, \infty) \to R^n$ satisfying:

$$(3.18) \quad \int_{k\tau}^{(k+1)\tau} \|\xi(t)\| dt \leq \delta, \quad k=0,1,2,\ldots \text{ (respectively } \|\xi(t)\| < \delta \text{ a.e. on } [0, T(x)])$$

for any $x \in G_0$, any F-solution $\varphi(.;x;\xi(.))$ of (3.2) (respectively, of (3.1) for outer perturbations) satisfies:

$$(3.19) \quad \|x_1 - \varphi(t;x;\xi(.))\| \leq \varepsilon \text{ for any } t \geq T(x) .$$

Definition 3.6 ([16])

The time-optimal feedback control v(.) of the system (2.9) is said to be stable with respect to the measurements if for any compact neighbourhood $G_0 \subset G$ of the target x_1, for any T_0, $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\xi(.) : [0, T_0] \to R^n$ is a measurable mapping satisfying: $\|\xi(.)\|_\infty \leq \delta$ and such that a Carathéodory solution $\varphi(.;x;\xi(.))$ of (3.1) exists on $[0, T_0]$ then $\|\varphi(.;x) - \varphi(.;x;\xi(.))\|_\infty \leq \varepsilon$.

For obvious reasons we introduce the following:

Condition 3.7: every F-solution of (2.10) is also a Carathéodory solution of (2.10).

Condition 3.8: through every point of G there passes a unique to the right F-solution of (2.10).

It is easy to prove that if the time-optimal feedback control v(.) of the system (2.9) is stable with respect to outer perturbations in the sense of Definition 3.4 then Conditions 3.7 and 3.8 are satisfied. On the other hand it is difficult to prove that stability to measurement in Definition 3.6 implies Condition 3.7 (see [16]).

From the Theorem 3.3 above it follows that if $\bar{f}(.):G \to R^n$ defined by (2.10) is measurable and bounded on compact subsets then the time-optimal feedback control is stable with respect to outer perturbations in the sense of Definition 3.4 if and only if Conditions 3.7 and 3.8 hold.

The stability to inner perturbations in Definitions 3.4 and 3.5 was not yet considered in the literature. In [13] it is proved that the outer perturbation $\xi(.)$ in (3.2) becomes an inner perturbation by the change of variable: $y = x - \int_0^t \xi(s)ds$ and therefore any absolutely continuous inner perturbation becomes a (measurable) outer perturbation by the reverse change of variable (if the vector field $\bar{f}(.)$ is continuous then the inner perturbation $\xi(.)$ defines the outer perturbation $\bar{f}(x+\xi(t)) - \bar{f}(x)$ but this is not the case if $\bar{f}(.)$ is discontinuous).

It seems though that Filippov's closure theorem 3.3 could be used to obtain the same type of stability results for inner perturbations as for the outer ones.

In order to apply Theorem 3.3 to systems of the form (3.1) it is sufficient to prove the following statement: if $\bar{f}(.)$ defined by (2.10) is measurable and locally essentially bounded then for any sequence of measurable bounded mappings $\xi_k(.):[0,T_o] \to R^n$ satisfying: $\|\xi_k(.)\|_\infty = \rho_k \to 0$ as $k \to \infty$, there exists $r_k > 0$, $r_k \to 0$ as $k \to \infty$ such that:

$$(3.20) \qquad \bar{f}(x+\xi_k(t)) \in \bigcap_{\mu(A)=0} \overline{co}\, \bar{f}(B_{r_k}(x) \setminus A) \quad \text{a.e. on } [0,T_o]\times G$$

Since according to [30], if $f(.)$ is locally essentially bounded $\bar{f}(x) \in F_{\bar{f}}(x)$ a.e. on G (where Filippov multifunction $F_{\bar{f}}(.)$ is defined by (3.8)) it follows that there exists a

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a null-set $A_0 \subset G$ such that $\bar{f}(x) \in \bigcap_{\mu(A)=0} \overline{co}\, \bar{f}(B_\varepsilon(x) \setminus A)$ for any $\varepsilon > 0$ and $x \in G \setminus A_0$. The statement above follows now from the fact that the set $\{(t,x)\ ;\ x + \xi_k(t) \in A_0\}$ is a null-set in $[0,T_0] \times G$ since in this case $\bar{f}(x + \xi_k(t)) \in$ $\in \bigcap_{\mu(A)=0} \overline{co}\, \bar{f}(B_{\varepsilon_k}(x + \xi_k(t)) \setminus A) \subset \bigcap \overline{co}\, \bar{f}(B_{\varepsilon_k+\rho_k}(x) \setminus A)$, $\gamma_k = \varepsilon_k + \rho_k \longrightarrow 0$

In fact, Filippov's closure Theorem 3.3 may be used to obtain results concerning stability or instability of the time-optimal feedback control with respect to inner and outer perturbations acting simultaneously.

In order to prove results concerning the strong stability in Definition 3.5 more properties than those in Conditions 3.7 and 3.8 are needed ([5], [14], [15]) : for instance, a necessary condition for strong stability in Definition 3.5 is that for any $x \in G$ the mapping defined by: $\varphi(t;x) = x_1$ for $t \geq T(x)$ is a unique to the right F-solution of (2.10).

While the results concerning the stability to perturbations of the equation (2.10) may be obtained more or less directly from the general theorems in the theory of discontinuous differential equations, the difficult problem that remains to be solved is to characterize the control systems (2.9) for which Conditions 3.7 and 3.8, necessary for stability, are satisfied. As we shall see in the next section, this problem is now only partially solved for some particular cases of linear systems of the form (2.20).

4. The case of linear stationary control systems

The simplest (and also, the best known) time-optimal control problem is that of steering any point $x \in R^n$ to the origin, $x_1 = 0 \in R^n$, in minimal time, through the linear system (2.20). The normality condition mentioned in Section 2 ensures the existence and uniqueness of a time-optimal open-loop control for any point x in a neighbourhood G of the origin, the continuity of the minimal-time function $T(.):G \longrightarrow R_+$, the fact that Pontryagin's Maximum Principle is a necessary and sufficient optimality condition, the existence of a unique time-optimal feedback control $\bar{v}(.):G \longrightarrow U$, the fact that it defines a regular synthesis ([6]) and some other properties ([2], [4],[5],[7], [12], [13], [17]-[19], [23], [26], [31], etc.).

H.Hermes was the first to remark in [16] that there are optimal control problems for which Condition 3.7 is not satisfied and therefore the optimal feedback control is not stable with respect to measurements (Definition 3.6) as well as in the sense of Definitions 3.4 or 3.5.

The first results characterizing classes of time-optimal control systems for which Conditions 3.7 and 3.8 are satisfied were proved by P.Brunovsky in [4]:

Theorem 4.1 ([4])

Let us consider the time-optimal control problem to the origin $x_1 = 0 \in R^2$ *for the normal linear system* (2.20) *in the case* $n = 2$.

Then Conditions 3.7 and 3.8 are verified if and only if does not exists a vertex w *of the polyhedron* U *such that its polar cone* $H(w,U) = \{ p \in R^2 ; \langle p,w \rangle \geq \langle p,u \rangle$ *for any* $u \in U \}$ *contains the eigenvector of* $-A^*$ *corresponding to its largest eigenvalue but does not contain the other eigenvector.*

From the results in [6] and [7] it follows the corresponding theorem for linear systems with one-dimensional inputs:

Theorem 4.2 ([6],[7])

The time-optimal feedback control satisfies Conditions 3.7 and 3.8, for the normal linear systems (2.20) *in the case* $U = [-1,1]$ *and* $B = b \in R^n$.

A weaker version of this theorem (in which Conditions 3.7 and 3.8 are satisfied only in a neighbourhood of the origin) was proved by other methods in [13].

Finally, two other theorems concerning Conditions 3.7 and 3.8 were recently proved in [19] for "minimally controllable" linear systems of the form:

(4.1) $\dot{x} = Ax + b^1u_1 + b^2u_2, \quad x \in R^3, \ |u_i| \leq 1, \ i=1,2.$

The conditions in [19] are expressed in terms of the determinants:

(4.2) $d(0)=\det[b^1, Ab^1, A^2b^1], \quad d(1)=\det[b^1, Ab^1, b^2],$
$d(2)=\det[b^1, b^2, Ab^2], \quad d(3)=\det[b^2, Ab^2, A^2b^2]$

and of their signs:

(4.3) $\delta(j) = \mathrm{SGN}\ d(j), \quad j = 0,1,2,3.$

<u>Theorem 4.3</u> ([19])

<u>The time-optimal feedback control for the strictly normal system</u> (4.1) <u>satisfies Conditions 3.7 and 3.8 iff</u> $\delta(0)\cdot\delta(2) = \delta(1)\cdot\delta(3) = 1$ <u>and</u> $\det[b^1, b^2, \exp(-At)Bu_{(2)}] \neq 0$ <u>where</u> $B = [b^1, b^2]$ <u>and</u> $u_{(2)} = (\delta(1), \delta(2))$.

<u>Theorem 4.4</u> ([19])

<u>The time-optimal feedback control for the minimally controllably system</u> (4.1) <u>satisfies Conditions 3.7 and 3.8 iff</u> $d(0).d(2) \geq 0$, $d(1).d(3) \geq 0$ <u>and either</u> $d(1) = 0$ or $d(2)=0$ <u>but not both or</u> $d(1) \neq 0$, $d(2) \neq 0$ <u>and</u> $\det[b^1, b^2, \exp(-At)Bu_{(2)}] \neq 0$.

Most of the above results were proved as preliminary steps in proofs of theorems stating stability to perturbations of the time-optimal feedback control ([4], [13], [19]).

We recall now the stability results proved so far:

Theorem 4.5 ([5])

The time-optimal feedback control of the normal system (2.20) in the case $n = 2$ for which Conditions 3.7 and 3.8 are satisfied (see Theorem 4.1) is L_∞ - strongly stable with respect to outer perturbations in the sense of Definition 3.5.

In the case the dimension of the set BU is 2 in [5] is proved a stronger theorem stating that the solutions of (3.2) reach the target point $x_1=0$ at a time close to the optimal one and remain there afterwards.

Theorem 4.6 ([13], [19])

The time-optimal feedback control of any linear system satisfying the hypotheses in Theorems 4.2-4.4 is stable with respect to the measurements (Definition 3.6) in a neighbourhood of the origin.

Theorem 4.7 ([14])

The time-optimal feedback control of a normal linear system (2.20) satisfying Conditions 3.7 and 3.8) is L_1^τ - strongly stable to outer perturbations (Definition 3.5).

From the above survey it follows that there remains much work to be done to characterize the normal linear systems for which Conditions 3.7 and 3.8 are verified and therefore are stable to perturbations in the sense of Definitions 3.4 - 3.6.

On the other hand, from Theorems 4.1,4.3,4.4 it follows that stability to perturbations of the time-optimal feedback control (in fact, Conditions 3.7 and 3.8) is not a "generic" property even for normal linear systems, i.e. there exists a significant class of systems that are not stable to perturbations.

It seems reasonable to suggest that at least for the unstable cases one should try to find suboptimal feedback controls that are stable to perturbations.

5. Applications

We review shortly the results obtained in [14], [15] and [24] concerning some control problems that lead in a natural way to perturbed systems of the form (3.2) and so stability results are needed.

5.1. Linear systems with slowly varying coefficients ([24])

Let us consider the time-optimal control problem to the origin $x_1=0$ for the system:

(5.1) $\dot{x} = A(\varepsilon t,\varepsilon)x + B(\varepsilon t,\varepsilon)u \;,\; x\in R^n,\; u\in U\subset R^p$

where $\varepsilon > 0$ is a small constant, $A(.,.), B(.,.)$ are continuous matrix-valued functions and $U\subset R^p$ is a compact convex polyhedron

In practice one takes $\varepsilon = 0$ and the time-optimal feedback control $v_0(.):G\subset R^n \longrightarrow U$ of the stationary system (2.20) where $A = A(0,0)$ and $B= B(0,0)$ is used to control the original system (5.1).

Therefore the dynamics of the controlled system is defined by the equation:

(5.2) $\dot{x} = A(\varepsilon t,\varepsilon)x + B(\varepsilon t,\varepsilon)v_0(x)$

which may be written also as follows:

(5.3) $\dot{x} = A(0,0)x + B(0,0)v_0(x) + A(\varepsilon t,\varepsilon)-A(0,0)\; x +$

$+ B(\varepsilon t,\varepsilon)-B(0,0)\; v_0(x)$

Since every Filippov solution of the system (5.3) is a (Filippov) solution of a perturbed system of the form (3.2) these solutions will be "close" to the corresponding optimal trajectories of the system (2.20) (so they will reach a certain neighbourhood of the origin) if the time-optimal feedback control $v_o(.)$ is stable to outer perturbations in the sense of Definition 3.4 and if for any Filippov solution $\varphi(.)$ of (5.3) the "perturbation" $t \longmapsto [A(\varepsilon t,\varepsilon)-A(0,0)]\varphi(t)$ + + $[B(\varepsilon t,\varepsilon)-B(0,0)]\, v_o(\varphi(t))$ is small enough.

Using the continuity with respect to ε of the minimal--time function of the system (5.1) it is proved that if $v_o(.)$ satisfies Conditions 3.7 and 3.8 and if $\varepsilon > 0$ is small enough then $v_o(.)$ is a suboptimal feedback control for the system (5.1) in the sense that any Filippov solution of the system (5.3) reaches a certain neighbourhood of the origin in a time close to the minimal one. However, these solutions remain on $[0,\infty)$ in the neighbourhood of the origin only if the functions $t \longrightarrow A(\varepsilon t,\varepsilon)$, $B(\varepsilon t,\varepsilon)$ have a certain boundedness property ([24]).

5.2. Singularly perturbed linear systems ([14])

Let us consider the time-optimal control problem to the origin $(0,0)\in R^n \times R^m$ for the system:

$$\begin{cases} \dot{x} = A_{11}x + A_{12}y + B_1u & (5.4) \\ \varepsilon\dot{y} = A_{21}x + A_{22}y + B_2u & (5.5) \end{cases} \quad , x\in R^n, y\in R^m,\ u\in U\subset R^p,$$

where $\varepsilon > 0$ is a small constant.

As in the preceding example, in practice one takes $\varepsilon = 0$ and, assuming that A_{22} is an inveritble matrix, one gets the "reduced system" of the form (2.20) where $A = A_{11}-A_{12}A_{22}^{-1}A_{21}$ and $B = B_1- A_{12}A_{22}^{-1}B_2$.

The time-optimal feedback control of the reduced system, $v_o(.): G\subset R^n \longrightarrow U$ is used to control the original system so one gets the differential system:

$$\begin{cases} \dot{x} = A_{11}x + A_{12}y + B_1v_o(x) & (5.6) \\ \varepsilon\dot{y} = A_{21}x + A_{22}y + B_2v_o(x) & (5.7) \end{cases}$$

where (5.6) may be written as follows:

(5.8) $\dot{x} = Ax + Bv_o(x) - \left[A - A_{11}\right]x + A_{12}y - \left[B - B_1\right]v_o(x)$

Therefore, the first component (the "slow" one) of the state variable, (x,y), will reach a certain neighbourhood of the origin in a time close to the minimal one if the time-optimal feedback control of the reduced system, $v_o(.)$, is stable to outer perturbations in the sense of Definitions 3.4 or 3.5, and if for any (Filippov) solution, $(\varphi(.), \psi(.))$, of the system (5.6)-(5.7), the perturbation $t \longrightarrow \left[A - A_{11}\right]\varphi(t) - A_{12}\psi(t) + \left[B - B_1\right]v_o(\varphi(t))$ is small enough. In [14] it is proved, essentially, that if $v_o(.)$ is L_1 - strongly stable to outer perturbations and if $\varepsilon > 0$ is small enough then the first component of any Filippov solution of the system (5.6)-(5.7) starting in a certain compact subset $G_o \subset G$ reaches a certain neighbourhood of the origin in a time close to the minimal one and remains there afterwards. Moreover, it is proved that if $B_2 = 0$ then the second component of the state variable has the same property.

5.3. Bang-bang state estimator for input-output control systems ([15])

Let us consider the input-output control system:

(5.9) $\dot{x} = Ax + Bu \quad , x \in R^n, \; u \in U \subset R^p$

(5.10) $y = C^* x \quad , y \in R^m$

and let us consider the problem of finding a state estimator (observer) for this system i.e. to find a matrix K and a mapping $v(.) : R^n \longrightarrow U$ such that $A - KC^*$ is a stable matrix and such that any solution of the system:

(5.11) $\dot{x} = Ax + Bv(z)$

(5.12) $\dot{z} = Az + KC^*(x-z) + Bv(z)$

reaches a certain neighbourhood of the origin as quickly as possible.

The second component, z, in the system (5.11)-(5.12) is an estimation of the state variable, x and can be computed from the equation (5.12) written as:

$$(5.13) \qquad \dot{z} = Az + K(y - C^{*}z) + Bv(z)$$

where the output y is available from observations.

In the classical engineering control theory a linear feedback control of the form $v(z) = -Lz$ is taken such that $A - BL$ is a stable matrix. If the control space $U \subset R^{p}$ is a bounded set then the linear regulator above will not be suitable for states outside a limited neighbourhood of the origin so we have to choose another kind of feedback control.

If we take v(.) to be the time-optimal feedback control of the system (5.9) and if we assume that it is stable to perturbations in the sense discussed in Section 3 then this feedback control will define a good state estimator provided we choose the matrix K such that for any Filippov solution $(\varphi(.),\psi(.))$ of (5.11)-(5.12) the "perturbation" $t \longmapsto KC^{*}(\varphi(t)-\psi(t))$ in (5.12) (which is an equation of the form (3.2)) is small enough ([15]).

The examples examined above show that there is still need for results concerning stability to perturbations of the time-optimal feedback control (so that we can avoid the unstable cases) though the main problem seems to be that of replacing altogether the time-optimal feedback control by a stable sub-optimal feedback control.

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