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by

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We introduce in this paper the notion of morphism of H-cones, and present some of its properties. §1 has an introductory scope and contains some general definitions and results from the theory of H-cones [1]. In § 2 a "cone of hyperharmonic" $\tilde{C}$, associated with an H-cone C, is constructed. Some properties of $\tilde{C}$ are studied, and $\tilde{C}$ is computed when C is the dual of an H-cone or is an H-cone of functions. The last case also justifies the terminology. The possibility of extending balayages and H-integrals from C to $\tilde{C}$ is presented.

§ 3 contains the definition of a morphism of H-cones, as an additive, monotone and continuous map from $\tilde{C}_1$ to $\tilde{C}_2$. We establish completely the relation with the notion of morphism of H-cone, as introduced in [7]. Next, we define the adjoint for a large class of morphisms. We show that this class is stable under composition and taking adjoint; also contains the semifinite morphisms. Further, some classes of morphisms are studied: semifinite, finite and isomorphisms. Next, are considered the morphisms between the H-cones of functions, which are induced by a correspondence between the spaces of representation. § 3 is ended with a result concerning the structure of the set of morphisms between two H-cones .

§ 4 treats the case when a morphism between two standard H-cones induces a correspondence between the canonical spaces of representation. We characterise here these morphisms, and prove that these morphisms are extremal and form a G_δ-set in a compact, convex set for the natural topology.

Finally, §5 contains a brief discution of morphisms of sheaf of H-cones, being an extention of analogous results from [7].

1. We recall some definitions and results about H-cones [1]. An ordered convex cone C is called an H-cone if:

H1 $s \in C \Longrightarrow s \geqslant 0$

H2 $s+u \leq t+u \Longrightarrow s \leq t$

H3 C is a lower complete lattice

H4 For any upper directed and dominated family (s_i) and any $s \in C$:

$$\bigvee_{i \in I}(s+s_i) = s + \bigvee_{i \in I} s_i$$

H5 For any family (s_i) and any $s \in C$:

$$\bigwedge_{i \in I}(s+s_i) = s + \bigwedge_{i \in I} s_i$$

H6 $s \leq s_1+s_2 \Longrightarrow \exists\, t_1, t_2$ such that $t_1 \leq s_1$, $t_2 \leq s_2$, $s=t_1+t_2$ (the Riesz decomposition property)

$D \subseteq C$ is called dense if, for any $s \in C$ there exists an upper directed family (s_i) from D, such that $s = \bigvee_{i \in I} s_i$.

A map $B: C \longrightarrow C$ is called a balayage on C if:

B1 $B(s+t)=Bs+Bt$

B2 $s \leq t \Longrightarrow Bs \leq Bt$

B3 $Bs \leq s$

B4 $B(Bs)=Bs$

B5 For any $s \in C$ and any upper directed family (s_i) such that $s = \bigvee_{i \in I} s_i$ we have: $Bs = \bigvee_{i \in I} Bs_i$

A map $\mu: C \longrightarrow \overline{\mathbb{R}}_+$ is called an H-integral if:

I1 $\mu(s+t) = \mu(s) + \mu(t)$

I2 $s \leq t \implies \mu(s) \leq \mu(t)$

I3 For any $s \in C$ and any upper directed family (s_i) such that

$s = \bigvee_{i \in I} s_i$ we have: $\mu(s) = \sup_{i \in I} \mu(s_i)$

I4 $\{s \in C \mid \mu(s) < +\infty\}$ is dense in C

C^* denotes the set of all H-integrals on C. With the pointwise opperations, C^* is also an H-cone. We have a natural map $C \longrightarrow C^{**}$, which is an injection iff C^* separates C.

An element $u \in C$ is called strictly positive if, for any $s \in C$ we have:

$$s = \bigvee_{n \in \mathbb{N}} (s \wedge nu)$$

If $u \in C$ is strictly positive, an element $s \in C$ is called u-continuous if, for any upper directed family (s_i) such that $s = \bigvee_{i \in I} s_i$ and any $\varepsilon > 0$, there exists i such that: $s \leq s_i + \varepsilon u$. $s \in C$ is called universally continuous if it is u-continuous for any strictly positive u. The set of all universally continuous elements of C is denoted by C_o.

The H-cone C is called standard if there exists a strictly positive element in C and there exists a countable, dense subset in C_o. If C is standard, then C^* is also standard. If C is standard, C^* separates C and C is dense and solid in C^{**}.

Let C be a standard H-cone. The coarsest topology on C^*, for which all the maps $\mu \longmapsto \mu(s)$ are continuous, for $s \in C_o$, is called the natural topology on C^*. If $u \in C$ is a strictly positive element, then:

$$K_u = \{\mu \in C^* \mid \mu(u) \leq 1\}$$

is a convex, compact and metrisable set in the natural topology.

We denote C_c the set of all $s \in C$, such that $\mu \longmapsto \mu(s)$ is a continuous map on K_u. C_c is clearly a convex subcone, and contains any u-continuous element.

Let X be a set and $\mathcal{G}$ a set of numerical, positive functions on X. $\mathcal{G}$ is called an H-cone of functions on X if:

F1 $\mathcal{G}$ is an H-cone, with the pointwise opperations and order.

F2 For any upper directed family (s_i) and any $s \in \mathcal{G}$ such that $s = \bigvee_{i \in I} s_i$, we have $s(x) = \sup_{i \in I} s_i(x)$, $\forall x \in X$

F3 For any $s, t \in \mathcal{G}$ we have $= (s \wedge t)(x) = \min\{s(x), t(x)\}$, $\forall x \in X$

F4 $\mathcal{G}$ contains the positive constants and separates X.

Let $\mathcal{G}$ be an H-cone of functions on X. The coarsest topology on X, for which all the functions from $\mathcal{G}$ are continuous, is called the fine topology. The coarsest topology on X, for which all the universally continuous elements of $\mathcal{G}$ are continuous, is called the natural topology.

F4 shows that, in an H-cone of functions, $\mathcal{G}^*$ separates $\mathcal{G}$.

We recall also that for any $A \subseteq X$ and $s \in \mathcal{G}$:

$$B^A(s) = \bigwedge\{t \in \mathcal{G} \mid t \geq s \text{ on } A\}$$

§2. We begin with the construction of an ordered convex cone $\widetilde{C}$, associated with an H-cone C. When C is the cone of positive superharmonic functions of a $\mathcal{B}$-harmonic space [6], then $\widetilde{C}$ is the cone of positive, hyperharmonic functions. It is this cone $\widetilde{C}$ which will be used in the definition of the morphisms of H-cones.

Let C be an H-cone. C^o will denote the set of all maps $\mu : C \longrightarrow \overline{\mathbb{R}}_+$, such that there exists an upper directed family $(\mu_i)_{i\in I}$ of H-integrals on C, for which:

$$\mu(s) = \sup_{i\in I} \mu_i(s) \quad , \forall s \in C$$

It is clear that any $\mu \in C^o$ satisfies the conditions I1 - I3 from the definition of an H-integral; but not any map $C \longrightarrow \overline{\mathbb{R}}_+$ which satisfies I1 - I3 is in C^o.

Proposition 2.1. With the operations and order defined pointwise (and the convention $0.\infty = 0$), C^o is a complete lattice and an ordered convex cone. C^o contains C^* as a solid and dense subcone.

Proof. We have to prove only that C^o is a complete lattice. Let $\mu_1, \mu_2 \in C^o$ and define, for $s \in C$:

$$\mu(s) = \sup\{\mu_1(s_1) + \mu_2(s_2) \mid s_1+s_2 \leq s\}.$$

Then $\mu = \mu_1 \vee \mu_2$ in C^o. Indeed, let $s \in C$ be such that $\mu(s) < +\infty$; let $\varepsilon > 0$ and let $(\mu_i)_{i\in I}$ and $(\nu_j)_{j\in J}$ be families from C^*, associated with μ_1 and μ_2. We can find $s_1, s_2 \in C$ such that $s_1+s_2 \leq s$ and:

$$\mu(s) \leq \mu_1(s_1) + \mu_2(s_2) + \varepsilon/3$$

Next, we can choose $i \in I$ and $j \in J$ such that:

$$\mu_1(s_1) \leq \mu_i(s_1) + \varepsilon/3 \quad \text{and} \quad \mu_2(s_2) \leq \nu_j(s_2) + \varepsilon/3$$

Recalling the construction of $\mu_i \vee \nu_j$ in $C^*[\,]$, it follows that : $\mu(s) \leq (\mu_i \vee \nu_j)(s) + \varepsilon$. If $\mu(s) = +\infty$ for each $\varepsilon > 0$ there exists $s_1, s_2 \in C$ such that : $\mu_1(s_1) + \mu_2(s_2) > 2\varepsilon$. We can find then $i \in I$ and $j \in J$ such that $\mu_i(s_1) + \nu_j(s_2) > \varepsilon$, which shows that $(\mu_i \vee \nu_j)(s)$ is greater than ε. Hence in both cases $\mu(s) = \sup\limits_{\substack{i\in I\\ j\in J}} (\mu_i \vee \nu_j)(s)$ $\forall s \in C$, which proves that $\mu \in C^o$. It is clear now that $\mu = \mu_1 \vee \mu_2$ in C^o. Since for any increasing family $(\mu_i)_{i\in I}$ from C^o we have obviously $\mu \in C^o$, where $\mu(s) = \sup\limits_{i\in I} \mu_i(s)$, $\forall s \in C$, it follows that C^o is a sup-complete lattice. Having the smallest element, C^o is in fact a complete lattice. //

We define now $\widetilde{C} = (C^*)^o$. Let $\varepsilon : C \longrightarrow \widetilde{C}$ be the composition of the natural maps $C \to C^{**} \longrightarrow (C^*)^o$.

<u>Corollary 2.2.</u> a) $\widetilde{C}$ is an ordered convex cone and a complete lattice.

b) $\forall s,t \in C$ and $\alpha \geq 0$ we have: $\varepsilon(s+t) = \varepsilon(s) + \varepsilon(t)$; $\varepsilon(\alpha s) = \alpha\,\varepsilon(s)$; $s \leq t \Longrightarrow \varepsilon(s) \leq \varepsilon(t)$.

c) For any family $(s_i)_{i\in I}$ from C we have $\varepsilon(\bigwedge s_i) = \bigwedge \varepsilon(s_i)$

If $(s_i)_{i\in I}$ is upper directed and bounded, then:

$$\varepsilon(\bigvee s_i) = \bigvee \varepsilon(s_i)$$

d) ε is injective iff C^* separates C. $\varepsilon(C)$ is solid in $\widetilde{C}$ iff C is solid in C^{**}. $\varepsilon(C)$ is dense in $\widetilde{C}$ iff C is dense in C^{**}. //

The greatest element in C^o is:

$$\mu(s) = \begin{cases} 0 & \text{if } \nu(s) = 0,\ \forall \nu \in C^* \\ +\infty & \text{if not} \end{cases}$$

Hence, the greatest element in $\tilde{C}$ is :

$$f(\mu) = \begin{cases} 0 & \text{if } \mu = 0 \\ +\infty & \text{if } \mu \neq 0 \end{cases}$$

<u>Proposition 2.3.</u> a) For any $s \in \tilde{C}$ and any increasing family $(s_i)_{i\in I}$ from $\tilde{C}$ we have $s + \bigvee s_i = \bigvee(s + s_i)$.

b) If C is an H-cone of functions and C is solid in C^{**}, then, for any $s \in \tilde{C}$ and any family $(s_i)_{i\in I}$ from $\tilde{C}$ we have:

$$s + \bigwedge s_i = \bigwedge(s + s_i)$$

<u>Proof.</u> a) Is clear.

b) Since any $s \in \tilde{C}$ is finely l.s.c., it is enough to prove that $\bigwedge s_i = \widehat{\inf s_i}$ (the regularisation being considered with respect to the fine topology). Let $x_0 \in X$ be such that:

$$(\bigwedge s_i)(x_0) < \widehat{\inf s_i}(x_0)$$

If we choose ρ, ρ' such that:

$$(\bigwedge s_i)(x_0) < \rho' < \rho < \widehat{\inf s_i}(x_0)$$

then $u_i \in C$, where $u_i = \rho \wedge s_i$. Hence [1]:

$$\bigwedge s_i \geqslant \bigwedge u_i = \widehat{\inf u_i}$$

But $\{x \mid u_i(x) > \rho'\}$ is a finely open set; we get then:

$$\widehat{\inf u_i}(x_0) \geqslant \rho' > (\bigwedge s_i)(x_0)$$

which is the desired contradiction. //

<u>Lemma 2.4.</u> If C is dense in C^{**}, then $C^* = C^{***}$.

<u>Proof.</u> Since C^{**} separates C^*, the natural map $C^* \longrightarrow C^{***}$ is injective. Let now $\varphi \in C^{***}$, hence $\varphi : C^{**} \longrightarrow \overline{\mathbb{R}}_+$. Let $\hat{\varphi} : C \longrightarrow \overline{\mathbb{R}}_+$ be the composition of φ with the natural map $C \longrightarrow C^{**}$. Now the hypothesis implies $\hat{\varphi} \in C^{**}$. It remains to

verify that $\widetilde{\widehat{\varphi}} = \varphi$. For any $s \in C$ we have:

$$\widetilde{\widehat{\varphi}}(s) = \widetilde{s}(\widehat{\varphi}) = \widehat{\varphi}(s) = \varphi(\widetilde{s}) \qquad //$$

Proposition 2.5. Suppose that C is dense in C^{**} and C^{*} separates C. Then $\widetilde{C^*}$ and C^o are isomorphic, as ordered convex cones and complete lattices.

Proof. Let $\mu \in C^o$. Using the hypothesis, μ can be prolonged to C^{**} by:

$$\widetilde{\mu}(\alpha) = \sup \{ \mu(\widetilde{s}) \mid s \in C, \widetilde{s} \leq \alpha \}$$

The lemma 2.4. proves that $\widetilde{\mu} \in \widetilde{C^*}$. Conversely, composing any $\mu \in \widetilde{C^*}$ with natural map $C \longrightarrow C^{**}$, we get an element from C^o. Since the above correspondences are clearly inverse one another, we have the desired isomorphism. //

Corollary 2.6. If C is dense in C^{**}, then $\widetilde{C^{**}}$ can be identified with $\widetilde{C}$.

Proof. Since C^{**} separates C^*, lemma 2.4. shows that we can apply prop. 2.5. in order to obtain the isomorphism between $\widetilde{C^{**}}$ and $(C^*)^o = \widetilde{C}$. //

Proposition 2.7. Let C be an H-cone of functions on X, and suppose moreover that C is dense in C^{**}. Then $\widetilde{C}$ is isomorphic, as an ordered convex cone and a complete lattice, with the cone of all the functions $f: X \longrightarrow \overline{\mathbb{R}}_+$, for which there exists an increasing family $(s_i)_{i \in I}$ from C such that:

$$f(x) = \sup_{i \in I} s_i(x), \quad \forall x \in X$$

Proof. Let $\mu \in \widetilde{C}$; hence $\mu: C^* \longrightarrow \overline{\mathbb{R}}_+$ and there exists an incresing family $(\mu_i)_{i \in I}$ from C^{**} such that:

$$\mu(\nu) = \sup_{i \in I} \mu_i(\nu), \quad \forall \nu \in C^*$$

The hypothesis show that we may suppose that $\mu_i = \widetilde{s}_i$, with $\widehat{s}_i \in C$. Let then $f(x) = \mu(\varepsilon_x)$. We have $f(x) = \sup_{i \in I} s_i(x)$,

$\forall\, x \in X$. Conversely, let $f : X \longrightarrow \overline{\mathbb{R}}_+$ and $(s_i)_{i\in I}$ be an increasing family from C, with $f(x) = \sup s_i(x)$, $\forall\, x \in X$. For any $\nu \in C^*$ let $\mu(\nu) = \sup\limits_{i\in I} \nu(s_i)$. Hence $\mu : C^* \longrightarrow \overline{\mathbb{R}}_+$ and by the construction $\widetilde{s}_i \uparrow \mu$. It is easy to prove that μ is independent of the choice of the family (s_i). Hence, the above correspondences are inverse one another, and give the desired identification. //

Remark. If C is dense in C^{**} and moreover contains a strictly positive element $u \in C$, then $s \in \widetilde{C}$ iff $s \wedge nu \in C$, $\forall\, n \in \mathbb{N}$. It follows that, if C is a standard H-cone, represented as an H-cone of functions on a set X, then C^{**} coincides with the set of all the functions C from $\widetilde{C}$ R_+ which are finite on a finely dense subset of X. [1]. Also, in the case of standard H-cone C, represented as an H-cone of functions on a set X, suppose that there exists a submarkovian resolvent of kernels $(V_\lambda)_{\lambda \geq 0}$ on the measurable space $(X, \mathcal{X})$, such that : V_0 is a proper kernel ; any $s \in C$ is $\mathcal{V}$-excessive ; for any $\mathcal{X}$-measurable, positive and bounded f, $V_0 f \in C$. Then $\widetilde{C}$ can be identified with the set of all $\mathcal{V}$-excessive functions.

We end by proving that any balayage and any H-integral on C extends uniquely to $\widetilde{C}$.

Proposition 2.8. Suppose that C is dense in C^{**}, and C^* separates C. Then:

a) There exists a unique bijection $B \longleftrightarrow \widetilde{B}$ between the set of balayages on C, and the set of all maps $\widetilde{B} : \widetilde{C} \longrightarrow \widetilde{C}$ having the properties: $\widetilde{B}(s+t) = \widetilde{B}s + \widetilde{B}t$; $s \leq t \Longrightarrow \widetilde{B}s \leq \widetilde{B}t$; $\widetilde{B}s \leq s$; $\widetilde{B}(\widetilde{B}s) = \widetilde{B}s$; for any increasing family $(s_i)_{i\in I}$ from $\widetilde{C}$ we have $\widetilde{B}(\bigvee s_i) = \bigvee \widetilde{B}s_i$, such that the following diagramm commutes:

$$\begin{tikzcd} C \arrow[r, "B"] \arrow[d, "\varepsilon"'] & C \arrow[d, "\varepsilon"] \\ \widetilde{C} \arrow[r, "\widetilde{B}"] & \widetilde{C} \end{tikzcd}$$

b) There exists a unique bijection $\mu \longleftrightarrow \widetilde{\mu}$ between C^* and the set of all maps $\widetilde{\mu} : \widetilde{C} \longrightarrow \overline{\mathbb{R}}_+$ having the properties:

$\widetilde{\mu}(s+t) = \widetilde{\mu}(s) + \widetilde{\mu}(t)$; $s \leq t \Longrightarrow \widetilde{\mu}(s) \leq \widetilde{\mu}(t)$; for any increasing family $(s_i)_{i \in I}$ from $\widetilde{C}$ we have $\widetilde{\mu}(\bigvee s_i) = \sup_{i \in I} \widetilde{\mu}(s_i)$

for each $s \in \widetilde{C}$ there exists an increasing family $(s_i)_{i \in I}$ from C, such that $s = \bigvee s_i$ and $\widetilde{\mu}(s_i) < +\infty$, $\forall\, i \in I$, such that the following diagramm commutes:

$$\begin{tikzcd} C \arrow[dr, "\mu"] \arrow[d, "\varepsilon"'] & \\ \widetilde{C} \arrow[r, "\widetilde{\mu}"'] & \overline{\mathbb{R}}_+ \end{tikzcd}$$

<u>Proof</u>. a) Let B be a balayage on C. For each $s \in \widetilde{C}$, let $\widetilde{B}(s) \in \widetilde{C}$ be defined by $\widetilde{B}(s)(\mu) = s(B^*\mu)$, $\forall\, \mu \in C^*$. Since $s = \bigvee \alpha_i$ with $\alpha_i \in C^{**}$, it follows that $\widetilde{B}s = \bigvee B^{**}\alpha_i$, hence $\widetilde{B}$ is well defined. The properties of $\widetilde{B}$ follow obviously from those of B. For any $s \in C$ we have:

$$\widetilde{B}(\varepsilon(s))(\mu) = \varepsilon(s)(B^*\mu) = B^*\mu(s) = \mu(Bs) = \varepsilon(Bs)(\mu)$$

hence the diagramm commutes.

Conversely, if $\widetilde{B}$ is given, from $\widetilde{B}s \leq s$ it follows that $\widetilde{B}(\varepsilon(s)) \in \varepsilon(C)$, $\forall\, s \in C$. Thus $\widetilde{B}|_C$ defines a balayage.

b) Let $\mu \in C^*$ and define, for each $s \in \widetilde{C}$: $\widetilde{\mu}(s) = s(\mu)$. $\widetilde{\mu}$ is well defined and clearly has the indicated properties. Since:

$$\widetilde{\mu}(\varepsilon(s)) = \varepsilon(s)(\mu) = \mu(s)$$

the diagramm commutes. Finally, $\widetilde{\mu}$ being given, μ is obtained as the restriction to $C \simeq \varepsilon(C)$. //

From now on, the extension to $\tilde{C}$ of any balayage B or H-integral μ on C, will be denoted by the same letter.

§3. Let C_1, C_2 be H-cones. We define a morphism of H-cones from C_1 to C_2 as a map $\varphi : \tilde{C}_1 \longrightarrow \tilde{C}_2$ with the properties:

$$\varphi(s+t) = \varphi(s) + \varphi(t)$$

$$s \leq t \implies \varphi(s) \leq \varphi(t)$$

for any increasing family $(s_i)_{i \in I}$ from $\tilde{C}_1$

$$\varphi(\bigvee s_i) = \bigvee \varphi(s_i)$$

<u>Remark.</u> Clearly we have thus a category, the objects being the H-cones and the morphisms those defined above. Acordingly, Hom (C_1, C_2) will stand for the set of all the morphisms from C_1 to C_2.

For any morphism $\varphi \in$ Hom (C_1, C_2) let us define the domain of φ as: $D(\varphi) = \{ s \in C_1 \mid \varphi(\varepsilon(s)) \in \mathcal{E}_2(C_2) \}$. $D(\varphi)$ can be void: if $\omega \in \tilde{C}_2$ is the greatest element, then $\varphi(s) = \omega$, $s \in \tilde{C}_1$ is a morphism and $D(\varphi) = \emptyset$.

We introduce a class of morphisms as follows: $\varphi \in$ Hom (C_1, C_2) is said to be a semifinite morphism if $D(\varphi)$ is dense in C_1.

<u>Exemples.</u> a) For any two H-cones C_1, C_2, the map $s \longmapsto 0$ is clearly a semifinite morphism, denoted by 0.

b) The semifinite morphisms from Hom$(C, \mathbb{R}_+)$ are exactly the H-integrals

c) The morphisms from $\mathbb{R}_+$ to C are exactly the maps $\lambda \longmapsto \lambda s$, with $s \in \tilde{C}$. Such a morphism is semifinite iff $s \in C$.

<u>Remark.</u> The composition of two semifinite morphisms need not be semi-finite.

When we consider standard H-cones, the semifinite morphisms admit a simple characterisation [7].

<u>Proposition 3.1.</u> Let C_1, C_2 be H-cones and $C' \subseteq \tilde{C}_1$ a dense, convex subcone. Let $\varphi_o : C' \longrightarrow \tilde{C}_2$ be a map such that:

$$\varphi_o(s+t) = \varphi_o(s) + \varphi_o(t)$$

$$s \leq t \implies \varphi_o(s) \leq \varphi_o(t)$$

The following are equivalent:

a) There exists a unique morphism $\varphi \in \mathrm{Hom}(C_1, C_2)$ such that $\varphi|_{C'} = \varphi_o$.

b) For any increasing family $(s_i)_{i\in I}$ from C' such that $\bigvee s_i \in C'$ we have $\varphi_o(\bigvee s_i) = \bigvee \varphi_o(s_i)$.

Proof. a $\implies$ b is obvious. b $\implies$ a. Let :

$$\varphi(s) = \bigvee \{\varphi(t) \mid t \in C',\ t \leq s\} \quad \forall s \in \tilde{C}_1$$

φ is a well defined map from $\tilde{C}_1$ to $\tilde{C}_2$, and $\varphi|_{C'} = \varphi_o$. Clearly $s \leq t \implies \varphi(s) \leq \varphi(t)$; and prop.2.3. shows that $\varphi(s+t) = \varphi(s) + \varphi(t)$. Let $(s_i)_{i\in I}$ be an increasing family from $\tilde{C}_1$ and let $s = \bigvee s_i$. It remains to prove that:

(*) $\bigvee\{\varphi(t) \mid t \in C',\ t \leq s\} = \bigvee_{i\in I} \bigvee \{\varphi(u) \mid u \leq s_i,\ u \in C'\}$

Let $t \in C'$, $t \leq s$. For each $i \in I$, there exists an increasing family $(u_j)_{j\in J}$ from C' such that $s_i \wedge t = \bigvee_{j\in J} u_j$. Since: $\bigvee(s_i \wedge t) = s \wedge t = t$, the non-obvious inequality in (*) is proved.//

Remark. Prop.3.1. shows that any morphism, as defined in [7], is also a morphism in our sense (and in fact a semifinite one).

Proposition 3.2. Let C be a standard H-cone, C_1 an H-cone and φ a semifinite morphism from C to C_1. Then $C_o \subseteq D(\varphi)$.

Proof. Let $D = \{t_n \in C_o \mid n \in \mathbb{N}\}$ be a countable, dense subset. Let $u \in C$ be strictly positive. Since $D(\varphi)$ is dense, let $s_n \in D(\varphi)$ be such that $s_n \leq t_n \leq s_n + n^{-1}u$. Let $(W_i)_{i\in I}$ be an increasing, countable family, which contains every s_n. This family is also dense, hence there exists $\alpha_i > 0$ such that :

$v = \sum \alpha_i w_i \in C^{**}$ and v is strictly positive. Hence, for each $t \in C_0$ there exists $i \in I$ such that $t \leq w_i + v$. By [7, lemma 1.2 it follows that $t \leq 2(w_i + \sum_{k \in K} \alpha_k w_k)$ with $K \subseteq I$ finite. Hence $\varphi(t) \in C_1$ and $t \in D(\varphi)$. //

Remark. Prop.3.2. shows that if C is a standard H-cone, then any semifinite morphism is a morphism in the sense of [7]. Combining the two preceding propositions with [7, prop.1.5.] we get :

Corollary 3.3. Let C_1, C_2 be standard H-cones and $\varphi_0 : (C_1)_0 \longrightarrow C_2$ a map. The following are equivalent:

a) $\varphi_0(s+t) = \varphi_0(s) + \varphi_0(t)$ and $s \leq t \Rightarrow \varphi_0(s) \leq \varphi_0(t)$.

b) There exists a (unique) semifinite morphism $\varphi \in \mathrm{Hom}(C_1, C_2)$ such that $\varphi|_{(C_1)_0} = \varphi_0$. //

Let C_1, C_2 be two H-cones and suppose that C_2 is dense in C_2^{**} and C_2^* separates C_2. We say that $\varphi \in \mathrm{Hom}(C_1, C_2)$ has an adjoit if, for each $\mu \in \widetilde{C_2^*}$ we have $\varphi^*(\mu) \in \widetilde{C_1^*}$, where:

$$\varphi^*(\mu)(s) = \mu(\varphi(s)), \ \forall s \in \widetilde{C_1}$$

It is obvious then, that $\varphi^* : \widetilde{C_2^*} \longrightarrow \widetilde{C_1^*}$ is in fact a morphism of H-cones. Moreover, if it exists, φ^* is uniquely defined by φ and will be called the adjoint of φ.

Proposition 3.4. [7] Let C_1, C_2 be standard H-cones and $\varphi \in \mathrm{Hom}(C_1, C_2)$ be a semifinite morphism. Then φ has an adjoint; moreover, φ^* is also semifinite.

Proof. Using cor.3.3., for any $\mu \in (C_2^*)_0$ we have $\varphi^*(\mu) \in C_1^*$ hence φ^* exists and is semifinite. //

Remark. Using the adjoint, we can give a simpler proof to cor.3.3. It suffices to observe that, for any morphism φ in the sense of

[7] we have :

$$\varphi^*(\bigvee \mu_i)(s) = (\bigvee \mu_i)(\varphi(s)) = \sup \mu_i(\varphi(s)) =$$
$$= \sup \varphi^*(\mu_i)(s) = (\bigvee \varphi^*(\mu_i))(s)$$

(where (μ_i) is an increasing family from $D(\varphi^*)$) and that $\varphi = \varphi^{**}$.

There exists morphisms which are not semifinite, but have an adjoint. For exemple, the morphism $\lambda \longmapsto \lambda s$ from $\mathbb{R}_+$ to C, where $s \in \tilde{C} \setminus C$. On the other hand, there exists morphimms (necesarilly not semifinite) which have no adjoint: it suffices to take any morphism from C to $\mathbb{R}_+$ which is not in C^o.

Suppose that C_1, C_2 are dense in C_1^{**}, C_2^{**} and that C_1^*, C_2^* separates C_1, C_2.

Proposition 3.5. If φ has an adjoint, then φ^* also has an adjoint and $(\varphi^*)^* = \varphi$.

Proof. It suffices to prove that φ is the adjoint of φ^*, which follows from:

$$\varphi(\tilde{s})(\mu) = \mu(\varphi(s)) = \varphi^*(\mu(s)) = \tilde{s}(\varphi^*(\mu)) =$$
$$= \varphi^{**}(\tilde{s})(\mu) \quad //$$

Proposition 3.6. Suppose that φ_1, φ_2 have adjoints and $\varphi_2 \circ \varphi_1$ is defined. Then $\varphi_2 \circ \varphi_1$ has an adjoint and :

$$(\varphi_2 \circ \varphi_1)^* = \varphi_1^* \circ \varphi_2^*$$

Proof. It suffices to show that $\varphi_1^* \circ \varphi_2^*$ is tha adjoint of $\varphi_2 \circ \varphi_1$. But :

$$(\varphi_2 \circ \varphi_1)^*(\mu)(s) = \mu((\varphi_2 \circ \varphi_1)(s))$$

and :

$$(\varphi_1^* \circ \varphi_2^*)(\mu)(s) = \varphi_1^*(\varphi_2^*(\mu)(s) = \varphi_2^*(\mu)(\varphi_1(s)) =$$
$$= \mu(\varphi_2(\varphi_1(s))) \quad //$$

It is easy to verify that, if $\varphi, \psi \in \mathrm{Hom}\,(C_1,C_2)$ have adjoints, then $\varphi + \psi$ has also adjoint and $(\varphi + \psi)^* = \varphi^* + \psi^*$. Moreover, $\varphi \le \psi \iff \varphi^* \le \psi^*$ which shows that the lattice opperations are preserved under the passage to the adjoint:

$$(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^* \ , \ (\varphi \vee \psi)^* = \varphi^* \vee \psi^*$$

Particularly, φ lies on an extreme ray iff φ^* lies on an extreme ray.

Hence, we have another category, in which the morphisms are those which have adjoint. Taking only standard H-cone as objects, we have a (contravariant) functor from this category into itself, defined by $F(C) = C^*$; $F(\varphi) = \varphi^*$. This functor is fully faithfull and cor.3.8. will show that it establishes in fact an equivalence between the category just considered and its dual.

In accordance with the general definition, we call isomorphism of H-cones, any morphism $\varphi \in \mathrm{Hom}\,(C_1,C_2)$ for which there exists $\psi \in \mathrm{Hom}\,(C_2,C_1)$ such that $\varphi \circ \psi = 1$, $\psi \circ \varphi = 1$. Obviously, the morphism φ is an isomorphism iff the map $\varphi : \tilde{C}_1 \longrightarrow \tilde{C}_2$ is bijective and $\varphi(s) \le \varphi(t) \implies s \le t$. There exists morphisms which are bijections, but not isomorphisms. For exemple, let C be a standard H-cone. Then $(C, \preceq)$ endowed with the specific order is still an H-cone. The identity $(C, \preceq) \longrightarrow (C, \le)$ is a morphism of H-cones, which is a bijection, but not isomorphism.

Let us call a morphism $\varphi \in \mathrm{Hom}\,(C_1,C_2)$ finite, if $D(\varphi) = C_1$ The balayages are exemples of finite morphisms. It is clear that the composition of two finite morphisms is again finite. Moreover, if φ is finite and ψ is semifinite, then $\varphi \circ \psi$ (provided it is defined) is again semifinite. However, if φ is semifinite and ψ is finite, $\varphi \circ \psi$ could be not semifinite: let $\psi : \mathbb{R}_+ \longrightarrow C$ be defined by $\psi(\lambda) = \lambda s$ with $s \in C$; and let φ be a semifinite morphism $\varphi \in \mathrm{Hom}(C,C_1)$ such that $\varphi(s) \notin C_1$.

An isomorphism need not be finite. However:

Proposition 3.7. Let C_1, C_2 be standard H-cones. Then the following are equivalent:

a) φ is an isomorphism between C_1 and C_2.

b) φ^* is a finite isomorphism between C_1^* and C_2^*.

Proof. a $\Longrightarrow$ b. Let $\mu \in C_2^*$; it suffices to prove that $\varphi^*(\mu)$ satisfies I4 from the definition of the H-integral. Let $s \in C_1$; there exists an increasing family $(t_i)_{i\in I}$ in C_2 such that $\varphi(s) = \bigvee t_i$ and $\mu(t_i) < \infty$. Since φ is isomorphism, we have $t_i = \varphi(u_i)$ with $u_i \in C_1$ and $s = \bigvee u_i$. Hence $\varphi^*(\mu)(u_i)$ is finite.

b $\Longrightarrow$ a. $\varphi^*: C_2^* \longrightarrow C_1^*$ extsnds clearly to an isomorphism between $\widetilde{C}_1 = (C_1^*)^o$ and $\widetilde{C}_2 = (C_2^*)^o$; //

Corollary 3.8. Let C be a standard H-cone. The natural map $C \longrightarrow C^{**}$ is an isomorphism. //

Remark. An H-cone could be isomorphic with a standard H-cone, without being itself a standard H-cone. Indeed, $\mathbb{R}_+^{(\mathbb{N})}$ is an H-cone, isomorphic with the standard H-cone $\mathbb{R}_+^{\mathbb{N}}$; but $\mathbb{R}_+^{(\mathbb{N})}$ is not standard, since it has no strictly positive element. However:

Proposition 3.9. Let C be an H-cone, C_1 a standard H-cone and suppose that C and C_1 are isomorphic. Then C is a standard H-cone iff C is dense in $\widetilde{C}$ and C contains a strictly positive element.

Proof. Denote $u \in C$ a strictly positive element and $\varphi: \widetilde{C} \longrightarrow \widetilde{C}_1$ an isomorphism. Let $s \in (C_1)_o$ be arbitrary. There exists $t \in \widetilde{C}$ such that $\varphi(t) = s$. It suffices to prove that t is u-continuous. But:

$$s = \varphi(t) = \varphi(\bigvee (t \wedge nu)) = \bigvee (s \wedge n.\varphi(u))$$

shows that there exists $\alpha > o$ such that $s \leq \alpha\varphi(u)$. It follows that $t \leq \alpha.u$, hence $t \in C$, simce C is also solid in $\widetilde{C}$. Now let $(t_i)_{i\in I}$ be an increasing family from C, such that $t = \bigvee t_i$. Since $s = \varphi(t) = \bigvee \varphi(t_i)$, we deduce that for any $\varepsilon > o$ there exists $i \in I$ such that $s \leq \varphi(t_i) + \varepsilon\varphi(u)$ It results $t \leq t_i + \varepsilon u$. //

Let C, C', C_1, C_1' be H-cones and $\varphi \in \mathrm{Hom}(C, C_1)$, $\psi \in \mathrm{Hom}(C', C_1')$. We say that φ and ψ are isomorphic if there exists isomorphisms $C \approx C'$ and $C_1 \approx C_1'$ such that the following diagramm is commutative:

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\varphi"] \arrow[d, phantom, "\wr\wr"] & \widetilde{C}_1 \arrow[d, phantom, "\wr\wr"] \\ \widetilde{C}' \arrow[r, "\psi"] & \widetilde{C}_1' \end{tikzcd}$$

With this defibition, one verifies easily that any semifinite morphism of standard H-cones is isomorphic with a finite one. Indeed, if C and C_1 are standard H-cones, then $\varphi \in \mathrm{Hom}(C, C_1)$ is semifinite iff there exists a strictly positive $u \in C$ such that $\varphi(u) \in C_1^{**}$. If φ is semifinite, there exists [7] $\alpha_n > o$ such that $\sum \alpha_n \varphi(s_n) \in C_1^{**}$ where $\{s_n \mid n \in \mathbb{N}\}$ is a dense part of C_o. We can take then $u = \sum \alpha_n s_n$. Conversely, if $\varphi(u) \in C_1$, it follows that $\varphi(s) \in C_1$ for any $s \in C_o$. Now, we can take C' as the subcone of u-bounded elements from C, and $C_1' = C_1^{**}$. Then C' is isomorphic with C; C_1' is isomorphic with C_1 and $\varphi|_{C'} : C' \longrightarrow C_1'$ is clearly finite.

We consider finally the morphisms between H-cones of functions. Let $\mathcal{F}$ and $\mathcal{F}'$ be H-cones of functions on the sets X and X'. We suppose moreover that $\mathcal{F}$ and $\mathcal{F}'$ are dense respectively in $\mathcal{F}^{**}$ and $\mathcal{F}'^{**}$. A map $\varphi : X' \longrightarrow X$ is called an H-map if, for any $s \in \widetilde{\mathcal{F}}$ we have $s \circ \varphi \in \widetilde{\mathcal{F}}'$.

Proposition 3.10. Let φ be an H-map. Then $s \longmapsto s\circ\varphi$ is a morphism (denoted also by φ) from $\mathcal{F}$ to $\mathcal{F}'$ which satisfies:

$$\varphi(1) = 1 \quad \text{and} \quad \varphi(s \wedge t) = \varphi(s) \wedge \varphi(t) \quad , \forall s,t \in \mathcal{F}$$

The map $\varphi : X' \longrightarrow X$ is fine-to-fine continuous and the morphism φ is semifinite. //

There exists morphisms (even finite isomorphisms) which are not induced by a H-map. If $\mathcal{F}$ is an H-cone of functions on X and $f : X \longrightarrow (o, +\infty)$ is not $\equiv 1$, then $\mathcal{F}$ and $f.\mathcal{F}$ are isomorphic, but there is no H-map which induces this isomorphism.

Let $\mathcal{F}$ and $\mathcal{F}'$ be H-cones of functions and $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{F}')$. If $\varphi(1)$ is a finite function, then φ^* exists. Moreover, for any $x \in X$, $\varphi^*(\varepsilon_x) \in C_2^*$.

Clearly, the morphisms which are induced by H-maps are closed under composition.

Proposition 3.11. Let φ be an H-map which is finely open. Then, if $\mathcal{F}'^{**} = \mathcal{F}'$, φ induces a finite morphism.

Proof. For any $s \in \mathcal{F}$ the set $A = \{x \mid s(x) < \infty\}$ is finely dense. Hence $\varphi^{-1}(A)$ is also finely dense. But $(s\circ\varphi)(x') < \infty$ for any $x' \in \varphi^{-1}(A)$, hence is finite on a finely dense set. Using prop.2.7. and [ ], so $\varphi \in \mathcal{F}'^{**} = \mathcal{F}'$. //

Proposition 3.12. Let $\mathcal{F}, \mathcal{F}'$ be standard H-cones of functions, φ be an H-map. Then φ is naturally measurable. Let $\mu' \in \mathcal{F}'^*$ be a representable H-integral with $\mu'(1) < \infty$. Then $\varphi^*(\mu')$ is also representable.

Proof. For any $s \in \mathcal{F}_o$, $s\circ\varphi$ is afinite, naturally l.s.c. function on X. Hence φ is naturally measurable. It is clear that $\varphi^*(\mu') \in \mathcal{F}^*$ is then representable by the measure m, defined on the naturally Borel sets by : $m(A) = m'(\varphi^{-1}(A))$, where m' is the representing measure for μ'. //

More generally, let S, S' be two standard H-cones of functions on X, X'. Let us call quasi-H-map any map $\varphi : X' \setminus A' \longrightarrow X$ where $A' \subseteq X'$ is a semi-polar set, such that, for each $s \in S$, there exists an element from $\widetilde{S'}$, denoted $\varphi(s)$, such that $\varphi(s) = s\circ\varphi$ on $X' \setminus A'$.

Proposition 3.13. Any quasi-H-map defines a morphism between S and S', which has the properties:

$$\varphi(1) = 1 .$$

$$\varphi(s \wedge t) = \varphi(s) \wedge \varphi(t) \quad , \forall s,t \in S$$

$\forall \mu' \in (S')^*_o$, $\varphi^*(\mu')$ is a representable H-integral

Proof. φ is well defined, since from $s = t$ on $X' \setminus X'$ it follows $\mu(s) = \mu(t)$, $\forall \mu \in S^*_o$ ([3]) hence $s = t$. The other statements are obvious. //

Remark. If A' is polar, then $\varphi^*(\mu')$ is representable for any universally bounded μ'.

For any two morphisms of H-cones $\varphi_1, \varphi_2 \in \mathrm{Hom}\,(C_1, C_2)$ we define :

$$(\varphi_1 + \varphi_2)(s) = \varphi_1(s) + \varphi_2(s) \;, \forall s \in \widetilde{C}_1$$

$$(\alpha.\varphi_1)(s) = \alpha.\varphi_1(s) \;, \forall s \in \widetilde{C}_1 \;, \alpha \geqslant 0$$

and the order relation $\varphi_1 \leq \varphi_2$ by $\varphi_1(s) \leq \varphi_2(s)$, $\forall s \in \widetilde{C}_1$. Thus $\mathrm{Hom}\,(C_1, C_2)$ becomes an ordered convex cone, in which 0 is the smallest element, and the morphism defined by $\varphi(s) = \omega$, $\forall s \in \widetilde{C}_1$ (where $\omega \in \widetilde{C}_2$ is the greatest element) is the greatest element.

The morphisms which have an adjoint; the semifinite morphisms; and the finite morphisms form solid, convex subcones. Moreover, the semifinite morphisms form a linearisable cone.

One verifies easily that, for any increasing family (φ_i)

from Hom (C_1,C_2), $\bigvee \varphi_i$ exists and:

$$(\bigvee \varphi_i)(s) = \bigvee \varphi_i(s) \ , \ \forall s \in \tilde{C}_1$$

Hence, for any $\varphi \in$ Hom (C_1,C_2) we have :

$$\varphi + \bigvee \varphi_i = \bigvee(\varphi + \varphi_i)$$

We recall [7] that, if C_1, C_2 are standard H-cones, and $(\varphi_i)_{i \in I}$ is any family of semifinite morphisms from Hom (C_1,C_2), then $\bigwedge \varphi_i$ exists. Moreover, for any semifinite $\varphi \in$ Hom (C_1,C_2) we have :

$$\varphi + \bigwedge \varphi_i = \bigwedge(\varphi + \varphi_i)$$

Hence:

Proposition 3.14. Let C_1, C_2 be standard H-cones. The semifinite morphisms between C_1 and C_2 form an ordered convex cone, which satisfies the properties H1 - H5 from the definition of an H-cone. //

§ 4. We call a pointed H-cone any pair (C,u) where C is a standard H-cone and $u \in C$ a strictly positive element. We recall [1] that the set:

$$K(C,u) = \{ \mu \in C^* \mid \mu(u) \leq 1 \}$$

is a convex, compact space in the natural topology. The set X, of non-zero extreme points of $K(C,u)$ is called the natural space of representation for (C,u). C is isomorphic with the standard H-cone of functions S on X, formed of the restrictions to X of the maps : $\mu \longmapsto \tilde{s}(\mu) \triangleq \mu(s)$.

Let (C,u), (C',u') be pointed H-cones. We call a morphism of pointed H-cones any $\varphi \in$ Hom (C,C') such that $\varphi(u) = u'$. Clearly, any such morphism is semifinite, hence it has an adjoint. We denote $K(\varphi)$ the restriction of φ^* to $K(C,u)$. Obviously, $K(\varphi)$ is a map from $K(C,u)$ to $K(C',u')$.

<u>Proposition 4.1.</u> $K(\varphi)$ is an affine map. $K(\varphi)$ is fine-to-fine continuous map. $K(\varphi)$ is naturally continuous iff $\varphi(C_o) \subseteq C'_c$.

<u>Proof.</u> All the continuity assertions are consequences of the formula:

$$\tilde{s}\left[K(\varphi)\mu\right] = \widetilde{\varphi(s)}\,\mu \qquad //$$

<u>Theorem 4.2.</u> Let φ be a morphism of pointed H-cones. The following are equivalent:

1) For each extreme point μ of $K(C',u')$, $K(\varphi)\mu$ is an extreme point in $K(C,u)$.

2) $\varphi(s \wedge t) = \varphi(s) \wedge \varphi(t)$, $\forall\, s,t \in C$

3) There exists a (unique H-) map, denoted $p(\varphi): X' \longrightarrow X$, such that $\varphi(s) = s \circ p(\varphi)$, $\forall s \in C$ (i.e. $p(\varphi)$ induces φ).

<u>Proof.</u> Let us remark that, if $\mu \neq o$ is extreme in $K(C',u')$, then $K(\varphi)\mu$ is also $\neq o$, since:

$$K(\varphi)\mu(u) = \mu(\varphi(u)) = \mu(u') = 1$$

<u>$1 \Longrightarrow 3 \Longrightarrow 2$</u> are now obvious.

<u>$2 \Longrightarrow 1$</u> Using the characterisation of extreme points of $K(C,u)$ as given in [4], we have, for any extreme $\mu \in K(C',u')$:

$$K(\varphi)(\mu)(s \wedge t) = \mu(\varphi(s \wedge t)) = \mu(\varphi(s) \wedge \varphi(t)) =$$

$$= \min\left\{\mu(\varphi(s)), \mu(\varphi(t))\right\} =$$

$$= \min\left\{K(\varphi)(\mu)(s), K(\varphi)(\mu)(t)\right\} \qquad //$$

<u>Remark.</u> The preceeding theorem, toghether with the definition of the isomorphism (§3) show that the correct setting for a "Banach-Stone" theorem (i.e. if two standard H-cone are isomorphic, then their natural spaces of representation are homeomorphic), is that of natural spaces of representation. Deleteing a polar part from such a space, the isomorphism between the standard H-cones is preserved ; in fact, this is the only way to produce "pathological"

exemples as in [9].

Corollary 4.3. Suppose that the equivalent conditions from th.4.2. hold. Then $p(\varphi)$ is a fine-to-fine continuous map, and is naturally measurable. $p(\varphi)$ is naturally continuous iff $\varphi(C_o) \subseteq C'_c$. //

Theorem 4.4. Suppose that $u = \sum s_n$, with $s_n \in C_o$. The set of all morphisms of pointed H-cones is a convex, compact set in the natural topology.

Proof. Let $\{\varphi_n\}$ be a sequence of morphisms of pointed H-cones. Let $\{s_n\}$ be a dense, contable subset of C_o. For each $n \in \mathbb{N}$, $\{\varphi_k(s_n)\}_{k\in\mathbb{N}}$ is a relatively compact set in the natural topology of C'. Using the diagonal procedure, we can choose a subsequnce (denoted still by φ_n) such that $\{\varphi_k(s_n)\}_{k\in\mathbb{N}}$ is naturally convergent, for each $n \in \mathbb{N}$. Let $\varphi(s_n)$ be the limit. We define, for each $s \in C$:

$$\varphi(s) = \bigvee \{\varphi(s_n) \mid s_n \leq s\}$$

It suffices to prove that $\mu(\varphi_n(s)) \longrightarrow \mu(\varphi(s))$, for each $\mu \in (C')^*_o$ and $s \in C_o$. Let $\varepsilon > 0$; we choose $m \in \mathbb{N}$ such that:

$$s_m \leq s \leq s_m + \varepsilon u$$

and: $\mu(\varphi(s_m)) \leq \mu(\varphi(s)) \leq \mu(\varphi(s_m)) + \varepsilon$. Then, for any $n \in \mathbb{N}$: $\mu(\varphi_n(s_m)) \leq \mu(\varphi_n(s)) \leq \mu(\varphi_n(s_m)) + \varepsilon . \mu(u')$

Since $\mu(\varphi_n(s_m)) \longrightarrow \mu(\varphi(s_m))$, there exists n_ε such that, for any $n \geq n_\varepsilon$:

$$|\mu(\varphi_n(s_m)) - \mu(\varphi(s_m))| \leq \varepsilon$$

Combining these inequalities, we get:

$$-2\varepsilon \leq \mu(\varphi_n(s)) - \mu(\varphi(s)) \leq \varepsilon(1 + \mu(u'))$$

Hence φ is a semifinite morphism by [7], and $\varphi_n \to \varphi$ in the natural topology. Moreover, $\varphi(u) = u'$. Indeed, μ being a fixed universally continuous H-integral, for each $\varepsilon > 0$ we can find n such that: $\mu(\varphi(\sum_{k=n+1}^{\infty} s_k)) < \varepsilon$. Then:

$$\mu(\varphi(\sum_{k=1}^{n} s_k)) \leq \liminf_{m\to\infty} \mu(\varphi_m(u)) \leq$$
$$\leq \limsup_{m\to\infty} \mu(\varphi_m(u)) \leq \mu(\varphi(\sum_{k=1}^{n} s_k)) + 2\varepsilon$$

Hence $\lim_{m\to\infty} \mu(\varphi_m(u))$ exists; it follows easily that this limit equals $\mu(\varphi(u))$. //

Remark. Without any hypothesis on u, the same proof shows that the set of all morphisms $\varphi \in \mathrm{Hom}(C,C')$ such that $\varphi(u) \leq u'$ is a convex, compact set in the natural topology.

Proposition 4.5. If C, C' are standard H-cones of functions, the set of all morphisms induced by H-maps is a G_δ-set in the natural topology.

Proof. Let $\{s_n\}$ be a countable, dense subset of universally continuous elements from C. Let us denote:

$$A_{n,m,k} = \{\varphi \in \mathrm{Hom}(C,C') \mid \varphi(1) = 1,\ \varphi(s_n \wedge s_m) + k^{-1} > \varphi(s_n) \wedge \varphi(s_m)\}$$

Every $A_{n,m,k}$ is a natural G_δ-set, and it is easy to show that $\bigcap_{n,m,k} A_{n,m,k}$ is exactly the set of all morphisms induced by H-maps. //

Proposition 4.6. Any H-map between the natural spaces of representation, induces an extreme element in the set of all morphisms of pointed H-cones.

Proof. Let φ be induced by a H-map, and suppose that:

$$\varphi = \alpha\psi + (1-\alpha)\xi\ ,\quad \alpha \in (0,1)$$

It follows that:

$$u' = \varphi(u) = \alpha\psi(u) + (1-\alpha)\xi(u) \leq \alpha u' + (1-\alpha)u' = u'$$

hence $\psi(u) = \xi(u) = u'$. Moreover, for any $s,t \in C$:

$$\varphi(s)\wedge\varphi(t) = \varphi(s\wedge t) = \alpha\psi(s\wedge t)+(1-\alpha)\xi(s\wedge t)\le$$

$$\le\alpha(\psi(s)\wedge\psi(t))+(1-\alpha)(\xi(s)\wedge\xi(t))\le\varphi(s)\wedge\varphi(t)$$

Then : $\psi(s\wedge t) = \psi(s)\wedge\psi(t)$ and $\xi(s\wedge t) = \xi(s)\wedge\xi(t)$.
Hence ψ and ξ are induced by H-maps (we use the same letters for the morphisms, as well as for the induced H-maps). Hence, for any $x\in X$ and any $s\in C'$:

$$s(\varphi(x)) = \alpha s(\psi(x))+(1-\alpha)s(\xi(x))$$

If $\varphi(x) = \psi(x)$ and $\varphi(x)\ne\xi(x)$, then we can choose $s\in C$ which separates the points, and this is a contradiction. Suppose that all three points $\varphi(x)$, $\psi(x)$ and $\xi(x)$ are distinct. If there exists $s\in C$ such that $s(\varphi(x)) > s(\psi(x))$ and $s(\varphi(x))\geq s(\xi(x))$, then we obtain again a contradiction. Hence, we may suppose that, for any $s\in C$:

$$s(\psi(x)) > s(\varphi(x)) > s(\xi(x))$$

Let $p\in C$ be a generator. We may choose convenably a balayage B on C such that the quantities :

$$\frac{s(\psi(x)) - s(\xi(x))}{s(\varphi(x)) - s(\xi(x))}$$

are not equal, when s is replaced by p and then by Bp. And this is the desired contradiction. //

<u>Remark</u>. In the case of H-integrals, the converse is also true : any extreme H-integral is induced by an H-map.

Let now S , S' be standard H-cones of functions on the sets X, X'. From th.4.2. it follows that H-maps extends uniquely to an H-maps between the natural spaces of representation. We characterise finally the morphisms induced by a quasi-H-map.

<u>Proposition 4.7</u>. Let $\varphi\in\mathrm{Hom}(S,S')$ be such that:

$\varphi(1) = 1$

$\varphi(s \wedge t) = \varphi(s) \wedge \varphi(t)$, $\forall s,t \in S$

$\forall \mu \in (S')^*_o$, $\varphi^*(\mu)$ is representable

Then φ is induced by a quasi-H-map.

Proof. We know that $p(\varphi)$ exists, and is a H-map between the natural spaces of representation. Let us denote:

$$A' = \{x \in X'_1 \mid p(\varphi)(x) \notin X\}$$

(X'_1 being the natural space of representation for S'). Then, for any compact $K \subseteq A'$:

$$\mu(K) \leq \mu(p(\varphi)^{-1}(p(\varphi)(K))) = \varphi^*(\mu)(p(\varphi)(K)) = 0$$

since $\varphi^*(\mu)$ is representable. Hence [3], A' is semipolar and $p(\varphi)$, when restricted to $X' \setminus A'$, takes its values in X. //

Remark. If we wish the set from prop. 4.7. to be polar, it suffices to ask that $\varphi^*(\mu)$ is representable, for any universally bounded H-integral μ on S'.

§ 5. We end with a brief discussion of morphisms between (pre-) sheafs of H-cones.

Let (X,τ) , (X',τ') be topological spaces, $\varphi: X' \longrightarrow X$ a continuous map , and $\mathfrak{F}$, $\mathfrak{F}'$ (pre)sheafs on X , X'. By a φ-morphism we mean a collection of morphisms $(\Phi_U)_{U\in\tau}$, such that , for each $U \in \tau$, $\Phi_U: \mathfrak{F}'(\varphi^{-1}(U)) \longrightarrow \mathfrak{F}(U)$, and the usual commutativity with the restrictions holds, i.e.: if $V \subseteq U$, then the following diagramm commutes:

$$\begin{array}{ccc} \mathfrak{F}'(\varphi^{-1}(U)) & \xrightarrow{\Phi_U} & \mathfrak{F}(U) \\ \rho\downarrow & & \downarrow\rho \\ \mathfrak{F}'(\varphi^{-1}(V)) & \xrightarrow{\Phi_V} & \mathfrak{F}(V) \end{array}$$

Proposition 5.1. Suppose that, for each $U \in \tau$, $U' \in \tau'$, $\mathfrak{F}(U)$

$\mathcal{F}'(U')$ are cones of numerical, positive $\overbrace{\text{functions}}^{\text{continuous}}$ on U, U'; and $1 \in \mathcal{F}(X)$, $1 \in \mathcal{F}'(X')$. Suppose also that Φ is a φ-morphism, such that $\Phi_X(1) = 1$ and for each $U \in \tau$, Φ_U is monotone and positively homogeneous. Then, for each $U \in \tau$ and $s \in \mathcal{F}'(\varphi^{-1}(U))$ we have: $\Phi_U(s) = s \circ \varphi s$.

Proof. Let $y \in \varphi^{-1}(U)$ be such that $s(\varphi(y)) > 0$. Let $\alpha, \beta > 0$ be such that $s(\varphi(y)) \in (\alpha, \beta)$. If we denote:

$$U_{\alpha,\beta} = \{x \in U \mid \alpha < s(x) < \beta\}$$

then $U_{\alpha,\beta} \in \tau$ and:

$$\alpha = \Phi(\alpha) < \Phi(s|_{U_{\alpha,\beta}}) = \Phi(s) \mid \varphi^{-1}(U_{\alpha,\beta}) < \Phi(\beta) = \beta$$

Hence $\Phi(s)(y) \in (\alpha, \beta)$. Since α, β were arbitrary, it follows that $\Phi(s)(y) = s(\varphi(y))$. The case $s(\varphi(y)) = 0$ is even simpler. //

Let now $(X, \mathcal{U})$, $(X', \mathcal{U}')$ be two harmonic spaces [6]. A continuous map $\varphi : X' \longrightarrow X$ is called a harmonic mapping [5] if, for any open set $U \subseteq X$ and any hyperharmonic function $s \in \mathcal{U}(U)$, we have $s \circ \varphi \in \mathcal{U}'(\varphi^{-1}(U))$.

Proposition 5.2. Let $(X, \mathcal{U})$, $(X', \mathcal{U}')$ be two $\mathcal{B}$-harmonic spaces with countable base, and $\varphi : X' \longrightarrow X$ be a harmonic mapping. For any regular open set $U \subseteq X$ and any positive superharmonic $s \in \mathcal{U}(X)$ we have:

$$B^{\complement \varphi^{-1}(U)}(s \circ \varphi) = (B^{\complement U}s) \circ \varphi$$

Proof. Since $B^{\complement U}s = s$ on $\complement U$, it follows that $B^{\complement \varphi^{-1}(U)}(s \circ \varphi) \leq (B^{\complement U}s) \circ \varphi$. Moreover, both functions are harmonic on $\varphi^{-1}(U)$. Now, $p = s \circ \varphi - B^{\complement \varphi^{-1}(U)}(s \circ \varphi)$ is a potential on $\varphi^{-1}(U)$, and dominates $p' = s \circ \varphi - (B^{\complement U}s) \circ \varphi$ on $\varphi^{-1}(U)$. The difference $(p - p')$ being harmonic on $\varphi^{-1}(U)$, we conclude the equality. //

Remark. A slight modification in the proof shows that the formula (*) holds for any open set $U \subseteq X$, provided $(X, \mathcal{U})$ satisfy the axiom of polarity. We give next a different proof of this fact, investigating the relation with the carrier theory in H-cones [2]

S, S' will be standard H-cones of functions on X, X'. τ, τ' will be finer than the natural topologies, and coarser than the fine topologies. Moreover, $\varphi : X' \longrightarrow X$ will be an H-map.

Proposition 5.3. If the equality (*) holds for any $s \in S$ and any $U \subseteq X$ from a base $\mathcal{B}$ for τ, then:

$$\operatorname{carr}(s \circ \varphi) \subseteq \varphi^{-1}(\operatorname{carr} s)$$

Proof. Using [2], let $x \in X'$ be such that $y = \varphi(x) \notin \operatorname{carr} s$. There exists an open neighbourhood $U \in \mathcal{B}$ for y, such that $B^{\complement U} s = s$. It follows that $B^{\complement \varphi^{-1}(U)}(s \circ \varphi) = (B^{\complement U} s) \circ \varphi = s \circ \varphi$, which proves that $x \notin \operatorname{carr}(s \circ \varphi)$. //

Proposition 5.4. Suppose that for any polar $A \subseteq X$, $\varphi^{-1}(A) \subseteq X'$ is polar. Let $U \subseteq X$ be open and such that $\complement U \setminus b(\complement U)$ is polar. If $s \in S$ and $\operatorname{carr}(s \circ \varphi) \subseteq \varphi^{-1}(\operatorname{carr} s)$, then (*) holds for U.

Proof. Since $s = B^{\complement U} s$ on $b(\complement U)$, it follows that $s \circ \varphi = (B^{\complement U} s) \circ \varphi$ on $\varphi^{-1}(b(\complement U))$. The hypothesis show that $\complement \varphi^{-1}(U) \setminus \varphi^{-1}(b(\complement U)$ is polar, hence $B^{\complement \varphi^{-1}(U)}(s \circ \varphi) \leq (B^{\complement U} s) \circ \varphi$. Now, since φ preserves the carrier, $B^{\complement U} s \circ \varphi$ is invariant under any balayage which dominated $B^{\complement \varphi^{-1}(U)U}$, proving thus the equality. //

Using now [6] and [8] we get:

Corollary 5.5. Let $(X, \mathcal{U})$ and $(X', \mathcal{U}')$ be $\mathcal{P}$-harmonic spaces with countable base, such that $(X, \mathcal{U})$ satisfy the axiom of polarity. Let $\varphi : X' \longrightarrow X$ be a harmonic mapping. Then, for any open set $U \subseteq X$, and any positive superharmonic function $s \in \mathcal{U}(X)$, the relation (*) holds. //

The above results suggested the following definitions. Let

$(C, \mathcal{B})$, $(C', \mathcal{B}')$ be pairs with C,C' H-cones, and $\mathcal{B}$, $\mathcal{B}'$ sets of balayages on C,C'.Between such pairs,we define two types of morphisms,as pairs (φ, ψ) where $\varphi \in \mathrm{Hom}\,(C,C')$ and:

I. for the first type $\psi : \mathcal{B}' \longrightarrow \mathcal{B}$ is a map,such that the following diagramm is commutative,for any $B' \in \mathcal{B}'$:

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\varphi"] \arrow[d, "\psi(B')"'] & \widetilde{C'} \arrow[d, "B'"] \\ \widetilde{C} \arrow[r, "\varphi"] & \widetilde{C'} \end{tikzcd} \qquad \text{i.e. } \forall s \in \widetilde{C} : \ B'(\varphi(s)) = \varphi(\psi(B')s)$$

II. for the second type $\psi : \mathcal{B} \longrightarrow \mathcal{B}'$ is a map such that the following diagramm is commutative,for any $B \in \mathcal{B}$:

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\varphi"] \arrow[d, "B"'] & \widetilde{C'} \arrow[d, "\psi(B)"] \\ \widetilde{C} \arrow[r, "\varphi"] & \widetilde{C'} \end{tikzcd} \qquad \text{i.e. } \forall s \in \widetilde{C} : \ \varphi(Bs) = \psi(B)(\varphi(s))$$

It is clear that,with each type of morphisms,we can define a category.

Exemples. $(0, \psi)$ is acceptable,for both types,for any ψ .Also, $(1,1)$ is acceptable in both cases.If $C = C'$ and φ is given, ψ can be choosen,in both cases,as the identity on the balayages greater than φ (this is the only case considered in [7]),Let us consider the case $C = \mathbb{R}_+$.If (φ, ψ) is a morphism of the first type,then ψ can be defined only on those balayages B' on C',for which $B's = 0$ or $B's = s$ (where $s = \varphi(1)$).However, if (φ, ψ) is of the second type,then for given φ, ψ can generally be chosen in a infinity of ways.Now,if $C' = \mathbb{R}_+$,then for each $\mu \in C^*$ there exists generally an infinity of maps ψ such that (μ, ψ) is a morphism of the first type.While,in order that (μ, ψ) be o morphism of the second type, ψ is defined only on those balayages which satisfy either: $\mu(s) = \mu(Bs$

or $\mu(Bs) = 0 , \forall s \in C$.

We define next the adjoint for the morphisms introduced above. This construction will justify also the consideration of the two types of morphisms. From now on, we suppose again that the H-cones in discussion have the properties: C is dense in C^{**}, and C^* separates C. Let (φ, ψ) be a morphism, and denote:

$$\psi^*(B) = [\psi(B^*)]^*$$

where B is a balayage on C^*.

<u>Proposition 5.6.</u> Let (φ, ψ) be a morphism of the first (resp. second) type, from $(C, \mathcal{B})$ to $(C', \mathcal{B}')$. If φ^* exists, then (φ^*, ψ^*) is a morphism of the second (resp. first) type, from $(C'^*, \mathcal{B}'^*)$ to $(C^*, \mathcal{B}^*)$.

<u>Proof.</u> Indeed, we have:

$$B(\varphi^*(\mu))(s) = \varphi^*(\mu)(B^*s) = \mu(\varphi(B^*s)) = \mu(\psi(B^*))(\varphi(s))$$

$$\varphi^*(\psi^*(B)\mu)(s) = \psi^*(B)\mu(\varphi(s)) = \mu(\psi(B))(\varphi(s))$$

for any $\mu \in C'^*$, $s \in C$, $B \in \mathcal{B}^*$. Respectively:

$$\varphi^*(B\mu)(t) = B\mu(\varphi(t)) = \mu(B^*(\varphi(t)) = \mu(\varphi(\psi^*(B)(t))$$

$$\psi^*(B)(\varphi^*(\mu))(t) = \varphi^*(\mu)(\psi(B^*)(t)) = \mu(\varphi(\psi(B^*)(t))$$

for any $\mu \in C'^*$, $t \in C$, $B \in \mathcal{B}'^*$. //

<u>Remark.</u> If the H-cones are standard, the adjoint establishes again an equivalence of categories.

We end with the construction of a morphism φ_B, which acts from $C_{\psi(B)}$ to C'_B [1]. Here also we extend the construction given in [7]. We consider pairs $(C, \mathcal{B})$ with C standard H-cone ; and morphisms (φ, ψ) of the first type, with φ semi-finite. For each $B \in \mathcal{B}$ we define $\varphi_B \in \mathrm{Hom}\,(C_{\psi(B)}, C'_B)$ by:

$$\varphi_B(s - Bs) = \varphi(s) - \psi(B)(\varphi(s)) = \varphi(s) - \varphi(Bs)$$

for each $s \in D(\varphi)$. Since $D(\varphi)$ contains a generator [7], it follows that φ_B is well defined and moreover is semifinite. We remark that, if φ is finite, then so is φ_B. Moreover, in this case, (φ_B) is a $p(\varphi)$-morphism, provided φ gives rise to a H-map $p(\varphi)$.

The same construction can be done if (φ, ψ) is of the second type. Now, φ_B acts from C_B to $C'_{\psi(B)}$ as:

$$\varphi_B(s - Bs) = \varphi(s) - \psi(B)(\varphi(s)) = \varphi(s) - \varphi(Bs)$$

for each $s \in D(\varphi)$.

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