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INVARIANCE PRINCIPLES FOR DEPENDENT RANDOM VARIABLES

by

Magda PELIGRAD

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INVARIANCE PRINCIPLES FOR DEPENDENT RANDOM VARIABLES

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Magda PELIGRAD*)

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*) Center of Mathematical Statistics, Str. Stirbei Voda 174, Bucharest, Romania

An invariance principle for dependent random variables

by

Magda Peligrad

Centre of Mathematical Statistics, Bucharest

In this paper we give a weak invariance principle for a class of dependent random variables which contains martingale-like sequences and φ-mixing sequences. Stationarity is not required.

1. Introduction and definitions.

McLeish (1975) proves an invariance principle (cf. Theorem (2.6), [8]) under assumption on the conditional expectations of variables with respect to the distant past. In section 2 of this paper we give an invariance principle similar to that of McLeish for another class of random variables under an "asymptotic martingale" type condition. In Section 3, this result ~~result~~ is used to extend the invariance principle obtained by McLeish (Theorem (5.1), [8]) for martingales, to martingale-like sequences. We also prove an invariance principle for φ- mixing sequences without the assumptions of stationarity and under a variety of conditions for the φ-mixing rate and for the moments. This improves the mixing rate required by Billingsley.

Proofs of the results of this paper are given in Section 4.

Let (Ω, F, P) be a probability space and $(F_n; n \geq 1)$ an increasing sequence of sub - σ-algebras of F. $(X_n, F_n;\ n \geq 1)$ is said to be a stochastic sequence if X_n is F_n-measurable for each n. We will denote the convergence in L_p and weak convergence by $\xrightarrow[L_p]{}$ and $\Rightarrow$ respectively. We will denote

$E(X_i \mid F_m)$ by $E_m X_i$ and $E^{1/p} |u|^p$ by $\|U\|_p$.

Let $(X_n, F_n;\ n \geq 1)$ be a stochastic sequence of square integrable random variables and put $S_n = \sum_{i=1}^{n} X_i$. We assume that

$$(1.1) \qquad E\,\frac{S_n^2}{n} \longrightarrow \sigma^2 \quad \text{as } n \to \infty,$$

where σ is a positive constant.

Consider the space $D[0,1]$, the set of all functions on the interval $[0,1]$ which have left hand limits and are continuous from the right at every point. We endow this space with Skorohod topology. Let β be the Borel σ-algebra in D and define a random function by

$$(1.2) \qquad W_n(t) = \frac{S_{[nt]}}{n^{1/2}\sigma} \quad , t \in [0,1]$$

where $[x]$ is the "greatest integer contained in x . This is a measurable map from (Ω, F) into (D,B), and we shall establish the weak convergence of W_n to the standard Brownian motion process on D .

(1.3) Definition. A sequence W_n of random elements of a metric space is said to be Renyi-mixing (R-mixing) with limiting process W if : $P(W_n \in \cdot \mid E)$ converges weakly to the measure $P(W \in \cdot)$ for every $E \in F$ such that $P(E) > 0$.

R-mixing is a useful concept when passing from non-random to random invariance principles.

2. The invariance principle

The following conditions are suggested by Gordin's condition (see [4]).

(2.1) For every fixed $m \geq 1$ the sequence $(U_{m,n} = \sum_{i>m}^{n} E_m X_i ; n > m)$ converges at U_m in $L_2(\Omega)$ norm as $n \longrightarrow \infty$. .

(2.2) The sequence $(U_m^2, m \geq 1)$ is uniformly integrable.

(2.3) Theorem. Let $(X_n, F_n ; n \geq 1)$ be a stochastic sequence of square integrable random variables which satisfies (2.1) and (2.2). If $(X_i^2, i \geq 1)$ is uniformly integrable, then $(W_n, n \geq 1)$ is tight in D and any limit process is a.s. continuous.

(2.4) Theorem. Suppose in addition to the conditions of theorem (2.3) that

$$(2.5) \qquad E_{k-m} \frac{(S_{k+n} - S_k)^2}{n} \xrightarrow[L_1]{} \sigma^2$$

as $\min (m \leq k, n) \longrightarrow \infty$.

Then W_n is R-mixing with limit W, a standard Brownian motion process on D.

3. Applications

One can apply the Theorem (2.4) to the martingale - like sequences.

(3.1) Definition [9] The stochastic sequence $(S_n, F_n ; n \geq 1)$ will be called a martingale in the L_2 limit if S_n is square integrable and $E_m(S_n - S_m) \xrightarrow[L_2]{} 0$ as $n \geq m \longrightarrow \infty$

This concept generalizes the notion of martingale ; conditions of this type have been considered in [2], [7].

The following theorem extends the Theorem (5.1) of [8].

(3.2) Theorem. Let $(X_n, F_n, n \geq 1)$ be a sequence of differences of a martingale in the L_2 limit such that the set $(X_n^2;\ n \geq 1)$ is uniformly integrable and

$$(3.3)\qquad \frac{1}{n}\sum_{i=1}^{n} E_{k-m}\, X_{k+i}^2 \xrightarrow[L_1]{} \sigma^2$$

as $\min(m \leq k, n) \longrightarrow \infty$.

Then W_n is R-mixing having as limit a standard Brownian motion process.

Another result is an invariance principle under φ-mixing condition.

Let $(X_n;\ n \geq 1)$ be a sequence of random variables and put $F_n^m = (X_i;\ n \leq i \leq m)$, $F_0 = (\phi, \Omega)$. For each $m \geq 0$ define

$$\varphi_m = \sup_n \sup_{(A \in F_0^n,\, P(A) \neq 0,\ B \in F_{n+m}^{\infty})} |P(B \mid A) - P(B)|$$

We say that $(X_n; n \geq 1)$ are φ-mixing if $\varphi_m \longrightarrow 0$. Obviously we can take φ_m nonincreasing.

(3.4) Theorem. Let $(X_n; n \geq 1)$ be a sequence of φ-mixing random variables centered at expectations and f an increasing function $f : R^+ \longrightarrow R^+$ such that $f(x) \geq x$ for every x. Let F be a primitive of $1/f$ on $(0, \infty)$. If

$$(3.5)\qquad \sum_k f(\varphi_k) < \infty$$

and

$$(3.6)\qquad \lim_{c \to \infty} \sum_{\{X_i^2 > c\}} X_i^2 (1 + |F(\varphi_i)|) = 0 \text{ uniformly in } i,$$

then $W_n \Rightarrow W$ on D, where W is a standard Brownian motion process on $[0,1]$.

This theorem gives a variety of conditions for the mixing rate related to the L_2 moment conditions.

If $\int_0^1 \frac{1}{f}$ converges, then the condition (3.6) in the above theorem becomes an usual condition, the sequence $(X_i^2,\ i \geq 1)$ being in this case uniformly integrable. We can take for example, for $x > 0$, $f(x) = x^{\alpha}$ with $\alpha < 1$, or $f(x) = x\,|\ln x|^{\beta}$ with $\beta \geq 1$, and we obtain the following :

(3.7) Corollary. If $(X_i;\ i \geq 1)$ is a sequence of random variables centered at expectations such that $(X_i^2,\ i \geq 1)$ is uniformly integrable and $\sum_{i=1}^{\infty} \varphi_i^{\alpha} < \infty$ where $\alpha < 1$ or $\sum_{i=1}^{\infty} \varphi_i\,|\ln \varphi_i|^{\beta} < \infty$ where $\beta > 1$, then $W_n \Rightarrow W$ on D , where W is a standard Brownian motion process on $[0,1]$.

This extends the Theorem 20.1 of Billingsley, [1] to non-stationary φ-mixing sequences and improves the mixing rate.

The following corollary is obtained from the Theorem (3.2) by taking $f(x) = x$ for every $x > 0$.

(3.8) Corollary. If $(X_i;\ i \geq 1)$ is a sequence centered at expectation satisfying

$$\text{a)}\ \sum_{k=1}^{\infty} \varphi_k < \infty$$

and

$$\text{b)}\ \lim_{c \to \infty} \int_{\{|X_i| > c\}} X_i^2\,|\ln \varphi_i| = 0 \quad \text{uniformly in } i ,$$

then $W_n \Rightarrow W$ on D , where W is a standard Brownian motion process on $[0,1]$.

4. Proofs

(4.1) Lemma. Let $(X_n, F_n; n \geq 1)$ be a stochastic sequence satisfying (2.1). Then we have

(4.2) $S_m = Z_m - U_m$ for every $m \geq 1$

where U_m is defined by (2.1) and $(Z_m, F_m; m \geq 1)$ is a martingale.

Proof. Put $Z_m = S_m + U_m$. Clearly

$$E_{m-1}(S_m + U_{n,m}) = S_{m-1} + U_{n,m-1}$$

But, by (2.1) the left side of the above equality converges in L_2 to $E_{m-1} Z_m$ and the right side to Z_m.

(4.3) Lemma. Let $(X_n, F_n; n \geq 1)$ be a stochastic sequence satisfying (2.1) and (2.2). If $(X_i^2; i \geq 1)$ is uniformly integrable then the set

(4.4) $$\left\{ \max_{j \leq n} \frac{(S_{j+k} - S_k)^2}{n} \; ; \; k \geq 1, \; n \geq 1 \right\}$$

is uniformly integrable.

Proof. Using Lemma (4.1) we have

$$\max_{j \leq n} \frac{(S_{j+k} - S_k)^2}{n} \leq 2\left(\max_{j \leq n} \frac{(Z_{j+k} - Z_k)^2}{n} + \max_{j \leq n} \frac{(U_{j+k} - U_k)^2}{n}\right)$$

But

$$\max_{j \leq n} \frac{(U_{j+k} - U_k)^2}{n} \leq 2\left(\sum_{j=k}^{k+n} \frac{U_j^2}{n} + U_k^2\right)$$

Therefore on account of Theorem 2o, p.36 of [6], it results that the set

$$\left\{ \max_{j \leq n} \frac{(U_{j+k} - U_k)^2}{n} \; ; \; k \geq 1, \; n \geq 1 \right\}$$

is uniformly integrable.

Now, again by (4.2) we have :

$$(Z_k - Z_{k-1})^2 \leq 3(X_k^2 + U_k^2 + U_{k-1}^2)$$

for all $k \geqslant 1$, whence by the hypothesis of this lemma it follows that the set $\{(Z_k - Z_{k-1})^2,\ k \geq 1\}$ is uniformly integrable.

The proof of the fact that

$$\left\{ \max_{j \leq n} \frac{(S_{k+j} - S_k)^2}{n} \ ;\ k \geqslant 1,\ n \geqslant 1 \right\}$$

is uniformly integrable is similar to that of Theorem 23.1 of Billingsley [1], where instead of stationarity, we use the uniform integrability of the martingale differences $\{(Z_k - Z_{k-1})^2, k \geq 1\}$.

Proof of theorem (2.3). By [1], Theorem 8.4 adapted to D, the tightness condition will follow if we prove

$$\lim_{\lambda \to \infty} \lambda^2 P(\max_{j \leq n} |S_{k+j} - S_k| > \lambda n^{1/2}) = 0$$

uniformly in (n,k). This follows from the uniform integrability of the set

$$\left\{ \max_{j \leq n} \frac{(S_{k+j} - S_k)^2}{n} \ ;\ k \geqslant 1,\ n \geqslant 1 \right\}$$

which is shown in lemma (4.3). Theorem 15.5 of [1] also shows that any weak limit process of W_n must be a.s. concentrated on the continuous functions.

Let d be the Skorohod's metric on D.

(4.5) Lemma. If Z_n is R-mixing with limiting process W and $d(W_n, Z_n) \xrightarrow{P} 0$, then W_n is R-mixing with the same limiting process W.

This is a minor extension of Theorem 4.1 of [1]. Its proof is similar to that of this theorem.

Let Z_n be defined by (4.2) and put

$$V_n(t) = \frac{Z_{[nt]}}{\sqrt{n}\,\sigma}, \quad t \in [0,1]$$

V_n is a random function from (Ω, F) into (D, B).

(4.6) <u>Lemma.</u> Let $(X_n, F_n; n \geq 1)$ be a stochastic sequence satisfying (2.1) and (2.2). If V_n is R-mixing with limiting process W, then W_n is R-mixing with the same limiting process W.

<u>Proof.</u> To use lemma (4.5) it is enough to show that for every $\varepsilon > 0$

$$P(d(W_n, V_n) > \varepsilon) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

Clearly

$$P(d(W_n, V_n) > \varepsilon) \leq P\left(\sup_{1 \leq k \leq n} \frac{|U_k|}{\sqrt{n}\,\sigma} > \varepsilon\right)$$

We have

$$P\left(\sup_{1 \leq k \leq n} \frac{U_k}{\sqrt{n}\,\sigma} > \varepsilon\right) \leq \sum_{k=1}^{n} P\left(\frac{|U_k|}{\sqrt{n}\,\sigma} > \varepsilon\right) \leq \sum_{k=1}^{n} \frac{1}{\varepsilon^2 n \sigma^2} \int_{\{U_k^2 > n\sigma^2\varepsilon^2\}} U_k^2$$

Because the set $(U_m^2; m \geq 1)$ is uniformly integrable we have ... for every $\varepsilon > 0$

$$P(d(W_n, U_n) > \varepsilon) < \varepsilon$$

<u>Proof of Theorem (2.4).</u> We shall prove that the conditions of Theorem (5.1) of McLeish (1975) are satisfied for martingale differences $(Z_n - Z_{n-1}; n \geq 1)$.

The first condition is

a) The set $\{(Z_n - Z_{n-1})^2, n \geq 1\}$ is uniformly integrable. This is already established in the proof of lemma (4.3).

We verify now the second condition.

b) $$\frac{1}{n} \sum_{i=1}^{n} E_{k-m}(Z_{k+i} - Z_{k+i-1})^2 \xrightarrow[L_1]{} \sigma^2$$

as $\min(m \leq k, n) \rightarrow \infty$.

By (2.1) and the Cauchy-Schwarz inequality, we have

$$\frac{1}{n} E\left|(Z_{k+n}-Z_k)^2-(S_{k+n}-S_k)^2\right| \leq \frac{1}{n} E(U_{k+n}-U_k)^2 +$$

$$+\frac{1}{n}\|U_{k+n}-U_k\|_2 \cdot \|S_{k+n}-S_k\|_2$$

On account of the fact that the set $(U_k^2;\ k\geq 1)$ is uniformly integrable $\frac{1}{n} E(U_{k+n}-U_k)^2 \longrightarrow 0$ as $n \longrightarrow \infty$, and by Lemma (4.3) the sequence $\left\{\frac{1}{n} E(S_{k+n}-S_k)^2;\ n\geq 1\right\}$ is bounded. It follows

(4.7) $\frac{1}{n} E\left|(Z_{k+n}-Z_k)^2-(S_{k+n}-S_k)^2\right| \longrightarrow 0$, as $n \longrightarrow \infty$

and b) follows by (2.5) and (4.7).

The conditions of Theorem (5.1) of [8] are so satisfied. Therefore V_n is R-mixing with limit W, where W is a standard Brownian motion process on D.

Proof of Theorem (3.2). If $(X_n;n\geq 1)$ is a sequence of differences of a martingal in the L_2 limit then $\|U_{m,n}\|_2 \longrightarrow 0$ as $n\geq m \longrightarrow \infty$. For fixed m and $n\leq n'$, we have

$$\|U_{m,n}-U_{m,n'}\|_2 \leq \|U_{n,n'}\|_2 \longrightarrow 0 \text{ as } n'>n\longrightarrow\infty$$

Therefore $(U_{m,n};\ n\geq m)$ converges in L_2 for m fixed, to a stochastic sequence $(U_m,F_m;m\geq 1)$ and so the condition (2.1) is satisfied. Because $\|U_{m,n}\|_2 \longrightarrow 0$ as $n>m\longrightarrow\infty$, it follows that $(U_m;m\geq 1)$ also converges in L_2 to 0, whence it follows (2.2).

It remains to verify the condition (2.5). By (2.1) and the Cauchy-Schwarz inequality we have

$$E\left|(Z_i-Z_{i-1})^2-X_i^2\right| \leq E(U_{i+1}-U_i)^2 + 2\|X_i\|_2\|U_{i+1}-U_i\|_2$$

Therefore $(Z_i-Z_{i-1})^2-X_i^2 \xrightarrow[L_1]{} 0$ and by (3.3) and (4.7), (2.5) holds.

From the proof of Serfling's lemma [12], we deduce the following:

<u>(4.8) Lemma.</u> If $(X_n; n \geq 1)$ is a sequence of random variables centred at expectations such that for some C, $|X_i| \leq C$ a.s., then for $m \leq i$

$$(4.9) \qquad \| I_A E(X_i \mid F_m) \|_2 \leq 2C\, P(A)^{1/2} \varphi_{m-i}$$

for every $A \in F$.

<u>(4.10) Lemma.</u> In the sequence $(X_i; i \geq 1)$ satisfies the conditions of Theorem (3.4) then the set $\left\{ \max_{i \leq n} \frac{1}{n} (S_{k+n} - S_k)^2 ; k \geq 1, n \geq 1 \right\}$ is uniformly integrable.

<u>Proof.</u> For positive C put

$$X_i^c = X_i \; I_{\{|X_i| \leq C\}}$$

$$Y_i = X_i^c - EX_i^c$$

and

$$V_i = X_i - X_i^c + E(X_i - X_i^c)$$

Note that $X_i = Y_i + V_i$.

Let us use the notation $E_y U = \int_{\{U > y\}} U \, dP$, $\overline{Y}_j = \sum_{i=1}^{j} Y_i$ and $\overline{V}_j = \sum_{i=1}^{j} V_j$.

Then

$$S_j^2 \leq 2(\overline{Y}_j^2 + \overline{V}_j^2)$$

and hence

$$E_y\left(\max_{k \leq n} \frac{S_k^2}{n}\right) \leq \frac{4}{n} E_{y/4}\left(\max_{k \leq n} \overline{Y}_k^2\right) + \frac{4}{n} E\left(\max_{k \leq n} \overline{V}_k^2\right)$$

On account of the fact that

$$V_i = \sum_{m=0}^{i-1} E_{i-m} V_i - E_{i-m-1} V_i$$

we have

$$\bar{V}_j = \sum_{m=o}^{j} Y_{j,m}$$

where

(4.11) $$Y_{j,m} = \sum_{i=m+1}^{j} E_{i-m}V_i - E_{i-m-1}V_i$$

Obviously for fixed m, $(Y_{j,m}, F_{j-m}; j \geq m)$ is a martingale. By Cauchy-Schwarz inequality we have :

$$E(\max_{j \leq n} \frac{\bar{V}_j^2}{n}) \leq \sum_{m=o}^{n} f(\varphi_m) \cdot \sum_{m=o}^{n} \frac{1}{f(\varphi_m)} E(\max_{j \leq n} \frac{Y_{j,m}^2}{n})$$

Using Doob's inequality ([3] p.317), (3.5) and (4.11) we have

$$E(\max_{j \leq n} \frac{\bar{V}_j^2}{n}) \leq \frac{k}{n} \sum_{m=o}^{n} \frac{1}{f(\varphi_m)} \sum_{i=m+1}^{n} (E(E_{i-m}V_i)^2 - E(E_{i-m-1}V_i)^2)$$

where K is some positive constant.

Simple computation shows that

$$E(\max_{j \leq n} \frac{\bar{V}_j^2}{n}) \leq \frac{K}{n} (\sum_{i=1}^{n} E(X_i - X_i^c)^2 (\frac{1}{f(\varphi_o)} + \sum_{m=1}^{i-1} \varphi_m (\frac{1}{f(\varphi_m)} - \frac{1}{f(\varphi_{m-1})})) \leq$$

$$\leq \frac{K}{n} (\sum_{i=1}^{n} E(X_i - X_i^c)^2 (\frac{1}{f(\varphi_o)} (1 + \varphi_1) + \frac{\varphi_{i-1}}{f(\varphi_{i-1})} + \sum_{m=1}^{i-1} \frac{\varphi_m - \varphi_{m-1}}{f(\varphi_m)})) \leq$$

$$\frac{K}{n} (\sum_{i=1}^{n} E(X_i - X_i^c)^2 (K' + F(1) - F(\varphi_i)))$$

where K' is a positive constant. By (3.6) we may choose and fix C such that this is less than $\varepsilon/8$. With these fixed values of C we apply lemma (4.3) for the sequence Y_i.

By lemma (4.8) for $n' \geq n \geq m$ we have

$$\| \sum_{i=n}^{n'} E(Y_i \mid F_m) \|_2 \leq 2C \sum_{i=n}^{n'} \varphi_{i-m}$$

Using (3.5) and the fact that $f(x) \geq x$, it follows that

$\left(\sum_{i=m+1}^{n} E(Y_i \mid F_m),\ n \geq m\right)$ is Cauchy in L_2 for m fixed, and therefore (2.1) is verified. By the same argument it follows that for all $m \leq n$

$$\left\| \sum_{i=m+1}^{n} E(Y_i \mid F_m) \right\|_2 \leq 2C \sum_{k=1}^{\infty} \varphi_k$$

which implies that the limit sequence $(U_m^2;\ m \geq 1)$ is uniformly bounded in L_1. This and (4.9) show that the sequence $(U_m^2, m \geq 1)$ is uniformly integrable.

By lemma (4.3) it follows that the sequence

$$\left\{ \max_{j \leq n} \frac{\bar{Y}_j^2}{n}\ ;\ n \geq 1 \right\},$$

is uniformly integrable, and therefore we may choose y sufficiently large such that

$$E_{y/4}\left(\max_{j \leq n} \frac{\bar{Y}_j^2}{n}\right) \leq \varepsilon/8$$

Then, for this value of y

$$E_y\left(\max_{j \leq n} \frac{S_j^2}{n}\right) \leq \varepsilon$$

where y, was chosen independently of our location in the sequence or the value of n, so :

(4.12) $\left\{ \max_{j \leq n} \frac{1}{n}(S_{k+j} - S_k)^2\ ;\ n \geq 1,\ k \geq 1 \right\}$

is uniformly integrable.

For the proof of Theorem (3.4) we need the following theorem (Theorem 19.2, [1]).

(4.13) <u>Theorem 1. Let</u> $(X_n; n \geq 1)$ <u>be a sequence of random functions in</u> D <u>with asymptotically independent increments such that</u> $(X_n^2(t);\ n \geq 1)$ <u>is uniformly integrable for each</u> t, $E(X_n(t)) \longrightarrow 0$ <u>and</u> $E(X_n^2(t)) \longrightarrow 1$ <u>as</u> $n \longrightarrow \infty$. <u>Suppose that for each positive</u> ε <u>and</u> η, <u>there exist a positive</u> δ <u>such that for all sufficiently large</u> n

(4.14) $$P(w(X_n, \delta) \geq \varepsilon) \leq \eta$$

where $w(x, \delta)$ is the modulus of continuity of x.

Then $X_n \Rightarrow W$, standard Brownian motion process on D.

Proof of theorem (3.4). We apply Theorem (4.13). The fact that W_n has asymptotically independent increments follows from the definition of φ-mixing sequences by induction as in the proof of the Theorem 20.1 of [1]. Evidently $E(W_n(t)) \longrightarrow 0$ and by (1.1) $E(W_n^2(t)) \longrightarrow t$ as $n \longrightarrow \infty$. By (4.12) it follows that $(W_n^2(t), n \geq 1)$ is uniformly integrable for each t and that $P(w(X_n, \delta) \geq \varepsilon) \leq \eta$ for all sufficiently large n. □

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Centre of Mathematical Statistics

174 Stirbei Voda St., 771o4 Bucharest

Romania

A Criterion for Tightness for a Class of Dependent Random Variables.

Magda Peligrad
Center of Mathematical Statistics.

Loynes (1976) proved that the finite-dimensional distributions of a sequence of martingales converge and if for each time t the variables are uniformly integrable, then weak convergence follows (in either C or D) provided the limiting process satisfies a certain condition; this condition is satisfied by the Wiener process. Using this result we prove a weak invariance principle for a class of dependent random variables , satisfying a Lindeberg-type condition. The weak invariance principle we obtain concerning φ-mixing sequences shows that the mixing rate used by McLeish (1975),(1977) can be improved provided the finite-dimensional distributions converge.

Let (X_i ; $i \geq 1$) be a sequence of square integrable random variables on the probability triple (Ω, F, P) and put

$$F_n^m = \sigma(X_i ; \; n \leq i \leq m)$$

For each $m \geq 0$, define

$$\varphi'_m = \sup_n \sup_{(A \in F_0^n, \; B \in F_{n+m}^{n+m}, \; P(A) \neq 0)} |P(B|A) - P(B)|$$

We denote $E(X_n | F_m)$ by $E_m X_n$, $\left(\sum_{i=1}^{n} \varphi'_i\right)^{-1}$ by a_n. We also assume that

(1) $EX_i = 0$ for all i , and $ES_n^2 = O(n)$

For each $t \in [0,1]$ put

$$W_n(t) = \frac{\sum_{i=1}^{[nt]} X_i}{n}$$

where $[x]$ is the greatest integer contaimd in x . The functio

$\omega \longrightarrow W_n(t,\omega)$ is a measurable map from (Ω, F) into $(D, \mathcal{B})$ where D is the set of all functions on the interval $[0,1]$ which have left hand limits and are continuous from the right at every point, and $\mathcal{B}$ the Borel σ-algebra on D induced by the Skorohod topology. We shall give sufficient conditions for the weak convergence of W_n to the standard Brawnian motion process on D, denoted by W in the sequel.

THEOREM Let $(X_i, i \geqslant 1)$ be a stochastic sequence satisfying (1) and assume, for every $\varepsilon > 0$

(2) $$\lim_{n\to\infty} \frac{1}{n\, a_n} \sum_{i=1}^{n} EX_i^2 \, I(|X_i| > \varepsilon a_n \sqrt{n}) = 0$$

and

(3) $W_n \underset{f.d.}{\longrightarrow} W$ (i.e. the finite-dimensional distributions converge) ;

Then W_n converges weakly to W.

In order to prove this theorem we need the following

LEMMA Let $(X_i, i \geqslant 1)$ be a stochastic sequence satisfying (1) and (2) . Then for every $t \in [0,1]$, $W_n(t) = \bar{Z}_n(t) + V_n(t)$, where for every n, $\bar{Z}_n(t)$ is a martingale and $V_n(t)$ converges to 0 in probability.

PROOF Condition (2) implies that there exists a sequence of positive numbers d_n convergeing to 0 as $n \to \infty$, such that

(4) $$\lim_{n\to\infty} \frac{1}{n\, a_n\, d_n} \sum_{i=1}^{n} EX_i^2 \, I(|X_i| > a_n d_n \sqrt{n}) = 0 .$$

For every $n \geqslant 1$ let us denote

$$X_i'(n) = X_i \, I_{(|X_i| \leq a_n d_n \sqrt{n})} - EX_i \, I_{(|X_i| \leq a_n d_n \sqrt{n})}$$

$$Y_i'(n) = X_i \, I_{(|X_i| > a_n d_n \sqrt{n})} - EX_i \, I_{(|X_i| > a_n d_n \sqrt{n})}$$

and note that $X_i = X_i'(n) + Y_i'(n)$.

Let us put

$$Z_i(n) = \sum_{j=1}^{n} E_i X_j'(n)$$

$$U_i(n) = \sum_{j=i+1}^{n} E_i X_j'(n)$$

$$Y_i(n) = \sum_{j=1}^{i} Y_j'(n)$$

and define the following random functions

$$\bar{Z}_n(t) = \frac{Z_{[nt]}(n)}{\sqrt{n}}$$

$$\bar{U}_n(t) = \frac{U_{[nt]}(n)}{\sqrt{n}}$$

and

$$\bar{Y}_n(t) = \frac{Y_{[nt]}(n)}{\sqrt{n}}$$

Obviously for every $t \in [0,1]$, $W_n(t) = \bar{Z}_n(t) + \bar{Y}_n(t) - \bar{U}_n(t)$ and for every n, $\bar{Z}_n(t)$ is a martingale. We shall prove the random elements $\bar{U}_n$ and $\bar{Y}_n$ converge to 0 in probability. By the properties of Skorohod's metric it is sufficient to prove that for every $\varepsilon > 0$

(5) $\quad P\left(\sup_{t\in[0,1]} |\bar{Y}_n(t)| > \varepsilon\right) \longrightarrow 0 \qquad$ as $n \to \infty$

and

(6) $\quad P\left(\sup_{t\in[0,1]} |\bar{U}_n(t)| > \varepsilon\right) \longrightarrow 0 \qquad$ as $n \to \infty$

We have

$$P\left(\sup_{t\in[0,1]} |\bar{Y}_n(t)| > \varepsilon\right) \leq P\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} |Y_i'(n)| > \varepsilon\right) \leq$$

$$\leq \frac{2}{\sqrt{n}} \sum_{i=1}^{n} E\,|X_i|\; I_{(|X_i| > a_n d_n \sqrt{n})} \leq \frac{2}{a_n d_n n} \sum_{i=1}^{n} E\, X_i^2\, I_{(|X_i| > a_n d_n \sqrt{n})}$$

whence by (4) we obtain (5).

In order to prove (6), note that, by Lemma 1.1.8 [2], it follows for $j \leq i$:

$$|E_i X_j'(n)| \leq 2 a_n d_n \sqrt{n}\; \varphi'_{i-j} \qquad \text{a.s.}$$

whence, for every $i \geq 1$,

$$|U_i(n)| \leq 2 d_n \sqrt{n} \qquad \text{a.s.}$$

and (6) follows. If we denote $V_n(t) = \bar{Y}_n(t) - \bar{U}_n(t)$ we have the desired result.

PROOF OF THEOREM

From the preceding Lemma it follows for every $t \in [0,1]$ $W_n(t) = \bar{Z}_n(t) + V_n(t)$, where for every n, $\bar{Z}_n(t)$ is a martingale and $V_n(t)$ converges to 0 in probability. By the Theorem 4.2, [1], in order to prove the theorem is sufficient to prove that $\bar{Z}_n$ converges weakly to W. For this, we shall verify the conditions of the Theorem of Loynes, [3], quoted at the begining of this paper.

Obviously from (3) it follows $\bar{Z}_n \xrightarrow[f.d.]{} W$. To verify that for each t, $\bar{Z}_n(t)$ are uniformly integrable it is sufficient to show that for some positive number C, not depending on n, $E\bar{Z}_n^2(t) \leq C$.

Obviously, for $t \in [0,1]$

$$E\bar{Z}_n^2(t) = E\frac{(\sum_{i=1}^{n} E_{[nt]} X_i'(n))^2}{n} \leq E\frac{(\sum_{i=1}^{n} X_i'(n))^2}{n}$$

and

$$(7)\quad E\frac{(\sum_{i=1}^{n} X_i)^2}{n} = \frac{E(\sum_{i=1}^{n} X_i'(n))^2 + E(\sum_{i=1}^{n} Y_i'(n))^2 + \sum_{i,j=1}^{n} EX_i'(n)Y_j'(n)}{n}$$

By Lemma 1.1.9, [2], for every $i \geq 1$, $j \geq 1$, we have

$$|EX_i'(n)Y_j'(n)| \leq 2a_n d_n\sqrt{n}\, E|Y_j'(n)|\, \varphi'_{|i-j|}$$

Therefore

$$\frac{\left|E\sum_{i,j=1}^{n} X_i'(n)Y_j'(n)\right|}{n} \leq \frac{4d_n}{\sqrt{n}}\sum_{j=1}^{n} E|Y_j'(n)| \leq \frac{8}{a_n n}\sum_{j=1}^{n} EX_j^2\, I_{(|X_j| > a_n d_n\sqrt{n})}$$

On account of (4)

$$\lim_{n\to\infty}\frac{\left|E\sum_{i,j=1}^{n} X_i'(n)Y_j'(n)\right|}{n} = 0$$

This fact and (7) imply for every n

$$\frac{E(\sum_{i=1}^{n} X_i^*(n))^2}{n} \leq \frac{E(\sum_{i=1}^{n} X_i)^2}{n} + C$$

where C is a positive constant, and the result follows by (1).

DEFINITION We say that the sequence of random variables is φ- mixing if $\varphi_n' \to 0$, as $n \to \infty$.

This condition is a weakening of the usual φ-mixing condition and it would be equivalent to it if we replaced F_{n+m}^{n+m} by F_{n+m}^{∞} in the definition of φ_m'.

For φ-mixing sequences defined in this way our theorem gives the following

COROLLARY Let (X_i , $i \geq 1$) be a φ-mixing sequences satisfying (1) and assume

(8) $\sum_i \varphi_i' < \infty$

(9) $W_n \xrightarrow[f.d.]{} W$

and for every $\varepsilon > 0$,

(10) $\lim \frac{1}{n} \sum_{i=1}^{n} EX_i^2 \, I_{(|X_i| > \varepsilon\sqrt{n})} = 0$

Then W_n converges weakly to W.

REMARK Observe that the condition (8) improves the mixing rate used by McLeish [4],[5], mixing coefficient being at the same time more general.

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Center of Mathematical Statistics

Stirbei Vodă 174 St.

77104 Bucharest/Romania