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STATISTICAL METHODS FOR PARABOLIC EQUATIONS

by

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0. Introduction

The present work, based upon the Monte Carlo method and the statistical study of time series, develops original numerical methods for solving the mixed problem for parabolic partial differential equations.

In order not to overload the formulae and the demonstrations we consider only equations of the type

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f$$

$$u(0,t) = 0 = u(1,t) \qquad 0 \leqslant x \leqslant 1$$

$$u(x,0) = u_o(x) \qquad 0 \leqslant t \leqslant T \qquad (0.1)$$

The first chapter presents original Monte Carlo numerical technics for solving the (0.1) problem.

Synthesizing the results of [7],[12] we present in the first part of § 1.1. various families of schemas with differences, in order then to built up, on the basis of technics used in [3],[4],[9], a probability model which joins to each family of schema a Markov absorbent process (there are given the states and the transition probabilities)

On the ground of these processes we built up unbiased estimators of the (0.1) finite differences problem solution.

We study the stability conditions of the estimators'mean and the convergence to the analytical solution.

The § 1.2. paragraph introduces iterative schemas of sequential Monte Carlo type. For each family of schemas is given the iterative method and is studied the variance and the convergence speed.

The § 1.3. paragraph studies the efficiency of the original Monte Carlo method from three points of view : the selection volume neccessary in obtaining a given error ; the average number of operations in order to estimate the finite differences solution ; minimizing methods for variance estimators.

The last paragraph of the chapter (§ 1.4.) gives two algorithms, EPMC and EPMCS, for estimating the (0.1) problem solution in a given point (i_o, j_o) using the Monte Carlo method built up in § 1.1., respectively for improving the estimator variance of the (0.1) problem solution using the iterative Monte Carlo method from § 1.2.

The second chapter develops original numerical methods for solving the (0.1) problem, based upon the statistical study of the time series.

After the transformation of the (0.1) problem, using the lines method (see [7]), into a bilocal problem of nonhomogeneous linear differential equation of the second degree, some filtration techniques for time series, enabled us to develop in § 2.1., starting from the results of [5], [6], [10], an original method for solving the bilocal problem for linear differential equations of the second degree with a stochastic nonhomogeneous term.

We also demonstrate that the solution of this problem is a time series and that by imposing some restrictions upon the mean of the nonhomogeneous stochastic term, the mean of the time serie converges towards the (0.1) problem solution.

In § 2.2. we study the efficiency of the differential filters method for time series in solving problem (0.1), from the point of view of the operations number needed to estimate the solution.

The last section of the chapter (§ 2.3.) presents the EPSD original algorithm for estimating the (0.1) problem soluton in a given (i_o, j_o) point, based on the time series differential filters method introduced above.

The 3rd chapter of the paper presents comparatively the results of the methods given in the first and the second chapter applied to a hydrogeology problem.

It refers to the forecasting of the behaviour during the exploitation of the hydrothermomineral fissural sytem of the cretaceous limestones of Băile Felix - 1 Mai (Romania).

Based on the mathematical - hydrogeologic model from [1], I elaborated a package of programmes for solving the planes parabolics equations sytem with nonzeros initial and limit conditions of Dirichlet type which describes the conservative and nonconservative hydraulic, dispersional and thermic diffusivity of a hydrothermomineral fissural system.

The paper ends by an interesting comparision, from the point of view of time computer, memory and accuracy, between the methods presented in the first two chapters of the paper.

Chapter I

§ 1.1. The probabilistic model. Unbiased estimators. Moments. Errors. Stability.

In the following we consider only nets with cartesian coordinates with constant steps.

Let be the field

$$D = (0 \leqslant x \leqslant 1 , 0 \leqslant t \leqslant T)$$

and the net in D

$$R_{hk} = \{ (ih, jk)/i=0,1,..,I , j=0,1,..,J \}$$

with the steps $h = 1/I$ and $k = T/J$.

We denote by $u(i,j)$ the exacte value of the (0.1) problem solution in the node (ih, jk) of the net R_{hk} and by $U(i,j)$ the value in point (i,j) of a net function U, defined on R_{hk}.

Derivatives $\partial u/\partial t$ and $\partial^2 u/\partial x^2$ are expressed as function of the centered differences $d_t, d_x^2, d_{\overset{\circ}{t}}$ and the following families of schemas with differences depending on a parameter "a" are considered

a) the scheme with differences of two levels with six points

$$\left.\begin{array}{lll} d_tU(i,j-1) = ad_x^2U(i,j) + (1-a)d_x^2U(i,j-1) + F(i,j-1) & & \\ U(0,j) = 0 = U(I,j) & 0 \leqslant i \leqslant I & \\ U(i,0) = u_o(ih) & 1 \leqslant j \leqslant J & \end{array}\right\} \quad (1.1.1)$$

b) the scheme with differences with three levels

b.1) symmetric of nine points

$$d_{\overset{\circ}{t}}U(i,j-1) = ad_x^2U(i,j) + (1-2a)d_x^2U(i,j-1) + ad_x^2U(i,j-2) + F(i,j-1)$$

$$0 \leqslant i \leqslant I, \quad 2 \leqslant j \leqslant J \qquad (1.1.2)$$

b.2) nonsymmetric of five points

$$d_tU(i,j-1) + akd_t^2U(i,j-1) = d_x^2U(i,j) + F(i,j-1)$$

$$0 \leqslant i \leqslant I, \quad 2 \leqslant j \leqslant J \qquad (1.1.3)$$

both, with limit and initial conditions

$$U(0,j) = 0 = U(I,j)$$

$$U(i,0) = u_o(ih)$$

$$U(i,1) = u_o(ih) + k(u_o''(ih) + f(ih,0)).$$

We denote by

$$d_tU(i,j) = (U(i,j+1)-U(i,j))/k , \quad d_{\overset{\circ}{t}}U(i,j) = (U(i,j+1)-U(i,j-1))/2k$$

$$d_x^2U(i,j) = (U(i+1,j)-2U(i,j)+U(i-1,j))/h^2$$

$d_t^2 U(i,j) = (U(i,j+1)-2U(i,j)+U(i,j-1))/k^2$, $F(i,j)=f(ih,1{,}5jk)$.

Let be a Markov process $\{z_m\}_{m=0,1,2,\ldots}$ with the set of states

$$S = \{0,1,\ldots,I\} \times \{0,1,\ldots,J\}$$

defined on the field of probability $(E,\mathcal{K},P)$ with

$$E = S \times S \times \ldots$$

$$\mathcal{K} = \mathcal{P}(S) \otimes \mathcal{P}(S) \otimes \ldots$$

$$P(\{s_0\}\times\{s_1\}\times \ldots \times\{s_m\}\times S \times S \times \ldots) = \text{init.prob.}(s_0)\, q_{s_0 s_1} \ldots q_{s_{m-1} s_m}$$

where $(q_{s_i s_j})$ are the transition probabilities of the process (the existence of P is given by the theorem of probabilities on product space) and "init.prob." is the initial probability of state s_0.

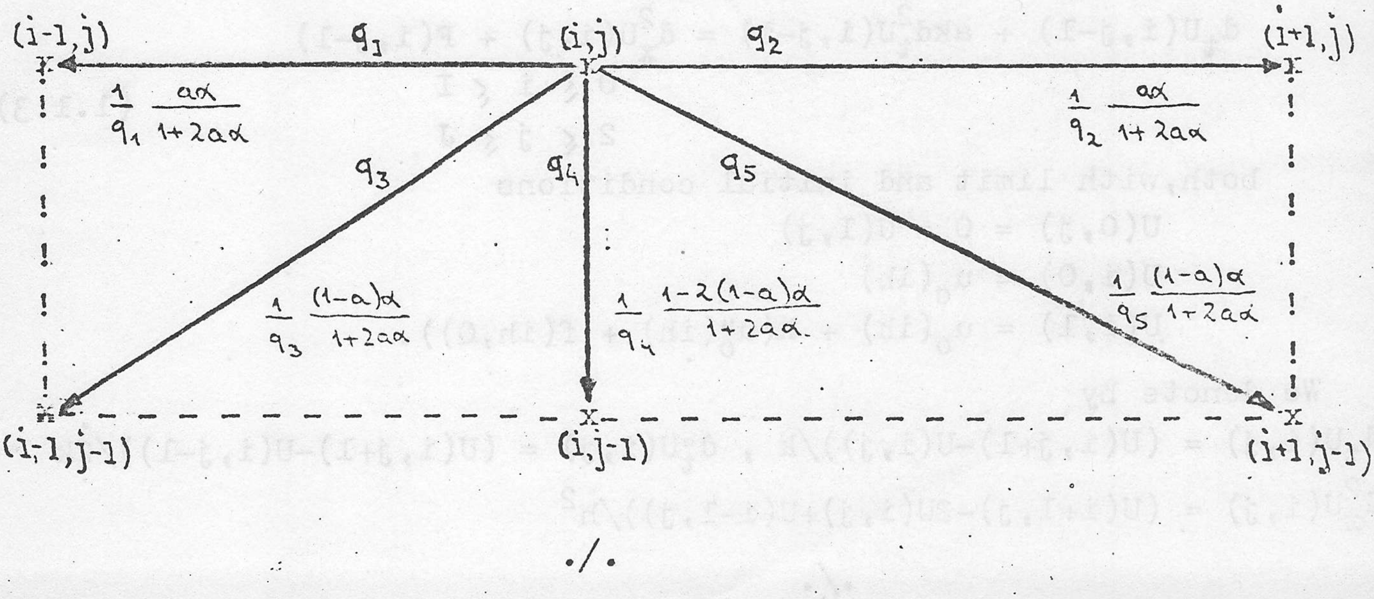
We denoted by $v_{s_{m-1}s_m}$ the weight coefficient of the Markov process trajectory when this one is passing from state s_{m-1} to state s_m, with $z_0 = (i_0,j_0)$ the point where we estimate the solution $U(i,j)$, and with

$$\tilde{F}_{z_m} = \begin{cases} 0 & \text{for } z_m = (0,j) \ \forall\ z_m = (I,j) \\ u_0(ih) & \text{" } z_m = (i,0) \\ u_0(ih) + k(u_0''(ih) + f(ih,0)) & \text{" } z_m = (i,1) \text{ and schemas } (1.1.2), (1.1.3) \\ \left.\begin{cases} \dfrac{k}{1+2a\alpha}F(i,j-1) & \text{for schema } (1.1.1) \\ \dfrac{2k}{1+4a\alpha}F(i,j-1) & \text{" " } (1.1.2) \\ \dfrac{k}{1+a+2\alpha}F(i,j-1) & \text{" " } (1.1.3) \end{cases}\right\} & \text{for } z_m = (i,j) \text{ with } 0<i<I,\ 1<j<J \end{cases}$$

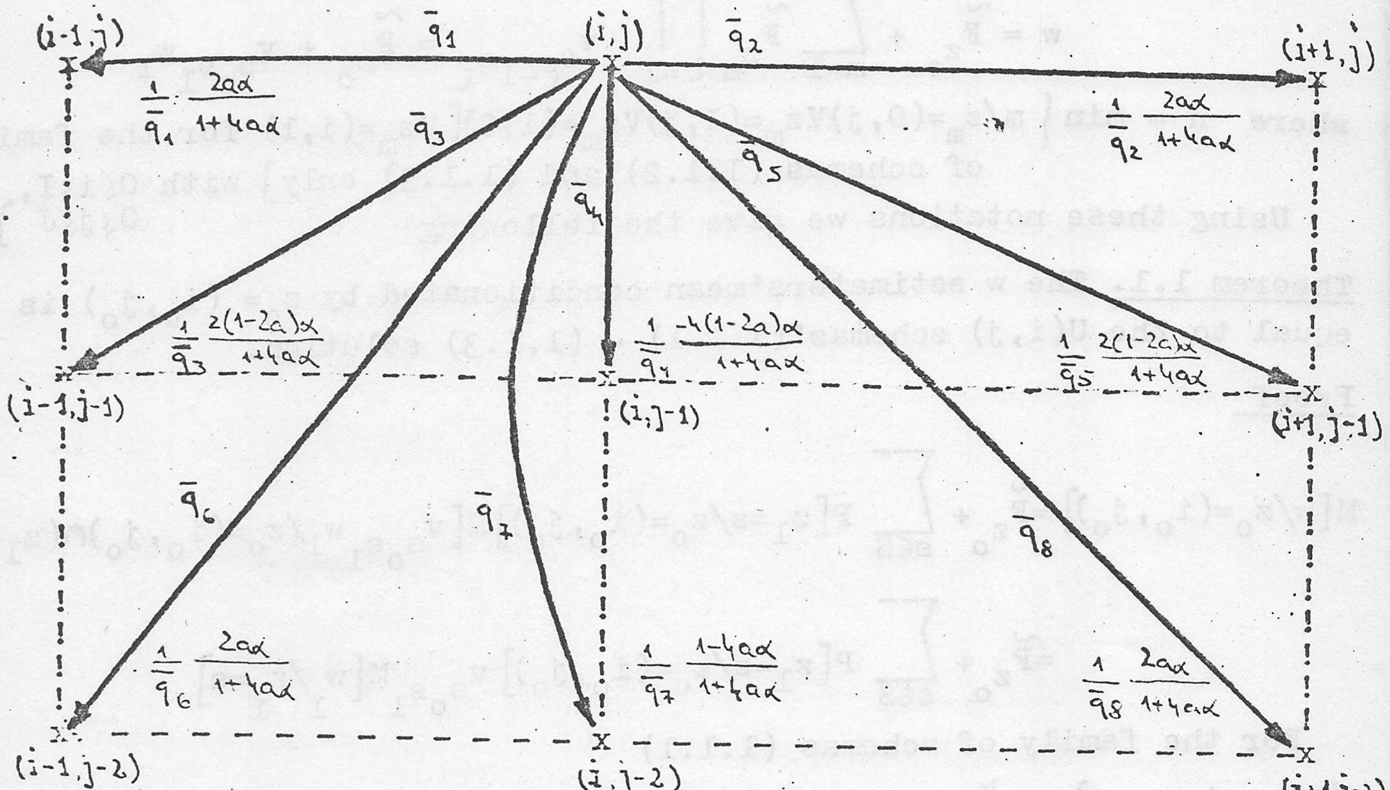
where $\alpha = k/h^2$.

Then, we define the transition probabilities and the weight coefficients

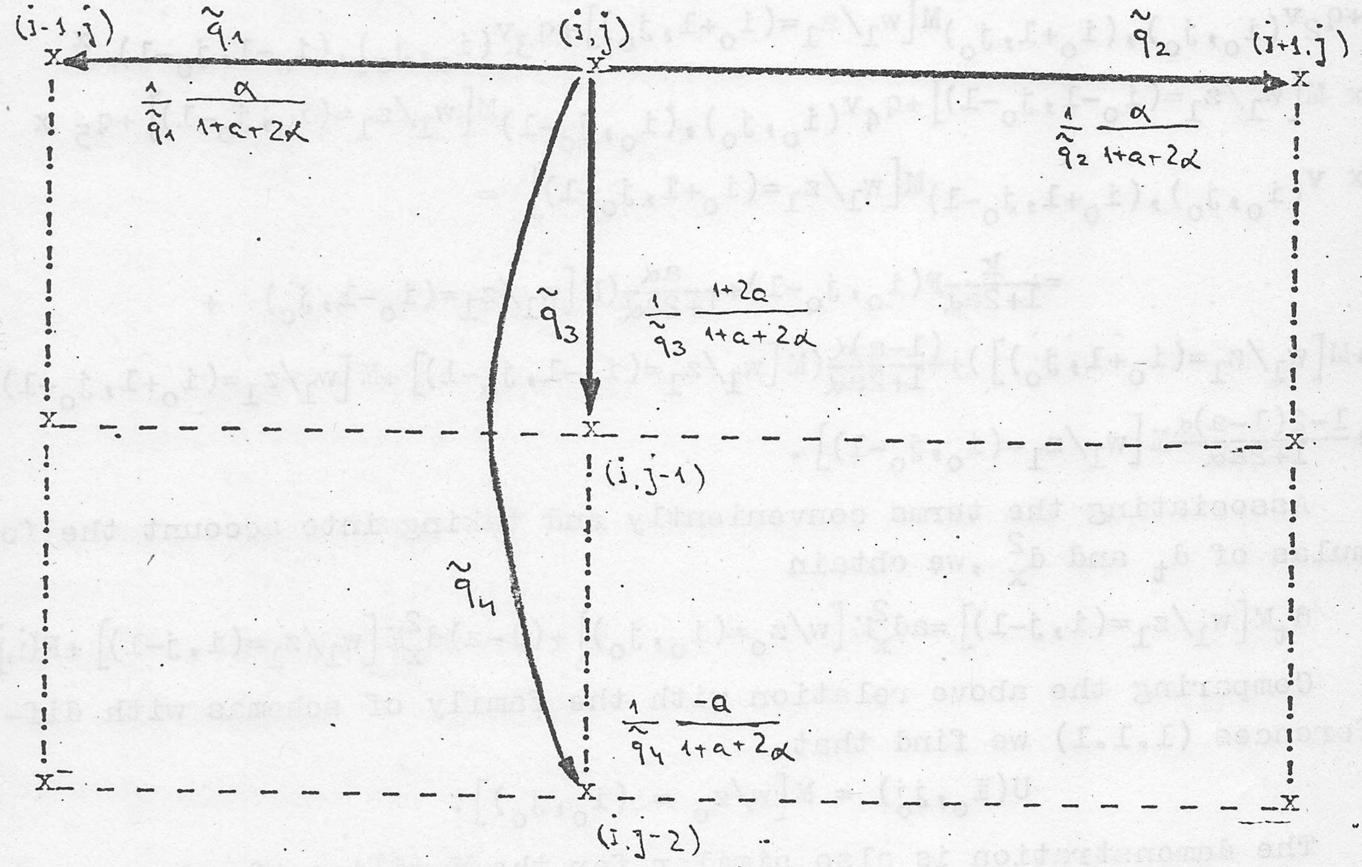
- for the family of schema (1.1.1)



- for the family of schema (1.1.2)



- for the family of schema (1.1.3)



On the basis of the Markov process $\{z_m\}_m$ attached as above to the family of schemas with differences (1.1.1) - (1.1.3), we denote the rough estimator w of the solution U(i,j):

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$$w = \tilde{F}_{z_0} + \sum_{m=1}^{n} \tilde{F}_{z_m} \prod_{\ell=1}^{m} v_{s_{\ell-1}s_\ell} = \tilde{F}_{z_0} + v_{s_0 s_1} w_1$$

where $n = \min\{ m / z_m=(0,j) \vee z_m=(I,j) \vee z_m=(i,0) [\vee z_m=(i,1)$ for the family of schemas (1.1.2) and (1.1.3) only$]$ with $0 \leq i \leq I,\ 0 \leq j \leq J \}$

Using these notations we give the following

<u>Theorem 1.1.</u> The w estimators'mean conditionated by $z_0 = (i_0, j_0)$ is equal to the U(i,j) schemas'(1.1.1) - (1.1.3) solution.

<u>Proof</u>

$$M[w/z_0=(i_0,j_0)] = \tilde{F}_{z_0} + \sum_{s\in S} P[z_1=s/z_0=(i_0,j_0)] M[v_{s_0 s_1} w_1/z_0=(i_0,j_0)\cap(z_1=s)]$$

$$= \tilde{F}_{z_0} + \sum_{s\in S} P[z_1=s/z_0=(i_0,j_0)]\, v_{s_0 s_1} M[w_1/z_1=s].$$

For the family of schemas (1.1.1)

$$M[w/z_0=(i_0,j_0)] = \frac{k}{1+2a\alpha}F(i,j-1) + q_1 v_{(i_0,j_0),(i_0-1,j_0)} M[w_1/z_1=(i_0-1,j_0)] +$$
$$+ q_2 v_{(i_0,j_0),(i_0+1,j_0)} M[w_1/z_1=(i_0+1,j_0)] + q_3 v_{(i_0,j_0),(i_0-1,j_0-1)} \times$$
$$\times M[w_1/z_1=(i_0-1,j_0-1)] + q_4 v_{(i_0,j_0),(i_0,j_0-1)} M[w_1/z_1=(i_0,j_0-1)] + q_5 \times$$
$$\times v_{(i_0,j_0),(i_0+1,j_0-1)} M[w_1/z_1=(i_0+1,j_0-1)] =$$
$$= \frac{k}{1+2a\alpha}F(i_0,j_0-1) + \frac{a\alpha}{1+2a\alpha}(M[w_1/z_1=(i_0-1,j_0)] +$$
$$+ M[w_1/z_1=(i_0+1,j_0)]) + \frac{(1-a)\alpha}{1+2a\alpha}(M[w_1/z_1=(i_0-1,j_0-1)] + M[w_1/z_1=(i_0+1,j_0-1)])$$
$$+ \frac{1-2(1-a)\alpha}{1+2a\alpha} M[w_1/z_1=(i_0,j_0-1)].$$

Associating the terms conveniently and taking into account the formulas of d_t and d_x^2 ,we obtain

$$d_t M[w_1/z_1=(i,j-1)] = a d_x^2 M[w/z_0=(i_0,j_0)] + (1-a) d_x^2 M[w_1/z_1=(i,j-1)] + F(i,j-1)$$

Comparing the above relation with the family of schemas with differences (1.1.1) we find that

$$U(i_0,j_0) = M[w/z_0 = (i_0,j_0)].$$

The demonstration is also similar for the families of schemas (1.1.2) - (1.1.3).

The moments of the rough estimators w satisfies schemas of parabolic nonhomogeneous types.

In order to simplify, we consider only the momentum of order two, the equations verified for the high moments being obtained similarly.

We denote by

$$M^m(i,j) = M\left[w^m/z_o=(i,j)\right]$$

Using this, we give the following

<u>Theorem 1.2.</u> For w estimator associate to the schema with differences (1.1.1), $M^2(i,j)$ is the solution of the family of the schemas with differences

$$M^2(i,j)=\left(\frac{k}{1+2a\alpha}\right)^2F^2(i,j-1)+\frac{2k}{1+2a\alpha}F(i,j-1)\left[U(i,j)-\frac{k}{1+2a\alpha}F(i,j-1)\right]+$$
$$+\frac{1}{q_1}\left(\frac{a\alpha}{1+2a\alpha}\right)^2M^2(i-1,j)+\frac{1}{q_2}\left(\frac{a\alpha}{1+2a\alpha}\right)^2M^2(i+1,j)+\frac{1}{q_3}\left(\frac{(1-a)\alpha}{1+2a\alpha}\right)^2M^2(i-1,j-1)+$$
$$+\frac{1}{q_4}\left(\frac{(1-a)\alpha}{1+2a\alpha}\right)^2M^2(i+1,j-1)+\frac{1}{q_5}\left(\frac{1-2(1-a)\alpha}{1+2a\alpha}\right)^2M^2(i,j-1).$$

with limit and initial conditions

$$M^2(i,0) = u_o^2(ih) \qquad 0 \leq i \leq I$$
$$M^2(0,j) = 0 = M^2(I,j). \qquad 0 \leq j \leq J$$

For w estimator associate to the schema with differences (1.1.2), $M^2(i,j)$ is the solution of the family of schemas with differences

$$M^2(i,j)=\left(\frac{2k}{1+4a\alpha}\right)^2F^2(i,j-1)+\frac{4k}{1+4a\alpha}F(i,j-1)\left[U(i,j)-\frac{2k}{1+4a\alpha}F(i,j-1)\right]+$$
$$+\frac{1}{\bar{q}_1}\left(\frac{2a\alpha}{1+4a\alpha}\right)^2M^2(i-1,j)+\frac{1}{\bar{q}_2}\left(\frac{2a\alpha}{1+4a\alpha}\right)^2M^2(i+1,j)+\frac{1}{\bar{q}_3}\left(\frac{2(1-2a)\alpha}{1+4a\alpha}\right)^2M^2(i-1,j-1)+$$
$$+\frac{1}{\bar{q}_4}\left(\frac{4(1-2a)\alpha}{1+4a\alpha}\right)^2M^2(i,j-1)+\frac{1}{\bar{q}_5}\left(\frac{2(1-2a)\alpha}{1+4a\alpha}\right)^2M^2(i+1,j-1)+\frac{1}{\bar{q}_6}\left(\frac{2a\alpha}{1+4a\alpha}\right)^2M^2(i-1,j-2)$$
$$+\frac{1}{\bar{q}_7}\left(\frac{1-4a\alpha}{1+4a\alpha}\right)^2M^2(i,j-2)+\frac{1}{\bar{q}_8}\left(\frac{2a\alpha}{1+4a\alpha}\right)^2M^2(i+1,j-2).$$

For w estimator associate to the schema with differences (1.1.3), $M^2(i,j)$ is the solution of the family of the schemas with differences

$$M^2(i,j)=\left(\frac{k}{1+a+2\alpha}\right)^2F^2(i,j-1)+\frac{2k}{1+a+2\alpha}F(i,j-1)\left[U(i,j)-\frac{k}{1+a+2\alpha}F(i,j-1)\right]+$$
$$+\frac{1}{\tilde{q}_1}\left(\frac{\alpha}{1+a+2\alpha}\right)^2M^2(i-1,j)+\frac{1}{\tilde{q}_2}\left(\frac{\alpha}{1+a+2\alpha}\right)^2M^2(i+1,j)+\frac{1}{\tilde{q}_3}\left(\frac{1+2a\alpha}{1+a+2\alpha}\right)^2M^2(i,j-1)+$$
$$+\frac{1}{\tilde{q}_4}\left(\frac{a}{1+a+2\alpha}\right)^2M^2(i,j-2).$$

The last two equations have limit and initial conditions

$M^2(i,0)=u_o^2(ih)$

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$$M^2(i,1)=[u_o(ih)+k(u''(ih)+f(ih,0))]^2.$$

$M^2(0,j)=0=M^2(I,j)$.

Proof

$$M^2(i,j)=M[(\tilde F_{z_o}+v_{s_os_1}w_1)^2/z_o=(i_o,j_o)]=\tilde F^2_{z_o}+2\sum_{s\in S}P[z_1=s/z_o=(i_o,j_o)]\ \mathrm{x}$$

$$\mathrm{x}\ v_{s_os_1}M[w_1/z_1=s]\tilde F_{z_o}+\sum_{s\in S}P[z_1=s/z_o=(i,j)]\,v^2_{s_os_1}M[w_1^2/z_1=s].$$

For the family of equations (1.1.1),using the technique of the demonstration from theorem 1.1. as well as its results,we obtain the innitial family of equations with differences,of schema (1.1.1) type with a more complicated nonhomogeneous term (in its calculus enters also the exact solution of schema (1.1.1)).

The demonstration is similarly for the w estimator associated to schemas (1.1.2) - (1.1.3).

Definition 1.3. It is called stable schema any schema for which exists the inequality

$$\|U^j\|_{(1)}\leqslant M_1\|u_o\|+M_2\max_{1\leqslant j'\leqslant j}\|F^{j'}\|\qquad 1\leqslant j\leqslant J$$

where U^j,u_o,F^j represents the vectors $U^j=(U(0,j),\ldots,U(I,j))$,

$u_o=(u_o(h),\ldots,u_o((I-1)h))$, $F^j=(F(1,j),\ldots,F(I-1,j))$,

M_1 and M_2 being positive constants non-depending on h and k, $\|\cdot\|_{(1)},\|\cdot\|$ are any norms on the stratum.

We call a conditioned stable schema,a schema of which stability is conditioned by the existence of a dependence between h and k.

We call an absolute stable schema,a schema which is stable for any h and k.

We denote by

$$V=\max_{0\leqslant\ell\leqslant n}v_{s_\ell s_{\ell+1}}\quad\text{and by}\quad F=\max_{0\leqslant\ell\leqslant n}\tilde F_{z_\ell}$$

and we have the following

Theorem 1.4. $M[w\ /\ z_o=(i_o,j_o)]$ associate to the schema (1.1.1) is conditioned stable for

$$0\leqslant a<\frac{1}{2}.$$

In this case α verifies the inequality

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$$0 \leqslant \alpha \leqslant \frac{1}{2(1-2a)} .$$

$M[w/z_o=(i_o,j_o)]$ associate to the schema (1.1.1) is absolute stable for

$$a > \frac{1}{2} .$$

$M[w/z=(i_o,j_o)]$ associate to the schema (1.1.2) is absolute stable for

$$a > \frac{1}{4} .$$

$M[w/z_o=(i_o,j_o)]$ associate to the schema (1.1.3) is absolute stable for $a > 0$.

<u>Proof</u> The theorem's proof requires two stages

1) stability towards the right term (we demonstrate $M[w/z_o=(i_o,j_o)]<\infty$) for w associate to the nonhomogeneous schemas (1.1.1) - (1.1.3) with null limit and initial conditions) ;

2) stability towards the initial conditions (we'll proof that $M[w/z_o=(i_o,j_o)]<\infty$ for w associate to the omogeneous schemas (1.1.1) — (1.1.3) with nonzero initial conditions).

1) Let be a nonhomogeneous schema with differences with null limit and initial conditions. In this case we'l have

$$\tilde{F}_{z_n} = 0 \quad \text{and then}$$

$$w = \tilde{F}_{z_o} + v_{s_o s_1}\tilde{F}_{z_1} + v_{s_o s_1} v_{s_1 s_2}\tilde{F}_{z_2} + \dots + v_{s_o s_1} v_{s_1 s_2} \dots v_{s_{n-2} s_{n-1}} \tilde{F}_{z_{n-1}} \leqslant$$

$$\leqslant \tilde{F}(1+V+V^2+\dots+V^{n-1}) .$$

With this

$$M[w/z_o=(i_o,j_o)] \leqslant \sum_{1}^{\infty} \tilde{F}(1+V+V^2+\dots+V^{m-1})P[n=m-1/z_o=(i_o,j_o)]$$

Therefore

$$M[w/z_o=(i_o,j_o)]<\infty \iff \sum_{1}^{\infty} V^m P[n=m/z_o=(i_o,j_o)]<\infty$$

$$M[V^m/z_o=(i_o,j_o)] < \infty .$$

But for the family of schema (1.1.1) results that

$$M[V^m/z_o=(i_o,j_o)] = \frac{a\alpha V}{1+2a\alpha}(M[V^m/z_1=(i-1,j)] + M[V^m/z_1=(i+1,j)]) + \frac{(1-a)\alpha V}{1+2a\alpha} \times$$

$$\times (M[V^m/z_1=(i-1,j-1)] + M[V^m/z_1=(i+1,j-1)]) + \frac{(1-2(1-a)\alpha V}{1+2a\alpha} M[V^m/z_1=(i,j-1)] \qquad (1.1.4)$$

We observe that the schema (1.1.4) is of the homogeneous schemas (1.1.1) type with nonzero initial conditions ($M[V^m/z_o=(i,0)]=1$).

Applieing the stability theorem for the type of schemas (1.1.1)

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(see [12]) to our (1.1.4) schemas, we obtain the results of the theorem 1.4. enunciation regarding the stability of $M[w/z_o=(i_o,j_o)]$ associated to schema (1.1.1).

Similarly for the estimator w associated to schema (1.1.2) and (1.1.3) we put an end to the right term stability demonstration of $M[w/z_o=(i_o,j_o)]$.

2) Let a homogeneous schema with differences with nonzero initial conditions. In this case

$$\tilde{F}_{z_m} = 0 \qquad 0 \leq m \leq n-1$$

$$w = v_{s_o s_1} v_{s_1 s_2} \dots v_{s_{n-1} s_n} \tilde{F}_{z_n} \leq \tilde{F} V^n$$

With this

$$M[w/z_o=(i_o,j_o)] \leq \sum_{1}^{\infty} \tilde{F} V^m P[n=m/z_o=(i_o,j_o)] .$$

Following a similarly jugement to the one of point 1) we show that

$$M[w/z_o=(i_o,j_o)]<\infty \iff M[V^m/z_o=(i_o,j_o)]<\infty .$$

Therefore we notice that the initial stability conditions do not impose additional conditions confronted by the ones obtained for the right term stability.

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Observation The stability conditions obtained through Monte Carlo method are the sams as the ones get for the schemas (1.1.1)-(1.1.3) in numerical analysis.

From the theorems 1.1 and 1.4 we can set forth

Theorem 1.5. The mean of the estimator w associated to schemas (1.1.1) (1.1.2) and (1.1.3) converges to the (0.1) problem solution.

Proof It results imediately from theorems 1.1 and 1.4.

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§ 1.2. The sequential Monte Carlo. Convergency.

In order to increase the convergency rate of the method and to obtain a better approximation of the solution we introduce a method of Monte Carlo sequential type.

Denoting by

$M^m[w/z_o=(i_o,j_o)]$ the approximation of $U(i_o,j_o)$ at the step m through the sequential Monte Carlo method,

and by

$$\hat{E}^m(i,j) = U(i,j) - M^m[w/z_o=(i,j)]$$

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we obtain iterative schemas of the following type

a) $$d_t\hat{E}^m(i,j-1) = ad_x^2\hat{E}^m(i,j) + (1-a)d_x^2\hat{E}^m(i,j-1) + T^m(i,j-1) \qquad (1.2.1)$$

where

$$T^m(i,j-1)=\frac{1+2a\alpha}{k}\Big(ad_x^2M^m[w/z_o=(i_o,j_o)]+(1-a)d_x^2M^m[w/z_1=(i,j-1)]+F(i,j-1)-$$
$$-d_tM^m[w/z_1=(i,j-1)]\Big);$$

b.1.)

$$d_t\hat{E}^m(i,j-1) = ad_x^2\hat{E}^m(i,j)+(1-2a)d_x^2\hat{E}^m(i,j-1)+ad_x^2\hat{E}^m(i,j-2)+T^m(i,j-1) \qquad (1.2.2)$$

where

$$T^m(i,j-1)=\frac{1+4a\alpha}{2k}\Big(ad_x^2M^m[w/z_o=(i_o,j_o)]+(1-2a)d_x^2M^m[w/z_1=(i,j-1)]+$$
$$+ad_x^2M^m[w/z_1=(i,j-2)]+F(i,j-1)-d_tM^m[w/z_1=(i,j-1)]\Big);$$

b.2.)

$$d_t\hat{E}^m(i,j-1)+akd_t^2\hat{E}^m(i,j-1) = d_x^2\hat{E}^m(i,j)+T^m(i,j-1) \qquad (1.2.3)$$

where

$$T^m(i,j-1)=\frac{1+a+2\alpha}{k}\Big(d_x^2M^m[w/z_o=(i_o,j_o)]+F(i,j-1)-d_tM^m[w/z_1=(i,j-1)]-$$
$$-akd_t^2M^m[w/z_1=(i,j-1)]\Big);$$

all of these with null limit and initial conditions.

Noting by $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ the tridiagonal square matrices with dimension $(I-1) \times (I-1)$ of the form

$$a_{ij} = \begin{cases} 1 & i=j \\ \left.\begin{array}{ll} \frac{-a\alpha}{1+2a\alpha} & \text{for the schema } (1.2.1) \\ \frac{-2a\alpha}{1+4a\alpha} & \text{" " " } (1.2.2) \\ \frac{-\alpha}{1+a+2} & \text{" " " } (1.2.3) \end{array}\right\} & j=i-1 \text{ or } j=i+1 \\ 0 & \text{in the rest} \end{cases}$$

$$b_{ij} = \begin{cases} \left.\begin{array}{ll} \frac{1-2(1-a)\alpha}{1+2a\alpha} & \text{for the schema } (1.2.1) \\ \frac{-4(1-2a)\alpha}{1+4a\alpha} & \text{" " " } (1.2.2) \\ \frac{1+2a\alpha}{1+a+2\alpha} & \text{" " " } (1.2.3) \end{array}\right\} & i=j \\ \left.\begin{array}{ll} \frac{(1-a)\alpha}{1+2a\alpha} & \text{" " " } (1.2.1) \\ \frac{2(1-2a)\alpha}{1+4a\alpha} & \text{" " " } (1.2.2) \\ 0 & \text{" " " } (1.2.3) \end{array}\right\} & j=i-1 \text{ or } j=i+1 \\ 0 & \text{in the rest} \end{cases}$$

$$c_{ij} = \begin{cases} \left.\begin{cases} \frac{1-4a\alpha}{1+4a\alpha} & \text{for schema (1.2.2)} \\ \frac{-a}{1+a+2\alpha} & \text{" " (1.2.3)} \end{cases}\right\} & i=j \\ \left.\begin{cases} \frac{2a\alpha}{1+4a\alpha} & \text{for schema (1.2.2)} \\ 0 & \text{" " (1.2.3)} \end{cases}\right\} & j=i-1 \text{ or } j=i+1 \\ 0 & \text{in the rest} \end{cases}$$

and by G the constant $\|A^{-1}B\|/(1-\|A^{-1}B\|)$ for schema (1.2.1), we give

<u>Theorem 1.6.</u> w^m is an unbiased estimator of $\hat{E}^m[w/z_o=(i_o,j_o)]$ and its dispersion is asymptotically bounded by a multiple of G^{m-1}.

For the family of schemas (1.2.1) the method converges geometrical if

$$\frac{1}{2(\sqrt{2}+2(1-a))} < \alpha .$$

<u>Proof</u> With the notations introduced above, we may notice that the solutions of schemas (1.2.1), (1.2.2) and (1.2.3) can be found out by solving the systems

$$A\hat{E}^{m,j} = B\hat{E}^{m,j-1} + \tilde{T}^{m,j-1}$$

where $\hat{E}^{m,j}, \hat{E}^{m,j-1}, \tilde{T}^{m,j-1}$ are vectors with components $\{\hat{E}^m(i,j)\}_i$, $\{\hat{E}^m(i,j-1)\}_i$, $\{\tilde{T}^m(i,j-1)\}_i$,
for schema (1.2.1) and

$$A\hat{E}^{m,j} = B\hat{E}^{m,j-1} + C\hat{E}^{m,j-2} + \hat{T}^{m,j-1}$$

for schemas (1.2.2) and (1.2.3),
and j varies in all cases between 0 and J.

With this observation we have shifted the problem in the linear system's field.

Applying the results got by Halton (see [8] the theorems 9,10 and lemma 6) to the above systems we obtain the first part of the theorem 1.6. enunciation (the convergency of w^m estimator).

As regarding the geometrical convergency condition of the schema (1.2.1), it results out of the following considerations

for linear systems the geometrical convergency is given (see [12]) by the inequality

$$\|A^{-1}B\| < 1/\sqrt{2} \approx 0{,}707 .$$

Taking $\|A\| = \rho(A)$ (the spectral radius of the matrix) and observing that the matrices of the system are tridiagonal, with the elements on the minor diagonals of the same sign, its results in the case of schemas (1.2.1) that

$$\rho(A) = 1+4a\alpha\sin^2\frac{\pi}{2I}\quad,\quad \rho(B) = 1-4(1-a)\alpha\sin^2\frac{\pi}{2I}$$

Bounding the above relation and accomplishing the calculus we get the geometrical convergency condition from the theorem 1.6. enunciation.

Observation Out of the obtained minimization for α in case of the schemas (1.2.1) we may conclude that, if an accelerated convergency of the method is wanted, then α cannot be chosen too small. (This thing disturbs the moment when we wish to obtain an increased accuracy by reducing the step k).

§1.3. The method efficiency

1.3.1. The volume of selection necessary to obtain a given error

Beeing given $\varepsilon > 0$ and $p > 0$ small, we estimate N so that

$$P[|U(i,j) - \bar{w}_N| < \varepsilon] > 1-2p$$

where we denoted by $\bar{w}_N = (w_1 + \dots + w_N)/N$.

Applying a technique used in [2] we obtain for p = 0,0455

$$N = \left[\frac{4}{\varepsilon^2}Di\right] + 1$$

where we noted by [.] the function - integer part, and by $Di = D^2[w/z_o=(i_o,j_o)]$.

To finalize the study (and to do the formula operative) we shell give an a priori evaluation of the conditioned dispersion Di (because of the fact that in the nonhomogeneous term' structure of the momentum of order two of the estimator w enters also the solution U(i, j) of the schema with differences, the theorem 1.2 cannot be used here).

Thus, we notice that

$$|w| \leq \frac{\tilde{F}}{1-V} \leq \frac{\tilde{F}}{1-\max(\|A\|,\|B\|,\|C\|)}$$

with V, F, A, B, C having the meanings of § 1.1 and § 1.2.

Using the inequality $M[X-M[X]]^2 \leq K^2-(M[X])^2$, where X is an aleatory variable bounded by K, results

$$Di \leq \frac{\tilde{F}^2}{(1-V)^2}$$

and then

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$$N = \left[\frac{4}{\varepsilon^2} \frac{\tilde{F}^2}{(1-V)^2} \right] + 1 \qquad (1.3.1)$$

§ 1.3.2. The average number of operations for estimate U(i,j)

The mean length of the introduced $\{z_m\}_m$ Markov process is noticing in the operations' number valuation necessary to the calculus of U(i,j)

Theorem 1.7. The mean length of the Markov process trajectory, starting from a point $z_o=(i_o,j_o)$, verifies families of schemas with differences, nonhomogeneous with null initial and limit conditions, of the type

$$d_t M[n/z_1=(i,j-1)] = a d_x^2 M[n/z_o=(i,j)] + (1-a) d_x^2 M[n/z_1=(i,j-1)] + \frac{1+2a\alpha}{k}$$

for the family of schemas (1.1.1),

$$d_t^o M[n/z_1=(i,j-1)] = a d_x^2 M[n/z_o=(i,j)] + (1-2a) d_x^2 M[n/z_1=(i,j-1)] + $$
$$+ a d_x^2 M[n/z_1=(i,j-2)] + \frac{1+4a\alpha}{2k}$$

for the family of schemas (1.1.2),

$$d_t M[n/z_1=(i,j-1)] + akd_x^2 M[n/z_1=(i,j-1)] = d_x^2 M[n/z_o=(i,j)] + \frac{1+a+2\alpha}{k}$$

for the family of schemas (1.1.3).

For the family of shemas (1.1.1) a bound of the mean length of the Markov process trajectory is

$$\| M[n/z_o=(.,j_o)] \| \leqslant j_o\sqrt{2} \; .$$

(where $\|.\|$ is the Euclidian norm and $M[n/z_o=(.,j_o)]$ represents the vector with components $\{M[n/z_o=(i,j_o)]\}_{i=\overline{1,I-1}}$

Proof We give the detailed demonstration for the family of schemas (1.1.1) the results for schemas (1.1.2),(1.1.3) being obtained similarly.

We notice that

$$P[n=m/z_o=(i_o,j_o)] = \frac{a\alpha}{1+2a\alpha}(P[n=m-1/z_1=(i-1,j)] + P[n=m-1/z_1=(i+1,j)]) +$$
$$+ \frac{(1-a)\alpha}{1+2a\alpha}(P[n=m-1/z_1=(i-1,j-1)] + P[n=m-1/z_1=(i+1,j-1)]) + \frac{1-2(1-a)\alpha}{1+2a\alpha} \times$$
$$\times P[n=m-1/z_1=(i,j-1)] .$$

Multiplieing by m-1, adding and taking into account the definition of the mean of a discrete random variable (such as n), we obtain the equation with differences

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$$M[n/z_o=(i,j)] = \frac{a\alpha}{1+2a\alpha}(M[n/z_1=(i-1,j)] + M[n/z_1=(i+1,j)]) + \frac{(1-a)\alpha}{1+2a\alpha} \times$$
$$\times(M[n/z_1=(i-1,j-1)] + M[n/z_1=(i+1,j-1)]) + \frac{1-2(1-a)\alpha}{1+2a\alpha}M[n/z_1=(i,j-1)] + 1 \qquad (1.3.2)$$

or minding the formulas of d_t and d_x^2 we obtain the family of schemas with differences out of the enunciation.

It is obviously that

$$M[n/z_o=(0,j)] = 0$$
$$M[n/z_o=(I,j)] = 0 \qquad j=\overline{1,J} \quad \text{and} \quad M[n/z_o=(i,0)] = 0 \quad i=\overline{0,I}$$

In order to settle a bound of $M[n/z_o=(i_o,j_o)]$ we solve the equation (1.3.2) with null limit and initial conditions through the separating variables method.

With the stability conditions imposed by the theorem 1.4 we obtain the bound

$$\| M[n/z_o=(.,j_o)] \| \leqslant \sum_{j'=0}^{j} \| F^{j'} \|$$

As

$$\| F^j \| = \sum_{m=1}^{I-1} F_m^2 \quad \text{with } F_m = \frac{2}{\pi m}(1-\cos m\pi)$$ the Fourier coefficients of function $F(i,j) = 1$, results through an usual calculus the inequality of the enunniation.

With this preliminary results we pass to the calculus of the average number of operations necessary to estimate $U(i,j)$ and we consider the following work hypotheses

a) compareing to the operations of multiplication-division, the operations of addition-subtracting are unsignificant as computering time and we won't take them into account at the calculation of the average number of operations necessary for estimate $U(i,j)$;

b) in order not to repete the evaluation of the functions F and u_o, we consider they are given punctually in each node of the net R_{hk} ;

c) the generation of a new state s_m of the Markov process $\{z_m\}_m$ is comparable as computer time to the duration of a multiplication-division operation.

Thus, the average number of operations necessary to estimate $U(i,j)$ by the above presented Monte Carlo method, is

- for the family of schemas (1.1.1)

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omogeneous | nonhomogeneous

- in the case $v_{s_m s_{m+1}} = 1$

$$M[n/z_o=(i,j)]\left(\left[\frac{4}{\varepsilon^2}\tilde{u}_o^2\right]+1\right)-\left[\frac{4}{\varepsilon^2}\tilde{u}_o^2\right]$$

$$j\sqrt{2}\left(\left[\frac{4}{\varepsilon^2}\tilde{u}_o^2\right]+1\right)-\left[\frac{4}{\varepsilon^2}\tilde{u}_o^2\right] \text{operations}$$

- in the case $v_{s_m s_{m+1}} \neq 1$

$$2M[n/z_o=(i,j)]\left(\left[\frac{4\tilde{u}_o^2}{\varepsilon^2(1-V)^2}\right]+1\right)- \quad ; \quad 3M[n/z_o=(i,j)]\left(\left[\frac{4\tilde{F}^2}{\varepsilon^2(1-V)^2}\right]+1\right)-$$

$$-\left[\frac{4\tilde{u}_o^2}{\varepsilon^2(1-V)^2}\right]+5 \leqslant 2\sqrt{2}j\left(\left[\frac{4\tilde{u}_o^2}{\varepsilon^2(1-V)^2}\right] \quad ; \quad -\left[\frac{4F^2}{\varepsilon^2(1-V)^2}\right]+6 \leqslant 3\sqrt{2}j\left(\left[\frac{4F^2}{\varepsilon^2(1-V)^2}\right]+1\right)-$$

$$+1)-\left[\frac{4\tilde{u}_o^2}{\varepsilon^2(1-V)^2}\right]+5 \text{ operations} \quad ; \quad -\left[\frac{4\tilde{F}^2}{\varepsilon^2(1-V)^2}\right]+6 \text{ operations}$$

- for the family of schemas (1.1.2)

we add 8 instead of 5 to the above formula ; we add 9 instead of 6 to the above formula

- for the family of schemas (1.1.3)

we add 4 instead of 5 to the above formula. ; we add 5 instead of 6 to the above formula.

§ 1.3.3. Methods of variance minimization

Having a transformations' ensemble relating to the primary w estimator, which preserves the mean value but changes the dispersion, we may construct, in general, more efficient variants of the Monte Carlo method than the initial one. The efficiency rise is connected with the consideration of some supplementary informations about the solution of problem (0.1).

1) The main part separation method

We suppose we have a ruogh approximation of the solutinn U(i, j), let it be G(i, j). Easily we may see that the difference e(i, j)=U(i, j)-G(i, j) verifies a schema with differences of the type (1.2.1)-(1.2.3) with almost null limit and initial conditions.

According to the maximum principle for the heat equation e(i, j) is very small and therefore combined with the theorem 1.6 results the $e(i,j)$ dispersion is small.

2) The essential selection method

We suppose we have a rough approximation of the solution $U(i,j)$ of the equation with omogeneous differences, let it be $G(i,j)$. We introduce the transition probabilities of the type

$$P^x_{(i,j),s} = v_{(i,j),s} P_{(i,j),s} \frac{G(s)}{G(i,j)}$$

where $s \in S$ - the set of states of the $\{z_m\}_m$ Markov process - and may take only the specific values of each families of schemas (1.1.1) - (1.1.3).

It is easy to show that, within the hypotheses we worked with up to now, the systems of probabilities $P^x_{(i,j),s}$ for each family of schemas (1.1.1) - (1.1.3) verify the conditions of a transition probabilities system.

<u>Lemma 1.8.</u> If the approximation $G(i,j)$ is the exact solution $U(i,j)$ of the families of omogeneous schemas (1.1.1)-(1.1.3), then the variance of w is zero.

<u>Proof</u> We give the proof for the family of schemas (1.1.1), the proof for the schemas (1.1.2)-(1.1.3) beeing similar.

From the theorem 1.1

$$M[w/z_0=(i,j)] = \sum_{s \in S_1} v_{(i,j),s} U(s) P_{(i,j),s}$$

with $S_1 = \{(i-1,j),(i+1,j),(i-1,j-1),(i,j-1),(i+1,j-1)/i=\overline{1,I-1}, j=\overline{1,J}\}$
or

$$M[w/z_0=(i,j)] = \sum_{s \in S_1} v_{(i,j),s} U(s) \frac{P_{(i,j),s}}{P^x_{(i,j),s}} P^x_{(i,j),s}$$

As the dispersion of w is

$$D^2[w/z_0=(i,j)] = \sum_{s \in S_1} \left(v_{(i,j),s} U(s) \frac{P_{(i,j),s}}{P^x_{(i,j),s}} P^x_{(i,j),s} - U(i,j)\right)^2 P^x_{(i,j),s}$$

it result that for

$$P^x_{(i,j),s} = v_{(i,j),s} P_{(i,j),s} \frac{U(s)}{U(i,j)} \quad \text{with } s \in S_1$$

this one is zero.

3) The stratification method

We suppose we know the values of U along the frontier of a subdomain of the net R_{hk} which contains the point (i, j) where we estimate the solution.

Thus, taking again the dispersion evaluation of § 1.3.1 we notice that we can get a "more tight" bound, such as

$$Di \leq \frac{\tilde{U}^2}{(1-V)^2} \leq \frac{\tilde{F}^2}{(1-V)^2}$$

because $\tilde{U}$ = max.val U on the frontier of the considered subdomain of R_{hk} is smaller the F accordingly to the maximum principle for the heat equation.

If the subdomain is chosen so that it would contain the point (i, j to which we add all the points to be reached within a step by the Markov process $\{z_m\}_m$ starting from (i, j) (for instance, for the family of schemas (1.1.1) the subdomain is of the type $\{(i-1, j), (i+1, j)$ $(i-1, j-1), (i, j-1), (i+1, j-1), (i, j)\}$ and the values of U along the frontier of this subdomain are the exact ones, then the variance of the primary estimator w through the stratification method is zero.

§ 1.4. Algorithms

The algorithm EPMC estimates the (0.1) problem solution in the given point (i, j) using the Monte Carlo method presented in § 1.1.

The algorithm steps are the followings

EPMCO Data input, initializations.

Input data read

h, k - steps of the net R_{hk} ;

I, J - the net R_{hk} sizes ;

$(u_o(ih))_{i=\overline{1,I-1}}$ - vector with the initial conditions of (0.1) problem in the net R_{hk} points ;

(i, j) - the point coordinates where we wish to estimate the solution ;

a - the family of schemas parameter ;

paraml - parameter which indicates if the equation is omogeneous (paraml=0) or nonhomogeneous (paraml=1) ;

$(F(i, j))_{i=\overline{1,I-1}}^{j=\overline{1,J}}$ - the matrix of the nonhomogeneous term values in the points (i, j) of the net R_{hk} (see the

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work hypothesis b) of § 1.3.2).When param1=0 we do not read the matrix ;

param2 - the parameter which indicates the schema to be used (param2=1 for schema (1.1.1);=2 for the schema (1.1.2) ;=3 for the schema (1.1.3)) ;

$(q_i)_i$ - vector which contains the transition probabilities of the $\{z_m\}_m$ Markov process ($i=\overline{1,5}$ if param2=1;$i=\overline{1,8}$ if param2=2;$i=\overline{1,4}$ if param2=3) ;

eps - the error we wish to obtain the soluton.

We initialize

SOL ←-- 0 , NRTR ←-- 0.

EPMC1 Initial calculus.
We compute α .
We compute the vector $v_{(i,j),s}$ (relied on the formulas of § 1.1 and on the value of param2).
We calculete V.
We compute $\tilde{u}_o = \|u_o\|_{max}$ if param1 = 0,

$\tilde{F} = \max\{\|u_o\|_{max}, \|F\|_{max}\}$ if param1 = 1.

We compute N the selection volume necessary to obtain the error eps with the probability 95,45% from the formula (1.3.1

EPMC2 We test the stability of the solution using the results of theorem 1.4. (In case of non-verifying conditions we list an error message and the algorithm is canceled).
We list the dispersion Di,the volume of selection N (see § 1.3.1).

EPMC3 If NRTR ≤ N then NRTR ←-- NRTR + 1 , S ←-- (i,j) , COEF ←-- 1 else go to step EPMC9.

EPMC4 We generate a new state S1 of the $\{z_m\}_m$ Markov process starting from the state S on the ground of the transition probabilities $(q_i)_i$ (about the generating algorithm see [13]).

EPMC5 COEF ←-- COEF x $v_{S,S1}$.

EPMC6 If S1 is an absorbing state (i,0) $i=\overline{1,I-1}$ then SOL ←-- SOL + + COEF x u_o(S1) and jump to step EPMC3 ; else if S1 is an absorbing state (0,j) or (1,j), $j=\overline{0,J}$,then go to step EPMC3.

EPMC7 If param1=1 (nonhomogeneous schema) then SOL ←-- SOL +

+ COEF x F(S1).

EPMC8 S ←-- S1 ; go to EPMC4.

EPMC9 SOL ←-- SOL/N ;

We list SOL, the algorithm result (the estimation of the (0.1) problem solution in the given point (i, j)).

EPMC10 STOP. *

The algorithm EPMCS improves the variance of the estimation of the (0.1) problem solution in a given point (i, j) through the sequential Monte Carlo method.

The theoretical ground of this algorithm is given in § 1.2. We use there, the nonhomogeneous (1.2.1) - (1.2.3) schemas with null limit and initial conditions.

We use here the algorithm EPMC to estimate the schemas solution in the nodes of the R_{hk} net.

The algorithm' steps are

EPMCS0 Data input, initializations.

Input data read

h,k I,J a (F(i,j)) param2 $(q_i)_i$ eps - have the same meaning as in the EPMC algorithm ;

$(SOLIN(i,j))_{i=\overline{1,I-1}}^{j=\overline{1,J}}$ - the initial approximations matrix of the solution in the nodes of the net R_{hk} (with the notation of § 1.2 $SOLIN(i,j)=M^{o}[w/z_o=(i,j)]$) ;

NRMAXIT - the maximum admited iterations number.

We initialize

SOL(i, j) ←-- SOLIN(i, j) $i=\overline{1,I-1}$, $j=\overline{1,J}$

$u_o(i,0)$ ←-- 0 , NRIT ←-- 0, param1 ←-- 1.

EPMCS1 If NRIT > NRMAXIT then go to EPMCS7 ;

else NRIT ←-- NRIT + 1.

EPMCS2 We compute the elements of matrix $T^{NRIT}(i,j)$ according to the formulae of § 1.2 (taking into account the value of param2).

We fill the matrix F(i, j) ←-- $T^{NRIT}(i,j)$ $i=\overline{1,I-1}$, $j=\overline{1,J}$

(the upper index NRIT of the notation T^{NRIT} do not appoint

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different matrices for different values).

EPMCS3 We estimate the solution of the chosen schema in each point (i,j) of the net R_{hk} ($i=\overline{1,I-1}$, $j=\overline{1,J}$) using the algorithm EPMC with adequate input parameters.
After each calling of the algorithm EPMC we fill the matrix
SOL ←-- SOL + the estimated value of the chosen schema solution in point (i,j).

EPMCS4 We compute $S_{max} = \| SOL - SOLIN \|_{max}$.

EPMCS5 If $S_{max} \leqslant eps$ then go to step EPMCS7 ;
else we fill the matrix SOLIN(i,j) ←-- SOL(i,j) $i=\overline{1,I-1}$, $j=\overline{1,J}$.

EPMCS6 Go to EPMCS1.

EPMCS7 We list the matrix $(SOL(i,j))_{i=\overline{1,I-1}}^{j=\overline{1,J}}$ of the solutions.

EPMCS8 STOP. ⁎

Chapter II

§ 2.1. The method of lines. Differential filters for solving the bilocal problem for differential equations with a stochastic nonhomogeneous term.

Let the domain

$$D = (0 \leqslant x \leqslant 1 , 0 \leqslant t \leqslant T)$$

and the network in D

$$R_k = \{ (x, jk)/0 \leqslant x \leqslant 1, j=0,1,.,J \}$$

with the step $k = T/J$.

Denotind by $U(x,j)$ the value in point (x,jk) of a network function U defined on R_k, with $b^2=1/k$ and by $p(x,j) = b^2U(x,j) + f(x,jk)$, and expressing the derivate $\partial u/\partial t$ in terms of the central difference d_{t}($d_tU(x,j) = (U(x,j+1) - U(x,j))/k$), problem (0.1) is transformed into a bilocal problem for differential equations of the second degree with the form

$$\frac{d^2U(x,j)}{dx^2} - b^2U(x,j) = -p(x,j)$$

$$U(0,j) = 0 = U(1,j) \qquad (2.1.1)$$

where $p(x,j)$ is modified for each level in comformity to the above formula.

It is known from the literature [7], that problem (2.1.1) approximates problem (0.1) with an $O(k)$ error and that the solution of problem (2.1.1) is absolutely stable in reference to initial data and the nonhomogeneous term.

These results enable us to consider the study of problem (0.1) from the point of view of the present study, as being the study of problem (2.1.1) based upon the transformation from above.

Let $Z_s(v)$ be a stochastic process, with the set of states

$$S = \{ s/s\in(x,jk), x\in[0,1], j=0,.,J \}.$$

When there is no possible confusion, the following notation will be made

$$Z_s(v) = Z_s = Z(x,j)$$

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and

$$M|Z_s|^2 < \infty$$

$M[Z_s] = m_s$ $M[Z_s - m_s][\overline{Z_p - m_p}] = R(s-p)$ will be considered.

Definition 2.1 Z_s is a weakly stationary stochastic process if

$$M[Z_s] = m \qquad R(s,p) = R(s-p)$$

(the mean is constant while the covariance function depends only on the difference s-p).

Definition 2.2 We call norm of the process Z_s denoted by $\|.\|$,

$$\|Z_s\| = \left\{M[|Z_s - m|^2]\right\}^{1/2}$$

Definition 2.3 The stochastic process Z_s is called differential in the quadratic mean in D if

$$\lim_{p \to 0} \left\| \frac{Z_{s+p} - Z_s}{p} \right\| \text{ exists and is unique.}$$

Definition 2.4 Let Z_s be a process with independent increases, f(s) such that $\int_R |f(s)|^2 ds$.

Then $\sum_m f(t_m)(Z_{s_m} - Z_{s_{m-1}})$ exists and is uniquely defined.

The random variable equal with the limit of the above sum is called stochastic integral,(the summing is done after the division of the interval).

We denote

$$\int_J f(s) dZ_s .$$

With the above definitions, the bilocal problem for linear differential equations of the second degree with nonhomogeneous stochastic term is introduced

$$\frac{d^2Z(x,j)}{dx^2} - b^2 Z(x,j) = -(p(x,j) + e(x,j))$$

$$Z(0,j) = Z(1,j) \qquad (2.1.2)$$

with e(x,j) a stochastic process with continuous parameter for which

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$M[e(x,j)e(y,j')] = \delta_{x,j}^{y,j'}$ (white noise), associated to problem (2.1.1)

Theorem 2.5 The solution of problem (2.1.2) is an unbiased estimator of the solution of problem (2.1.1).

Proof We formally solve problem (2.1.2) neglecting the nature of e(x, j). Hence, by using the reasoning followed for solving problem (2.1.1) (see [10]), we obtain

$$Z(x,j)=\int_0^1 G(x,y)(p(y,j)+e(y,j))dy=\int_0^1 G(x,y)p(y,j)dy+\int_0^1 G(x,y)e(y,j)dy \quad (2.1.3)$$

with G(x,y) Green's function from the solution of problem (2.1.1).

Since $\int_0^1 G(x,y)e(y,j)dy$ has no mathematical sense, e(y,j) being much too iregular for integration in any sense, we define

$$\int_0^1 G(x,y)e(y,j)dy = \int_0^1 G(x,y)dv(y) \quad (2.1.4)$$

where v(y) is a Wiener process (a gaussian process with independent increments, with $M[v(x)] = 0$ and $D^2[v(x)] = x$)(see [5]).
Hence the solution of the stochastic bilocal problem is

$$Z(x,j) = \int_0^1 G(x,y)p(y,j)dy + \int_0^1 G(x,y)dv(y) \quad (2.1.5)$$

From the properties of stochastic integrals and of Wiener's processes it immediately results that

$$M\left[\int_0^1 G(x,y)dv(y)\right] = 0 \text{ hence } M[Z(x,j)] = \int_0^1 G(x,y)p(y,j)dy$$

and with it

$$M[Z(x,j)] = U(x,j)$$

Lemma 2.6

$$M[Z^2(x,j)] = \left\{\int_0^1 G(x,y)p(y,j)dy\right\}^2 + \int_0^1 G^2(x,y)dy \quad (2.1.6)$$

Proof From the properties of stochastic integrals and of Wiener's Processes it is known that

$$M\left[\int f(x)dv(x)\right] = 0 \qquad M\left[\left\{\int f(x)dv(x)\right\}^2\right] = \int f^2(x)dx .$$

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Using these two relations and the result (2.1.5), using a routine calculus formula (2.1.6) is obtained.

§ 2.2 The efficiency of the method

§ 2.2.1. Numerical evaluation of the stochastic integral

$$\int_0^1 G(x,y)dv(y).$$

The use of the method presented in §2.1. needs numerical evaluation method, for the stochastic integral $\int_0^1 G(x,y)dv(y)$ from formula (2.1.5) of $Z(x,y)$, that will be easily implemented on an electronic computer. Hence we give the following

Theorem 2.7 If $G(x,y)p(y,j) \in C_3^o([0,1])$ then

$$Z(x,j)=\sum_{i=1}^{I-1} (hG(x,ih)p(ih,j) + \frac{h^2}{2}\frac{\partial}{\partial y}(G(x,ih)p(ih,j)) + \frac{h^3}{6} x$$

$$x \frac{\partial^2}{\partial y^2}(G(x,ih)p(ih,j)) + G(x,ih)N_{1i} + \frac{\partial}{\partial y}G(x,ih)N_{2i}) +$$

$$+ O(h^2) + O_p(h^2) \qquad (2.2.1)$$

where $h = 1/I$,

$N_{1i} = \int_{ih}^{(i+1)h} dv(y) \sim N(0,h)$ (random normal variable of mean 0 and variance h),

$N_{2i} = \int_{ih}^{(i+1)h} (y-ih)dv(y) \sim N(0,\frac{2}{3}h^3)$ (normal random variable of mean 0 and variance $(2/3)h^3$).

with $M[N_{1i}N_{2i}] = h^2/2$.

$O_p(h^2)$ means $\lim_{h\to 0} (1/h^2)P[|x|>\varepsilon] = 0$ $(\forall)$ $\varepsilon>0$.

and $O(h^2)$ means here $\lim_{h\to 0} (1/h^2)O(h^2) = 0$.

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Proof We use a demonstration technique suggested by [11] .From the hypothesis of the theorem, it results that a Taylor series can be developed around point (x,ih), the functions $G(x,y)p(y,j)$ and $G(x,y)$, obtaining

$$G(x,y)p(y,j)=G(x,ih)p(ih,j)+(y-ih)\frac{\partial}{\partial y}(G(x,ih)p(ih,j))+\frac{(y-ih)^2}{2}\ x$$

$$x\ \frac{\partial^2}{\partial y^2}(G(x,ih)p(ih,j)) + O(|y-ih|^2)$$

and

$$G(x,y)= G(x,ih)+(y-ih)\frac{\partial}{\partial y}G(x,ih)+\frac{(y-ih)^2}{2}\frac{\partial^2}{\partial y^2}G(x,ih)+O(|y-ih|^2).$$

Considering the division of the interval $[0,1]$ given by step h, by approximating the integrals from (2.1.5) through the corresponding Riemann sums and replacing functions $G(x,y)p(y,j)$, and respectively $G(x,y)$ by the above developments in Taylor series, formula (2.2.1) is obtained with the round-off error

$$\int_{ih}^{(i+1)h} O(|y-ih|^2)dy + \int_{ih}^{(i+1)h} O(|y-ih|^2)dv(y) = O(h^2) + O_p(h^2)$$

(for details concerning this assertion see [11]).
There remains only to evaluate the stochastic integrals

$$N_{1i} = \int_{ih}^{(i+1)h} dv(y) \qquad N_{2i} = \int_{ih}^{(i+1)h} (y-ih)dv(y)$$

$$N_{3i} = \int_{ih}^{(i+1)h} (y-ih)^2 dv(y)$$

By definition and from the properties of stochastic integrals, (see for [5]), it results that

$N_{1i} \sim N(0,h)$, $N_{2i} \sim N(0,h^3/3)$, $N_{3i} \sim O_p(h^2)$ with $M[N_{1i}N_{2i}] = h^2/2$

Based on theorem 2.7 follows the calculus of the number of operations needed to estimate the mean $M[Z(x,j)]$ taking into account the following work hypotheses

a) in order to avoid the repetition of the evaluation of $G(x,y)p(y,j)$ and $G(x,y)$ these will be done only once for the network nodes, and will be stored in the corresponding vectors ;

b) we consider only multiplication-division, since operation of addition-subtraction can be neglected from the point of view of the com-

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puter time ;

c) the generation of a single normal radom variable N_{1i} or N_{2i} needs the time necessary for one multiplivation-division.

Hence, the number of hecessary operations to estimate the mean $M[Z(x,j)]$, under the work-hypothesis a) - c), based on formula (2.2.1) is the following

$$2(I-1)(2N+7) + 1 \text{ operationa}$$

where N is the selection volume needed to estimate the mean of the process Z(x,j) with a given error $\varepsilon > 0$ and the probability 95,45%.

Refering to the efficiency of the method presented in § 2.1, we remark the following

a) the stochastic equation (2.1.2) can be easily solved by means of an analog computer filtering a given stochastic process, with a filter with differential operator (see [6]) ;

b) the stochastic equation (2.1.2) has also a known physical significance ; it represents Langevin's equation governing the movement of a harmonic unidimensional oscilator, without friction, under a random impulse me(x,j).

§ 2.3. The algorithm

The EPSD algorithm estimates the solution of problem (0.1) in given point (x,j), using the differential filter for time series from § 2.1.

The steps of the algorithm (easy to program in a high level programming language for an electronic computer) are the following

EPSDO Data input, initializations.

Input data read

h - the step of the network R_h ;

I - dimension of the network R_h ;

x - the coordinate of the point level j for which the estimation is needed ;

$G(x,y)p(y,j), \frac{\partial}{\partial y}(G(x,y)p(y,j)), \frac{\partial^2}{\partial y^2}(G(x,y)p(y,j)), \frac{\partial}{\partial y}G(x,y)$ -

external functions evaluating on the basis of formulae

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given in § 2.1 ;

N - the selection volume upon Z(x, j) to obtain a given error O with a probability greater than 95,45%.

Initialiazition of

SOL ←-- 0 NRGEN ←-- 0.

EPSD1 Based upon external functions, the vectors are evaluated and loaded

$G_0(i) \leftarrow G(x,ih)$ $G_1(i) \leftarrow \frac{\partial}{\partial y}G(x,ih)$

$G_2(i) \leftarrow hG(x,ih)p(ih,j)$ $G_3(i) \leftarrow \frac{h^2}{2}\frac{\partial}{\partial y}(G(x,ih)p(ih,y))$

$G_4(i) \leftarrow \frac{h^3}{6}\frac{\partial^2}{\partial y^2}(G(x,ih)p(ih,j))$ for $i=\overline{1,I-1}$

EPSD2 If NRGEN ⩽ N then NRGEN ←-- NRGEN + 1 , i ←-- 1 ;

else go to EPSD6.

EPSD3 The selection variables V_1 and V_2 are generated upon the random variables N_{1i} and N_{2i} (for the generation algorithm see [13]) ;

EPSD4 We compute

$SOL \leftarrow SOL+G_2(i)+G_3(i)+G_4(i)+G_0(i)V_1+G_1(i)V_2$

EPSD5 If i ⩽ I-1 then i ←-- i+1, go to EPSD3 ;

go to EPSD2.

EPSD6 SOL ←-- SOL/N

The result of the algorithm is listed, SOL (estimation of the mean of Z(x, j), solution of problem (2.1.1)).

EPSD7 STOP. *

Chapter III

A hydrogeology problem

In the hydrothermomineral deposits, the propagation by filtration of the interaction of the hydrodynamic active water particles, the propagation by convection and by conduction of the heat and the propagation by hydrodynamic dispersion of the components concentration in the fluid mixture of the hydrodynamic active water respect the adequate diffusivity equations (see [1]), which may be accepted for the great majority of these deposits under the formulae

- the hydraulic diffusivity

$$\frac{\partial H}{\partial t} = \mathcal{A}\,\mathrm{divgrad}\, H + \frac{W}{S_e} \qquad (3.1)$$

- the thermic diffusivity

$$\frac{\partial T^o}{\partial t} = \tilde{\lambda}\,\mathrm{divgrad}\, T^o - \frac{\varsigma c}{\tilde{\varsigma}\tilde{c}}\vec{v}\,\mathrm{grad}\, T^o + \frac{\mathfrak{M}}{\tilde{\varsigma}\tilde{c}} \qquad (3.2)$$

- the dispersional diffusivity

$$\frac{\partial C}{\partial t} = \mathfrak{D}\,\mathrm{divgrad}\, C - \frac{\vec{v}}{m_e}\,\mathrm{grad}\, C + \frac{\mathfrak{N}}{m_e} \qquad (3.3)$$

where $\vec{v} = -K\,\mathrm{grad}\, H$ represents the filtration speed expressed, accor-ding Darcy's law, through the product between the hydraulic conductibility K and the piezometric charge gradient with changed sign, -gradH; $\partial H/\partial t$ and divgrad H, $\partial T^o/\partial t$ and divgrad T^o, $\partial C/\partial t$ and divgrad C represents the local derivative and the laplacian of the piezometric charge H, temperature T^o and concentration C ;

$\mathcal{A}$, $\tilde{\lambda}$ and $\mathfrak{D}$ represent the hydraulic diffusivity coefficient, the temperature diffusivity coefficient through the rock and water stored, and the dispersional diffusivity coefficient ;

ς and c represent the specific mass and namely the specific heat of the hydrodynamic active water ;

$\tilde{\varsigma}$ and $\tilde{c}$ represent the specific mass and the specific heat of the rock of eficace porosity m_e ;

grad T^o and grad C represent the temperature gradient, namely the concentration gradient.

For $\partial H/\partial z = 0$, $\partial T^o/\partial z = 0$ and $\partial C/\partial z = 0$, that is for areal diffusivity, the equations (3.1) - (3.3) are reduced to the formulae

$$\frac{\partial H}{t} = \mathcal{A}\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right) + \frac{W}{s_e} \tag{3.4}$$

$$\frac{\partial T^o}{\partial t} = \tilde{\chi}\left(\frac{\partial^2 T^o}{\partial x^2} + \frac{\partial^2 T^o}{\partial y^2}\right) + \frac{\rho c}{\tilde{\rho}\tilde{c}} K\left(\frac{\partial H}{\partial x}\frac{\partial T^o}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial T^o}{\partial y}\right) + \frac{\mathfrak{M}}{\hat{\rho}\tilde{c}} \tag{3.5}$$

$$\frac{\partial C}{\partial t} = \mathcal{D}\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) + \frac{K}{m_e}\left(\frac{\partial H}{\partial x}\frac{\partial C}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial C}{\partial y}\right) + \frac{\mathfrak{N}}{m_e} \tag{3.6}$$

with initial and limit conditions of Dirichlet type.

With the programmes elaborated in the high level language FORTRAN, rolled on a computer from the University of Bucharest Computer Centre (the FELIX C-256 and IMB 360/30) and applied to the data provided by the hydrothermomineral fissural system of the Băile Felix - 1 Mai cretaceous limestones, I obtained the following results

r.1) the schemas (3.4) - (3.6) application for a number of 469 points lasted almost 30 minutes time computer on FELIX C-256 and 50 minutes on IBM 360/30 ;

r.2) the application of the method from the second chapter involved a necessary of memories at least of the three times bigger then the adequate Monte Carlo method (chapter I) ;

r.3) taking into account the very short Markov process trajectory (corroborated with the observation from § 1.3.3 regarding the variance minimizing) the solutions' variance obtained through Monte Carlo method was satisfactory, imposing no more the use of the sequential methods (see § 1.2) or the variance minimization technics ;

r.4) the Monte Carlo method proved to be much faster, almost one hundred times, then the classic numerical analysis methods regarding the estimation in one point alone, of the schemas (3.4)-(3.6) solution; but in the whole, considering that, in this case, we needed the schema solutions in 469 points, and that the Markov process trajectories were not memorated in view of a multiple use like is suggested by a technic from [13], the time requested by the Monte Carlo method comparatively the methods from chapter II was sensible higher (almost three times)

The method resumed in this chapter, new and original by the corelated study of the three fields (hydrogeologic, hydrothermodinamic and hydrochemical) as well as by the used mathematic apparatus, is serving to the specialists practical activity in the research and revaluation of the underground mineral, thermal, drinkable and industrial waters fields, contributing to this activity growing efficiency and to the enlargement of the precision degree of the reserves determined.

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