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ISSN 0250-3638

THE NULLSTELLENSATZ OVER t-ORDERED FIELDS:A t-ADIC ANALOGUE OF THE THEORY OF FORMALLY p-ADIC FIELDS

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PREPRINT SERIES IN MATHEMATICS
No.50/1980

BUCURESTI

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Contents

§ 1. Introduction

§ 2. t-orderings

§ 3. The space of t-orderings

§ 4. t-adically closed fields

§ 5. The Kochen ring

§ 6. The Riemann space of places

§ 7. The Nullstellensatz for holomorphy rings - a weak form.

§ 8. The restricted Riemann space.

§ 9. The Nullstellensatz for holomorphy rings

§ 1o. The Nullstellensatz for the coordinate ring of a variety over a t-ordered field.

§ 1. Introduction

The theory of formally p-adic fields was initiated by Kochen in [11] in a complete analogy to the classical theory of formally real fields. Important developments of the theory were achieved by Roquette in [16-18], and together with Jarden in [10]. The most interesting result of this theory is the Nullstellensatz over p-adically closed fields proved by Jarden and Roquette in [10]. A particular form of this result was proved by Kochen in [11].

On the other hand the possibility to extend the framework of the theory of formally p-adic fields was suggested in [11] as in [10]. Such extensions were obtained by Transier [22] and by the author [3]. As an application of the general theory developed in [3], we considered in [3] Section 8, among other cases, the case where the base field is a valued field with a real closed residue field. As examples of such fields let us mention the field of formal power series $\mathbb{R}((t))$ and the field of Puiseaux series $\mathbb{R}((t^{1/n}/n \geq 1))$.

A different approach to a Nullstellensatz over $\mathbb{R}((t))$ is considered by Jacob in [9]. The aim of the present paper is to develop the theory of so called formally t-adic fields and t-ordered fields defined in § 3 and § 4, and to prove a Nullstellensatz in this context which can be seen as a generalization of Jacob's principal result from [9]. Our results and their proofs are presented in the same spirit as the principal results and techniques from the paper [10] of Jarden and Roquette. The main facts used in proof are Zariski's local uniformization theorem [23] and some model-theoretic results (see § 4).

The present work is an improved version of the work [5]. This latter work was prepared while the author had the opportunity to be a Humboldt fellow at the University Heidelberg. The present version is based on some Roquette's remarks contained in a letter [19]. It is a great pleasure for me to express here my warmest thanks to Professor Peter Roquette for his advices and permanent encouragement in my work at the University Heidelberg.

§ 2. t-orderings

This section is integrally based on Roquette's letter [19].

Let K be an ordered field. Its ordering $\leq$ is completely determined by the positive cone $P = \{x \in K \mid x \geq 0\}$ of K. Let $P^{\bullet} = P \setminus \{0\}$.

A subset $\mathcal{O}$ of K is called convex if the following condition is satisfied:

$$\left.\begin{array}{l} a, b \in \mathcal{O} \\ \\ a \leq x \leq b \end{array}\right\} \Rightarrow x \in \mathcal{O}$$

If $\mathcal{O}$ is a subring of K, $1 \in \mathcal{O}$, then $\mathcal{O}$ is convex iff $|x| \leq 1 \Rightarrow x \in \mathcal{O}$. Here $|x| = \max(x, -x)$.

There is a smallest convex subring of K, namely the ring of finite elements of K:

$$K_{fin} = \{x \in K \mid \bigvee_{n \in \mathbb{N}} |x| \leq n\}$$

K_{fin} is a valuation ring in K and its maximal ideal is the set of infinitesimal elements of K:

$$K_{inf} = \{x \in K \mid \bigwedge_{n \in \mathbb{N}} |nx| \leq 1\}$$

It follows easily that the following assertions are equivalent for a subring $\mathcal{O}$ of K :

i) $\mathcal{O}$ is convex

ii) $\mathcal{O}$ contains K_{fin}

iii) $\mathcal{O}$ is a valuation ring and the ordering P of K is compatible with the valuation v attached to $\mathcal{O}$, i.e. $1 + \mathfrak{m}_v \subset P$. Here $\mathfrak{m}_v$ denotes the maximal ideal of $\mathcal{O}$.

Therefore the set of convex subrings of K is linearly ordered with respect to inclusion. There is a canonical bijection from the set of prime ideals of K_{fin} onto the set of convex subrings of K: $\mathfrak{p} \longmapsto (K_{fin})_{\mathfrak{p}}$.

Let $t \geqslant 0$ be an infinitely small element of K. Let $\mathfrak{m}_t$ denote the radical of the ideal in K_{fin} generated by t, i.e. $\mathfrak{m}_t = \{x \in K_{fin} \mid \bigvee_{n \in \mathbb{N}} x^n \in t\, K_{fin}\}$. It follows easily that $\mathfrak{m}_t$ is a prime ideal and $\mathfrak{m}_t$ coincides with the set of all $x \in K$ subject to: $x^{2n} \leqslant t$ for some $n \in \mathbb{N}$. The corresponding valuation ring $\mathcal{O}_t$, i.e. the localization of K_{fin} with respect to the prime ideal $\mathfrak{m}_t$, consists of those elements $x \in K$ which satisfy the condition $x^{-1} \notin \mathfrak{m}_t$, i.e. $t\, x^{2n} < 1$ for all $n \in \mathbb{N}$.

<u>Definition.</u> t is called <u>prime</u>, if $\mathfrak{m}_t$ is generated by t as ideal in $\mathcal{O}_t$, i.e. $\mathfrak{m}_t = t\, \mathcal{O}_t$. For the corresponding valuation v_t of $\mathcal{O}_t$, this means, if $t \neq 0$, that the value $v_t(t)$ is the smallest positive element of the value group $v_t(K)$.

Observe that the null element $t = 0$ is prime and $\mathcal{O}_0 = K$.

<u>Definition</u>. Let t be an arbitrary element of K. The ordering P of K is called a <u>t-ordering</u> if t is a prime element with respect to P, i.e. $t \in P$, t is infinitely small, and $\mathfrak{m}_t = t\, \mathcal{O}_t$.

In particular, for $t = 0$, each ordering is a 0-ordering.

In the following we consider a field K and a fixed element t of K. If P is a t-ordering of K we denote by $\mathcal{O}(P) = \{x \in K \mid \bigwedge_{n \in \mathbb{N}} 1 - t\, x^{2n} \in P\}$ the valuation ring afore denoted by $\mathcal{O}_t$, and by $\mathfrak{m}(P) = t\, \mathcal{O}(P)$ the corresponding maximal ideal.

(2.1) <u>Lemma.</u> If P is a t-ordering of K then $\mathcal{O}(P) = \{x \in K \mid 1 - t\, x^2 \in P\}$.

<u>Proof</u>. The case $t = 0$ is trivial, so we may assume $t \neq 0$. Let $x \notin \mathcal{O}(P)$. We have to show that $1 - t\, x^2 \notin P$. Since $x \notin \mathcal{O}(P)$ it follows $x^{-1} \in \mathfrak{m}(P) = t\, \mathcal{O}(P)$, i.e. $\frac{1}{xt} \in \mathcal{O}(P)$ and hence $1 - t\, (\frac{1}{xt})^{2m} \in P$ for all $m \in \mathbb{N}$. In particular, for $m = 1$, we obtain $1 - t\, x^2 \in -P$. Since t is a generator of the maximal ideal $\mathfrak{m}(P)$ we have $1 - t\, x^2 \neq 0$ and hence $1 - t\, x^2 \notin P$. Q.E.D.

(2.2). Theorem. Let K be a field and t an element of K. Then a necessary and sufficient condition for an ordering P of K to be a t-ordering is that the following conditions are satisfied:

(1) $t \in P$, $1-t \in P$, $t \neq 1$

(2) $(1-tx^2)(1-tx^4) \in P$ for all $x \in K$.

Proof. Since the theorem is trivial in the case $t = 0$, we may assume that $t \neq 0$.

If P is a t-ordering then by definition $t, 1-t \in P$. Let $x \in K$. If $x \in \mathcal{O}(P)$ it follows $1-tx^2 \in P$ and $1-tx^4 \in P$, and hence $(1-tx^2)(1-tx^4) \in P$. If $x \notin \mathcal{O}(P)$, then $x^2 \notin \mathcal{O}(P)$ and by (2.1), $1-tx^2$ and $1-tx^4$ are not contained in P, and hence $(1-tx^2)(1-tx^4) \in P$.

The converse is more difficult. The condition (2) means that $\mathrm{sgn}(1-tx^2) = \mathrm{sgn}(1-tx^4)$ for all $x \in K$, with the possible exception $tx^2 = 1$ or $tx^4 = 1$.(In fact these situations are not possible as we shall see in the following, but for the moment we consider these cases as possible).

By induction it follows that:

(3) $\mathrm{sgn}(1-tx^2) = \mathrm{sgn}(1-tx^{2k})$ for $k \geq 1$,

with a possible exception when

(4) $t = x^{-2k}$ for some $k \geq 1$.

Thus the condition (2) is equivalent with:

(5) $tx^2 < 1$ iff $tx^{2k} < 1$ for all $x \in K$, $k \geq 1$,

with the possible exception (4) for some $x \in K^{\bullet} = K \setminus \{0\}$, and also with the condition:

(6) $t < x^2$ iff $t < x^{2k}$ for all $x \in K$, $k \geq 1$,

with a possible exception when

(7) $t = x^{2k}$ for some $x \in K^{\bullet}$, $k \geq 1$.

We shall use in the following the condition (2) in the form (5) as well as in the form (6).

First let us show that t is infinitely small with respect to P. Let us put:

$$\frac{1}{t} = 1 + h \quad \text{where } h > 0 \text{ by (1)}$$

$$x = 1 + \frac{h}{4}$$

and let us compute:

$$x^4 > 1 + h = \frac{1}{t}$$

$$tx^4 > 1$$

$$tx^2 \geq 1 \quad \text{by (2)}$$

$$x^2 \geq \frac{1}{t}$$

$$1 + \frac{h}{2} + \frac{h^2}{16} \geq 1 + h$$

$$h \geq 8$$

$$\frac{1}{t} = 1 + h \geq 9$$

(8) $$t \leq \frac{1}{9}$$

$$t < \frac{1}{4}$$

$$t < \frac{1}{2^{2k}} \quad \text{for all } k \geq 1 \quad \text{by (6)}$$

Therefore t is infinitely small, excepting the case given by (7):

(9) $$t = \frac{1}{2^{2k}} \text{ for some } k \geq 2.$$

In this case it follows from (8):

$$t < \frac{1}{9}$$

$t < \frac{1}{3^{2k}}$ for all $k \geq 1$, by (6).

Thanks to (9), the exception $t = \frac{1}{3^{2k}}$ is not possible. We conclude that t is infinitely small with respect to the ordering P.

Now let us show that t is a prime element with respect to P. We have to show that the ideal

$\mathfrak{m}_t = \{x \in K \mid \bigvee_{n \in \mathbb{N}} t - x^{2n} \in P\}$ is generated by t in the ring

$$\mathcal{O}_t = \{x \in K \mid \bigwedge_{n \in \mathbb{N}} 1 - tx^{2n} \in P \setminus \{0\}\} .$$

Let $0 \neq x \in \mathfrak{m}_t$. We have to show that $x \in t\,\mathcal{O}_t$. Since $x \in \mathfrak{m}_t$ we have $x^{2n} < t$ for some $n \in \mathbb{N}$. It follows by (6) that $x^2 < t$ (we ommit here the exception (7); we shall consider this case at the end of the proof).

We write the preceeding relation in the form:

$$t\,(\tfrac{x}{t})^2 < 1$$

Then, by (5) applied to $\frac{x}{t}$, we get:

$$(10) \quad t\,(\tfrac{x}{t})^{2k} < 1 \quad \text{for all } k \in \mathbb{N}$$

The exception is by (4):

$$(11) \quad t = (\tfrac{x}{t})^{-2k} \quad \text{for some } k \in \mathbb{N},$$

and it will be further discussed.

We conclude by (10) that $\frac{x}{t} \in \mathcal{O}_t$ and hence $x \in t\,\mathcal{O}_t$, as contended.

It remains to discuss only the cases (7) and (11). These cases may be considered together, by writing:

$$(12) \quad t^{r} = x$$

where the exponent r is some rational number. If (12) holds, let us replace x by $\frac{x}{2}$. If $x \in \mathfrak{m}_t$ then $\frac{x}{2} \in \mathfrak{m}_t$ and hence $\frac{x}{2} \in t\, \mathcal{O}_t$, $x \in t\, \mathcal{O}_t$, with the possible exception:

$$(13) \qquad t^s = \frac{x}{2} \quad \text{for some } s \in \mathbb{Q}.$$

But (12) and (13) cannot be satisfied in the same time, because, if this is the case, $t^u = \frac{1}{2}$ with $u = s-r \in \mathbb{Q}$; this fact is not possible since t is infinitely small. Q.E.D.

§ 3. The space of t-orderings

Let K be a field of characteristic zero, and $t \neq 1$ be a fixed element of K. Let us look at the set of all t-orderings of K. Denote this one by X_K. If $t = 0$, X_K coincides with the set of all orderings of K.

Definition. K is called formally t-adic if there is at least one t-ordering of K, i.e. X_K is non-empty. For $t = 0$ we recover the concept of a formally real field.

It follows easily that X_K is a closed subset of the space of orderings of K. With the induced topology, X_K is a boolean space with the subbase of clopen sets $\{H(a) \mid a \in K\}$, where $H(a) = \{P \in X_K \mid a \in P\}$.

Definition. Let $\mathfrak{p}$ be a place of K and v be the corresponding valuation. $\mathfrak{p}$ (respectively v) is called a t-place (respectively a t-valuation) if the residue field K_v is formally real and t generates the maximal ideal $\mathfrak{m}_v$ of the corresponding valuation ring $\mathcal{O}_v$.

For $t = 0$, a valuation v of K is a 0-valuation iff v is trivial and K is formally real. If $t \neq 0$, and v is a t-valuation, v(t) is the smallest positive element of the value group v(K). As usual we identify the ordered group of integers with subgroup of the value group v(K) by putting $v(t) = 1$ and, consequently, $v(t^n) = n$ for every $n \in \mathbb{Z}$. After this identification the number 1 is the smallest positive element in v(K) and hence $\mathbb{Z}$ is an isolated subgroup of v(K), in the sense of ordered groups.

If v is a t-valuation, let us denote by X_K^v the set of those orderings P of K which are compatible with v, i.e. $1+\mathfrak{m}_v \subset P$, and satisfy the condition $t \in P$. In particular, for $t = 0$, $X_K^v = X_K$ coincides with the set of all orderings of K.

(3.1.) <u>Lemma</u>. If $P \in X_K^v$ then P is a t-ordering.

<u>Proof.</u> The case $t = 0$ is trivial, so we may assume $t \neq 0$.

According to (2.2) we have to show that $1-t \in P$ and $(1-tx^2)(1-tx^4) \in P$ for all $x \in K$. Since $1 + m_v \subset P$ and $m_v = t\,\mathcal{O}_v$ it follows $1-t \in P$. Now let $x \in K$. We may assume $x \neq 0$. If $v(x) \geq 0$ then $1-tx^2$, $1-tx^4 \in 1 + m_v \subset P$, and hence $(1-tx^2)(1-tx^4) \in P$. If $v(x) < 0$ then $1 - \frac{1}{tx^2}$ and $1 - \frac{1}{tx^4}$ belong to $1 + m_v \subset P$ and hence $(1-tx^2)(1-tx^4) = (1-\frac{1}{tx^2})(1-\frac{1}{tx^4})t^2x^6 \in P$. Q.E.D.

(3.2) <u>Lemma.</u> If $P \in X_K^v$ then $\mathcal{O}_v = \mathcal{O}(P) = \{x \in K \mid 1-tx^2 \in P\}$.

<u>Proof.</u> The case $t = 0$ is trivial, so we may assume $t \neq 0$.

The inclusion $\mathcal{O}_v \subset \mathcal{O}(P)$ is immediate since $1+m_v \subset P$. Conversely, let $x \in K \setminus \mathcal{O}_v$. Then $1-\frac{1}{tx^2} \in 1 + m_v \subset P$ and $1-tx^2 = -tx^2(1-\frac{1}{tx^2}) \in -P^{\circ}$. It follows $x \notin \mathcal{O}(P)$. We conclude that $\mathcal{O}_v = \mathcal{O}(P)$. Q.E.D.

3.3. <u>Proposition</u>. Suppose $t \neq 0$ and let v be a t-valuation of K. Denote by Y^v the set of orderings of the residue field K_v and by Z^v the group of characters of the factor group $v(K)/_{\mathbb{Z}+2v(K)}$. Then there is a (non-canonical) bijective map from the set X_K^v onto the cartesian product $Y^v \times Z^v$. Moreover this bijection is also a homeomorphism, if we consider the canonical topology on Y^v and the topology induced on Z^v by the product topology on the set of maps from $v(K)/_{\mathbb{Z}+2v(K)}$ into $\{\pm 1\}$. In particular, the set X_K^v is non-empty.

<u>Proof.</u> Denote by $\tilde{v} : K^{\bullet}/_{K^{\bullet 2} \cup tK^{\bullet 2}} \longrightarrow v(K)/_{\mathbb{Z}+2v(K)}$ the surjective morphism induced by v, and let $\mu : v(K)/_{\mathbb{Z}+2v(K)} \longrightarrow K^{\bullet}/_{K^{\bullet 2} \cup tK^2}$ be a $\mathbb{F}_2$ - linear map subject to $\tilde{v} \circ \mu = 1$. We

define a map $f_\mu : X_K^v \to Y^v \times Z^v$ dependent on μ by putting $f_\mu(P) = (\bar{P}, \sigma)$ where $\bar{P}$ is the ordering of K_v induced by P, and σ is the composite morphism $v(K)/\mathbb{Z}+2v(K) \xrightarrow{\mu} K^\bullet/K^{\bullet 2} \cup tK^{\bullet 2} \to K^\bullet/P^\bullet \cong \{\pm 1\}$.

The map f_μ has an inverse g_μ defined as follows. Let $(\bar{P}, \sigma) \in Y^v \times Z^v$, and $x \in K^\bullet$. We have $x = y.z$ with $y.(K^{\bullet 2} \cup tK^{\bullet 2}) = \mu(v(x) + \mathbb{Z} + 2v(K))$ and $z \in \mathcal{O}_v^\bullet$. We put $x \in g_\mu(\bar{P}, \sigma)$ iff $\bar{z}.\sigma(v(x) + \mathbb{Z} + 2v(K)) \in \bar{P}$. If $x = y'.z'$ is another representation of the same type of x, we have $z.z'^{-1} \in \mathcal{O}_v^\bullet \cap (K^{\bullet 2} \cap tK^{\bullet 2}) = \mathcal{O}_v^{\bullet 2}$, and hence $z.z'^{-1} \in \bar{P}^\bullet$. Thus $g_\mu(\bar{P}, \sigma)$ is well defined. It follows easily that $g_\mu(\bar{P}, \sigma)$ is a t-ordering of K, $f_\mu \circ g_\mu = 1$ and $g_\mu \circ f_\mu = 1$. Q.E.D.

(3.4) <u>Corollary.</u> The space of t-orderings X_K is the disjoint union of the non-empty sets X_K^v, where v ranges over the set of t-valuations of K.

<u>Proof.</u> If $P \in X_K$ then $P \in X_K^v$ where v is the valuation attached to $\mathcal{O}(P) = \{x \in K \mid 1 - tx^2 \in P\}$. Conversely, by (3.1), $X_K^v \subset X_K$ for each t-valuation v. By (3.3), X_K^v is non-empty.

It follows $X_K = \bigcup X_K^v$. If $P \in X_K^{v_1} \cap X_K^{v_2}$ then by (3.2), $\mathcal{O}(P) = \mathcal{O}_{v_1} = \mathcal{O}_{v_2}$ and hence $X_K^{v_1} \cap X_K^{v_2} = \emptyset$ for $v_1 \neq v_2$. Q.E.D.

(3.5) <u>Lemma.</u> Let K be a field and $t \neq 0$ be an element of K. If K is formally t-adic then t is transcendental over the prime subfield $\mathbb{Q}$ of K.

<u>Proof.</u> Let P be a t-ordering of K, and assume that t is algebraic over $\mathbb{Q}$, i.e. $t^n + a_1 t^{n-1} + \dots + a_n = 0$ for some $n \in \mathbb{N}$, $a_i \in \mathbb{Q}$, $a_n \neq 0$. Then $a_n = -t(a_{n-1} + \dots + t^{n-1}) \in t\,\mathcal{O}(P) \cap \mathbb{Q}$, and hence $a_n = 0$, which is absurd. Q.E.D.

Now let us describe the space of t-orderings X_K in some particular cases.

1) Consider the special case $K = \mathbb{Q}(t)$, where t is transcendental over $\mathbb{Q}$. There is a unique t-valuation v of K, defined as follows: if $f \in \mathbb{Q}[t]$, let v(f) be the smallest natural number n such that $(f.t^{-n})(0) \neq 0$; for f, $0 \neq g \in$

$\in \mathbb{Q}[t]$ let $v(\frac{f}{g}) = v(f) - v(g)$. The valuation ring $\mathcal{O}_v$ associated to v is the local ring $\mathbb{Q}[t]_{t\,\mathbb{Q}[t]}$ and its maximal ideal $\mathfrak{m}_v$ is generated by t. The valuation v is discrete, i.e. the value group is $\mathbb{Z}$, and $v(t) = 1$. The residue field of v may be identified with $\mathbb{Q}$.

Moreover, by (3.3), there is a unique t-ordering P of K, namely $P = \{x \in K^{\cdot} \mid \overline{(xt^{-v(x)})} = (xt^{-v(x)})(0) > 0\} \cup \{0\}$. By (2.2), it follows that P coincides with the semiring of K generated by $\{t, 1-t\} \cup \{(1-tx^2)(1-tx^4) \mid x \in K\} \cup K^2$.

Let us observe that $\mathcal{O}_v$ equals the ring $K_{fin}(P) = \{x \in K \mid \bigvee_{n \in \mathbb{N}} n \pm x \in P\}$ of finite elements with respect to P, and $\mathfrak{m}_v$ equals the ideal $K_{inf}(P) = \{x \in K \mid \bigwedge_{n \in \mathbb{N}} n \pm x \in P\}$ of infinitely small elements of K.

2) Let $K = \mathbb{R}((t))$ be the field of formal power series in t with coefficients in the field $\mathbb{R}$ of reals. There is a unique t-valuation v of K: for $f = \sum a_n t^n \in \mathbb{R}((t))$, let $v(f) = \min\{n \in \mathbb{Z} \mid a_n \neq 0\}$. The corresponding valuation ring $\mathcal{O}_v$ is $\mathbb{R}[[t]]$ with its maximal ideal $\mathfrak{m}_v = t\,\mathbb{R}[[t]]$. $\mathcal{O}_v$ is discrete and complete, and the residue field K_v is isomorphic to $\mathbb{R}$.

As in the previous case, there is a unique t-ordering of K, namely $P = K^2 \cup t\,K^2$.

§ 4. t-adically closed fields

<u>Definition</u>. A field K equipped with a t-ordering P, where t is a fixed element of K, is called a <u>t-ordered field</u>.

For the special case $t = 0$, we recover the concept of an ordered field. If $t \neq 0$, and (K,P) is a t-ordered field, then by (3.5), K may be identified with a field extension of the field $\mathbb{Q}(t)$ of rational functions, and P extends the unique t-ordering of $\mathbb{Q}(t)$.

Denote by $\mathcal{L}$ the first order language of ordered fields extended with an individual constant <u>t</u>. Let W be the theory in $\mathcal{L}$ obtained by adding to the usual axioms of ordered fields the following sentences:

$$0 \leq \underline{t} < 1$$

$$(\forall x)\ (1 - \underline{t}\, x^2)(1 - \underline{t}\, x^4) \geqslant 0.$$

The models of W are exactly the t-ordered fields. In particular each ordered field (K,P) may be seen as a model of W if we interpret the constant $\underline{t}$ on K as the null element 0 of K. If (K,P) is a t-ordered field where $0 \neq t \in K$, we must distinguish between the model (K,P) of W where the constant $\underline{t}$ is interpreted as t and the model (K,P) of W where the constant $\underline{t}$ is interpreted as 0. We say that (K,P) is a <u>proper</u> t-<u>ordered field</u> if (K,P) is a t-ordered field and $t \neq 0$.

If (K,P) is a t-ordered field, let $v = v_P$ denote the t-valuation attached to P, $\mathcal{O}_v = \mathcal{O}(P) = \{x \in K \mid 1 - tx^2 \in P\}$ the corresponding valuation ring, $\mathfrak{m}_v = t\,\mathcal{O}_v$ its maximal ideal, $K_v = \mathcal{O}_v/\mathfrak{m}_v$ the residue field, and v(K) the value group. If $t \neq 0$, v(t) is the smallest positive element of v(K) and we may identify the ordered group of integers $\mathbb{Z}$ with an isolated subgroup of v(K) by putting v(t) = 1.

Now let us denote by $\widetilde{W}$ the theory in $\mathcal{L}$ whose models are the t-ordered fields (K,P) which satisfy the following conditions:

i) the valuation v_P is henselian

ii) the residue field K_{v_P} is real closed

iii) if $t \neq 0$, then the value group $v_P(K)$ is a $\mathbb{Z}$-group, i.e. the factor group $v_P(K)/\mathbb{Z}$ is divisible.

Since the valuation ring $\mathcal{O}_{v_P}$ is described in terms of the language $\mathcal{L}$ we may easily write the corresponding axioms of the theory $\widetilde{W}$. In particular, if t = 0, the t-ordered field (K,P) is a model of $\widetilde{W}$ iff K is real closed.

First we are interested to describe the algebraically maximal models of the theory W.

(4.1) <u>Proposition</u>. Let (K,P) be a t-ordered field. If (K,P) is an algebraically maximal model of W then (K,P) is a model of $\widetilde{W}$.

<u>Proof.</u> The result is well known if t = 0, so we may

assume that $t \neq 0$. Suppose that the t-valuation $v = v_P$ is not henselian, and let $(\tilde{F},w)$ be the Henselization of (K,v). Since the extension $(F,w) \mid (K,v)$ is immediate, it follows by (3.3), that P can be extended to a t-ordering T of F. Thus (F,T) becomes a proper algebraic t-ordered field extension of (K,P), which is absurd, because by hypothesis, (K,P) is an algebraically maximal model of W. We conclude that the valuation v_P is henselian.

Now assume that the residue field K_v is not real closed, and let $(\bar{F}, \bar{T})$ be the real closure of $(K_v, \bar{P})$, where $\bar{P}$ is the ordering of K_v induced by P. Denote by (F,w) an unramified algebraic extension of (K,v) whose residue field F_w is isomorphic over K_v with $\bar{F}$. Let $\mu: v(K)/_{\mathbb{Z}+2v(K)} \longrightarrow K^{\bullet}/_{K^{\bullet 2}} \cup tK^{\bullet 2}$ be a section of the $\mathbb{F}_2$ - linear map $\tilde{v}: K^{\bullet}/_{K^{\bullet 2}} \cup tK^{\bullet 2} \longrightarrow v(K)/_{\mathbb{Z}+2v(K)}$ induced by v. Since $w(F) = v(K)$, μ is also a section of the map $\tilde{w} : F^{\bullet}/_{F^{\bullet 2}} \cup tF^{\bullet 2} \longrightarrow w(F)/_{\mathbb{Z}+2w(F)}$ induced by w. By (3.3), the t-ordering P is completely determined by $\bar{P}$ and some character $\sigma: v(K)/_{\mathbb{Z}+2v(K)} \longrightarrow \{\pm 1\}$. According to (3.3), the ordering $\bar{T}$ of F_w and the character $\sigma: w(F)/_{\mathbb{Z}+2w(F)} \longrightarrow \{\pm 1\}$ induce by means of μ a t-ordering T of F which extends P. Thus (F,T) is a proper algebraic t-ordered field extension of (K,P), in contrast with the maximality condition satisfied by (K,P). It follows that the residue field K_v is real closed.

Now assume that v(K) is not a $\mathbb{Z}$-group, i.e. there exist a prime number p and some $\alpha \in v(K)$ such that α is not p-divisible in v(K) modulo $\mathbb{Z}$. In this situation we can construct a t-ordered field extension $(F,T) \mid (K,P)$ of degree p, contradicting the fact that (K,P) is an algebraically maximal model of W. The construction is as follows:

Let $a \in K$ be such that $v(a) = \alpha$ and let b be a p-th root of a. Let us put $F = K(b)$. Then $[F:K] \leq p$. Let w be a valuation of F which extends v, and put $\beta = w(b)$. From $b^p = a$ it follows $p\beta = \alpha$. Since α ist not p-divisible in v (K) and p is a prime number we conclude that $\beta \in W(F)$ is of order p modulo v(K). Hence $(w(F) : v(K)) \geq p \geq [F:K]$. On the other hand it is known from valuation theory that the index of value groups is not larger than the field degree. We conclude that

$(w(F) : v(K)) = p = [F:K]$. Moreover it follows that w is the unique extension of v to F and the residue field F_w coincides with K_v. We claim that w is a t-valuation of F; for this it remains to show that $1 = v(t)$ is the smallest positive element in $w(F)$. Suppose that there exists $\gamma \in w(F)$ such that $0 < \gamma < 1$; then $0 < p\gamma < p$.

Since $(w(F): v(K)) = p$, it follows $p\gamma \in v(K)$. As $\mathbb{Z}$ is isolated in $v(K)$ we conclude that $p\gamma = n \in \mathbb{Z}$. On the other hand, since β is of order p modulo $v(K)$ it follows that β generates $w(F)$ modulo $v(K)$; hence $\gamma = k\beta + \lambda$ with $k \in \mathbb{Z}$, $\lambda \in v(K)$, $0 \le k < p$. Since $\gamma \notin v(K)$, we have $k \neq 0$. It follows $p\gamma = n = k\alpha + p\lambda$, and hence $k\alpha$ is p-divisible in $v(K)$ modulo $\mathbb{Z}$. As p is a prime number and $0 < k < p$, k is relatively prime to p, and hence α is also p-divisible in $v(K)$ modulo $\mathbb{Z}$, contrary to the choice of α.

Now we extend P to a t-ordering T of F, contradicting the algebraic maximality of (K,P). First let us assume $p \neq 2$. Let $\mu: v(K)/_{\mathbb{Z}+2v(K)} \longrightarrow K^\circ/_{K^{\circ 2} \cup tK^{\circ 2}}$ be a section of the $\mathbb{F}_2$ - linear map $\tilde{v} : K^\circ/_{K^{\circ 2} \cup tK^{\circ 2}} \longrightarrow v(K)/_{\mathbb{Z}+2v(K)}$ induced by the valuation v. Using μ, the t-ordering P is completely determined by the ordering $\bar{P}$ of K_v induced by P and by certain character $\sigma: v(K)/_{\mathbb{Z}+2v(K)} \longrightarrow \{\pm 1\}$. Let us consider the commutative diagram

$$\begin{array}{ccc} K^\circ/_{K^{\circ 2} \cup tK^{\circ 2}} & \xrightarrow{\tilde{v}} & v(K)/_{\mathbb{Z}+2v(K)} \\ \downarrow & & \downarrow \\ F^\circ/_{F^{\circ 2} \cup tF^{\circ 2}} & \xrightarrow{\tilde{w}} & w(F)/_{\mathbb{Z}+2w(F)} \end{array}$$

Since $[F:K] = (w(F): v(K)) = p \neq 2$, we have $K^\circ \cap (F^{\circ 2} \cup tF^{\circ 2}) = K^{\circ 2} \cup tK^{\circ 2}$ and $v(K) \cap (\mathbb{Z} + 2w(F)) = \mathbb{Z} + 2v(K)$, and hence the vertical maps are injective. It follows that we can extend μ to a $\mathbb{F}_2$ - linear map $\mu' : w(F)/_{\mathbb{Z}+2w(F)} \longrightarrow F^\circ/_{F^{\circ 2} \cup tF^{\circ 2}}$ subject to $\tilde{w} \circ \mu' = 1$. On the other hand we cam extend σ to a character

$\sigma' : w(F)/_{\mathbb{Z}+2\,w(F)} \longrightarrow \{\pm 1\}$. Using μ', it follows by (3.3) that $\bar{P}$ and σ' induce a t-ordering T of F extending P.

It remains to consider the case $p = 2$. Let us consider the commutative diagram

$$
\begin{array}{ccc}
1 & & 0 \\
\downarrow & & \downarrow \\
K^{\bullet} \cap (F^{\bullet 2} \cup tF^{\bullet 2})/_{K^{\bullet 2} \cup tK^{\bullet 2}} & \underset{\mu''}{\overset{--\to}{\leftarrow --}} & v(K) \cap (\mathbb{Z}+2w(F))/_{\mathbb{Z}+2v(K)} \\
\downarrow & & \downarrow \\
K^{\bullet}/_{K^{\bullet 2} \cup tK^{\bullet 2}} & \underset{\mu}{\overset{\tilde{v}}{\rightleftarrows}} & v(K)/_{\mathbb{Z}+2v(K)} \longrightarrow 0 \\
\downarrow & & \\
F^{\bullet}/_{F^{\bullet 2} \cup tF^{\bullet 2}} & \underset{\mu'}{\overset{\tilde{w}}{\rightleftarrows}} & w(F)/_{\mathbb{Z}+2w(F)} \longrightarrow 0
\end{array}
$$

It follows easily that $K^{\bullet} \cap (F^{\bullet 2} \cup tF^{\bullet 2})/_{K^{\bullet 2} \cup tK^{\bullet 2}} =$

$= \{K^{\bullet 2} \cup tK^{\bullet 2},\ a[K^{\bullet 2} \cup tK^{\bullet 2}]\} \cong \mathbb{Z}/_{2\mathbb{Z}}$, $w(F)/_{v(K)} =$

$= \{v(K),\ \beta + v(K)\} \cong \mathbb{Z}/_{2\mathbb{Z}}$, $v(K) \cap (\mathbb{Z}+2w(F)) = \mathbb{Z}+2w(F)$, and

$\mathbb{Z} + 2\,w(F)/_{\mathbb{Z}+2v(K)} = \{\mathbb{Z} + 2v(K),\ \alpha + \mathbb{Z}+2v(K)\} \cong \mathbb{Z}/_{2\mathbb{Z}}$. Thus

v induces a bijective map $K^{\bullet} \cap (F^{\bullet 2} \cup tF^{\bullet 2})/_{K^{\bullet 2} \cup tK^{\bullet 2}} \longrightarrow$

$\longrightarrow v(K) \cap (\mathbb{Z}+2w(F))/_{\mathbb{Z}+2v(K)}$ and hence we may choose the corresponding sections μ'', μ, μ' such that the preceeding diagram can be completed as shown.

Now let us observe that we may assume from the begginning that $a \in P$. Using μ, the t-ordering P induces the ordering $\bar{P}$ of K_v and a character $\sigma : v(K)/_{\mathbb{Z}+2v(K)} \longrightarrow \{\pm 1\}$. Since

$\mu(v(a)+\mathbb{Z}+2v(K)) = \mu(\alpha+\mathbb{Z}+2v(K)) = a\,(K^{\cdot 2}\cup tK^{\cdot 2})$ and $a\in P$, it follows $\sigma(1+\mathbb{Z}+2v(K)) = 1$, and hence σ can be extended to a character $\sigma' : w(F)/_{\mathbb{Z}+2w(F)} \longrightarrow \{\pm 1\}$. Thus the ordering $\widetilde{P}$ of $F_w = K_v$ and the character σ' determine by means of μ' a t-ordering T of F extending P. Q.E.D.

Now let us investigate the model-theoretic relation between the theories W and $\widetilde{W}$. First let us observe that the category $C_{\widetilde{W}}$ of models of $\widetilde{W}$ is equivalent with the category C_{W^*} of models of the theory W^* defined as follows. Let L^* denote the language of valued fields extended with an individual constant $\underline{t}$. Denote by W^* the theory in L^* having as models the systems (K,t,v) where K is a field equipped with the valuation v, t is an element in K, and the following conditions hold:

i) v is henselian

ii) the residue field K_v is real closed.

iii) t generates the maximal ideal m_v

iv) if $t \neq 0$, the value group $v(K)$ is a $\mathbb{Z}$ - group.

In particular, every system (K,t,v), where K is a real closed field, $t = 0$, and v is the trivial valuation of K, is a model of W^*.

If (K,t,P) is a model of $\widetilde{W}$ then the field K equipped with the t-valuation $v = v_P$ associated to P becomes a model of W^*. Conversely, if (K,t,v) is a model of W^* then there is a unique t-ordering of K which is compatible with v. If $t = 0$, we have nothing to show. Assume $t \neq 0$. Since $v(K)$ is a $\mathbb{Z}$-group, we have $v(K) = \mathbb{Z} + 2\,v(K)$, and hence the $\mathbb{F}_2$ - linear map

$$\tilde{v} : K^{\cdot}/_{K^{\cdot 2}\cup\, t\,K^{\cdot 2}} \longrightarrow v(K)/_{\mathbb{Z}+2v(K)} \cong 0$$ has a unique section μ.

On the other hand there exists a unique character σ of the group $v(K)/_{\mathbb{Z}+2\,v(K)} \cong 0$. By (3.3) there is only one t-ordering P of K which is compatible with the valuation v : for each $x\in K^{\circ}$, we have $x = t^{\varepsilon}\, y^2\, u$ with $\varepsilon\in\{0,1\}$, $u \in O_v^{\circ}, y\in K^{\circ}$, and $x \in P$ iff $\overline{u}$ belongs to $\widetilde{P}$, the unique ordering of the real closed field K_v. Thus we have for every $x\in K^{\circ}$: $x\in P$ iff K satisfies the following sentence:

$$(\exists y)(\exists u)\left[v\left(\frac{x}{y^2}\right) = 0 \wedge v\left(\frac{x}{y^2} - u^2\right) > 0\right] \vee \left[v\left(\frac{x}{ty^2}\right) = 0 \wedge \right.$$
$$\left. \wedge\ v\left(\frac{x}{ty^2} - u^2\right) > 0\right].$$

As a consequence we obtain the following result:

(4.2.) Theorem. The theory $\widetilde{W}$ is model-complete.

Proof. Let $f : (K,t,P) \to (F,t,T)$ be an embedding between models of $\widetilde{W}$. We have to show that this embedding is elementary.

Let $f^* : (K,t,v) \to (F,t,w)$ be the corresponding embedding between models of W^*. To show that f is elementary it suffices to verify that f^* is elementary. The elementarity of f^* is a consequence of the model-completeness of W^*.

Indeed, the model-completeness of real closed fields [15], Theorem 4.3.5, and the model-completeness of $\mathbb{Z}$-groups [20] Exercise 17.7., imply by [8] Theorem 1 the model-completeness of W^*. Q.E.D.

Moreover we have the following result:

(4.3). Theorem. $\widetilde{W}$ is the model-companion of W.

Proof. Since $\widetilde{W}$ is model-complete, it remains to show that each model of W can be embedded into a model of $\widetilde{W}$. Let (K,P) be a t-ordered field, and $(\widetilde{K}, \widetilde{P})$ the real closure of (K,P). If t = 0, then $(\widetilde{K}, 0, \widetilde{P})$ is a model of $\widetilde{W}$ extending (K, 0, P) and we have nothing to prove. So let us assume $t \neq 0$. Then we consider the family $\underline{F}$ of the subfields N of $\widetilde{K}/K$ subject to: $\widetilde{P} \cap N$ is a t-ordering of N. The family $\underline{F}$ is non-empty ($K \in \underline{F}$) and inductively ordered with respect to inclusion. By Zorn's lemma there exists a maximal member F of this family. Let $T = \widetilde{P} \cap F$ be the corresponding t-ordering. Then (F, T) is an algebraically maximal model of W. Indeed, if we assume the contrary, there exists a proper algebraic t-ordered field extension (F', T') of (F,T). Since $(\widetilde{K}, \widetilde{P})$ is also the real closure of (F,T), (F',T') can be embedded over (F,T) into $(\widetilde{K}, \widetilde{P})$. Thus we obtain a member of the family $\underline{F}$, which is a proper extension of F, contradicting the maximality of F. Therefore (F,T) is an algebraically maximal model of W and hence, by (4.1), (F,T) is a model of $\widetilde{W}$. We conclude that $\widetilde{W}$ is the model-companion of W. Q.E.D.

(4.4.) Corollary. Let (K,P) be a t-ordered field. Then (K,P) is an algebraically maximal model of W iff (K,P) is a model of $\widetilde{W}$.

Proof. If (K,P) is an algebraically maximal model of W then (K,P) is a model of $\widetilde{W}$ by (4.1).

Conversely, by (4.3), every model (K,P) of $\widetilde{W}$ is existentially complete in each t-ordered field extension of (K,P), and hence (K,P) is an algebraically maximal model of W. Q.E.D.

(4.5.) <u>Corollary.</u> Let (K,P) be a model of $\tilde{W}$. Then the following hold:

i) $P = K^2 \cup t\,K^2$ and hence P is the unique t-ordering of K.

ii) K has at most two orderings.

iii) K^2 is a trivial fan, i.e. K^2 is a fan and $(K^\bullet : K^{\bullet 2}) \leq 4$.

iv) K is hereditarily $\mathbb{N}$ - pythagorean.

<u>Proof.</u> The statement is trivial for $t = 0$, so we may assume $t \neq 0$.

i) Let $a \in P^\bullet$ and $v(a) = \alpha$, where v denotes the t-valuation attached to P. Since $v(K)/_{2\,v(K)} \cong \mathbb{Z}/_{2\mathbb{Z}}$, there exist only two possibilities: either $\alpha \in 2\,v(K)$ or $\alpha \in 1 + 2\,v(K)$. If $\alpha \in 2\,v(K)$ then $a = y^2 u$ with $y \in K^\bullet$ and $u \in O_v^\bullet$. As $a \in P^\bullet$ and K_v is real closed it follows $\bar{u} \in K_v^{\bullet 2}$. Since v is henselian we conclude that $u \in K^{\bullet 2}$ and hence $a \in K^2$. If $\alpha \in 1 + 2\,v(K)$ then $a = ty^2 u$ with $y \in K^\bullet$ and $u \in O_v^\bullet$. In the same way as above it follows $u \in K^{\bullet 2}$ and hence $a \in t\,K^2$. Thus $P = K^2 \cup t\,K^2$.

ii) K is pythagorean, i.e. $K^2 + K^2 = K^2$. Indeed, let $x \in K^\bullet$. We have to show that $1 + x^2 \in K^2$. We have $1 + x^2 \in P = K^2 \cup t\,K^2$. If $v(x) \neq 0$ then $1 + x^2 \in K^2$ since $v\,(1+x^2) \in 2v(K)$ and $v(t) = 1$. If $v(x) = 0$ then $v(1 + x^2) = 0$, because otherwise $1 + \bar{x}^2 = 0$ which is absurd since K_v is real closed. It follows $1 + x^2 \in K^2$ too.

Since K is pythagorean and $K^\bullet/_{K^{\bullet 2}} = \{\pm K^{\bullet 2}, \pm\, t\,K^{\bullet 2}\}$ we conclude that P and $P' = K^2 \cup -tK^2$ are the only orderings of K.

iii) K^2 is a fan as intersection of the orderings P and P'. K^2 is a trivial fan since $(K^\bullet : K^{\bullet 2}) = 4$.

iv) We have to show that K is hereditarily n-pythagorean for every natural number $n \geq 1$, i.e. $F^{2n} + F^{2n} = F^{2n}$ for each $n \geq 1$ and for every formally real algebraic field extension F of K. Let F be such a field extension of K and let $x \in F^\bullet$. We have to show that $1 + x^{2n} \in F^{2n}$. Let v be the t-valuation attached to P. Since v is henselian, v extends uniquely to a valuation w of F. We may assume $w(x) \geq 0$. Then $w(1+x^{2n}) = 0$ because otherwise $1 + \bar{x}^{2n} = 0$, which is absurd since $F_w = K_v$

is real closed. Moreover we have $1+\bar{x}^{2n}=\bar{y}^{2n}$ for some $\bar{y}\in K_v^{\circ}$. Since w is henselian too we conclude that $1+x^{2n}\in F^{2n}$. Q.E.D.

Now let K be a formally t-adic field. Among the algebraic field extensions of K which are formally t-adic there exists a maximal one by Zorn's lemma; this is called the t-adic closure of K. The t-adic closure, say F, of K is equipped with a unique t-ordering $T=F^2\cup tF^2$. In general the t-adic closure is not unique. If P is a given t-ordering of K then there exists a t-adic closure F of K whose canonical t-ordering extends P. We proceed as follows: let $(\tilde{K},\tilde{P})$ be the real closure of (K,P). If $t=0$ then $(\tilde{K},\tilde{P})$ satisfies the desired property. If $t\neq 0$ we choose by Zorn's lemma a maximal subfield of $\tilde{K}/K$ with the property: $T=F\cap\tilde{P}$ is a t-ordering. Then (F,T) is a model of $\tilde{W}$, $T=F^2\cup tF^2$, F is a t-adic closure of K, and T extends the given t-ordering P of K. Moreover we have the following result:

(4.6). <u>Proposition.</u> Let (K,P) be a t-ordered field and v be the t-valuation associated to P. If either $t=0$ or the value group v(K) is a $\mathbb{Z}$-group then the t-adic closure F of K, subject to: the unique t-ordering $T=F^2\cup tF^2$ of F extends P, is unique up to an ordered field isomorphism over (K,P).

<u>Proof.</u> Since for $t=0$ the result is well known we shall assume that $t\neq 0$ and v(K) is a $\mathbb{Z}$-group. We have to show that there exists a model (F,T) of $\tilde{W}$ which extends (K,P) and can be embedded over (K,P) into each model (F',T') of $\tilde{W}$ extending (K,P). We proceed as follows. Let v be the t-valuation attached to P and (K',v') the henselization of (K,v). If the residue field $K'_{v'}=K_v$ is real closed then (K',P'), where $P'=K'^2\cup tK'^2$ satisfies the desired conditions. If K_v is not real closed let (F,w) be an unramified algebraic extension of (K',v') having the residue field F_w isomorphic over K_v with the real closure $(\tilde{K}_v,\tilde{\bar{P}})$ of $(K_v,\bar{P})$, where $\bar{P}$ is the ordering of K_v induced by P. Such an extension is uniquely determined up to isomorphism of valued field extensions of (K',v'). Moreover $T=F^2\cup tF^2$ is the unique t-ordering of F and (F,T) satisfies the desired conditions.

On the other hand, if (F',T') is an arbitrary model of $\tilde{W}$ extending (K,P), F can be identified with the algebraic closure of K in F' and T is induced by the t-ordering T' of F'. Indeed, let N be the algebraic closure of K in F'. Since the valuation w' associated to T' is henselian, it follows that the field N equipped with the valuation w" induced by w' is henselian too. The residue field $N_{w''}$ can be identified with the algebraic closure of K_v in $F'_{w'}$, which is isomorphic with the real closure of $(K_v, \tilde{P})$. Since every monomorphism of $\mathbb{Z}$-groups is pure and N is algebraic over K it follows that the extension $(N,w'')\rfloor(K,v)$ is unramified, i.e. $w''(N) = v(K)$. N becomes a model of $\tilde{W}$, with the t-ordering $T'' = N^2 \cup t N^2 = T' \cap N$, and (N,T'') is isomorphic with (F,T) over (K,P). Q.E.D.

(4.7) Corollary. Let (K,P) be a t-ordered field and v be the t-valuation associated to P. Assume that either $t = 0$ or the value group $v(K)$ is a $\mathbb{Z}$-group. Then any two models of $\tilde{W}$ extending (K,P) are elementarily equivalent over (K,P). In particular there exist only two complete extensions of the theory $\tilde{W}$: one is obtained by adding to the axioms of $\tilde{W}$ the sentence $\underset{\sim}{t} = 0$, the other one by adding the sentence $\underset{\sim}{t} \neq 0$.

Proof. Let (F,T) and (F',T') be arbitrary models of $\tilde{W}$ which extend (K,P). By (4.6), there is a model $(\hat{K}, \hat{P})$ of $\tilde{W}$ extending (K,P) which can be embedded over (K,P) into (F,T) and (F',T'). As $\tilde{W}$ is model-complete we conclude that (F,T) and (F',T') are elementarily equivalent over $(\hat{K}, \hat{P})$ and hence over (K,P).

It remains to show that $\tilde{W}$ has only two complete extensions. Any two models of $\tilde{W}$ which satisfy the condition $t = 0$ are real closed and hence elementarily equivalent. On the other hand every t-ordered field satisfyng the condition $t \neq 0$ is an extension of the field of rational functions $\mathbb{Q}(t)$ equipped with the unique t-ordering P_0 described in § 3.

Let v_0 be the unique t-valuation of $\mathbb{Q}(t)$. Since v_0 is discrete, i.e. $v_0(\mathbb{Q}(t)) = \mathbb{Z}$, it follows that the theory $\tilde{W} \cup \{\underset{\sim}{t} \neq 0\}$ has a prime model, namely the field of algebraic power series $K = \tilde{\mathbb{Q}}\langle t \rangle$ with coefficients in the field $\tilde{\mathbb{Q}}$ of real algebraic numbers, and the t-ordering of K is $P = K^2 \cup t K^2$. Since $\tilde{W}$ is model-complete we conclude that $\tilde{W} \cup \{\underset{\sim}{t} \neq 0\}$ is complete. Q.E.D.

(4.8). Remark. The class of those t-ordered fields (K,P) which satisfy the additional condition: if $t \neq 0$, the value group $v(K)$ of the t-valuation v attached to P is a $\mathbb{Z}$-group, is axiomatizable in the first order language L. Denote by W' the corresponding theory. Then by (4.3) and (4.7) we conclude that $\widetilde{W}$ is the model-completion of W^{t}.

(4.9) We end this section with a parallel between the theory of formally t-adic fields considered in the present work and the theory of formally p-adic fields developed by Ax, Kochen, Roquette and others. The t-ordered fields correspond to the valuedfields (K,v) subject to: the residue field K_v is finite with p elements, where p is a prime number, and $v(p)$ is the smallest positive element of the value group $v(K)$. The respective prototypes are on the one hand the ordered field of rationals $\mathbb{Q}$ if $t = 0$, and the t-ordered field of rational functions $\mathbb{Q}(t)$ if $t \neq 0$, and on the other hand the field $\mathbb{Q}$ of rationals with the p-adic valuation. The t-adically closed fields correspond to the p-adically closed fields. The corresponding minimal models are on the one hand the field $\widetilde{\mathbb{Q}}$ of algebraic real numbers if $t = 0$, and the field of algebraic power series $\widetilde{Q}\langle t\rangle$ if $t \neq 0$, and on the other hand the field of algebraic p-adic numbers. Among other remarkable models we mention on the one hand the field $\mathbb{R}$ of reals if $t = 0$, the fields $\widetilde{\mathbb{Q}}((t))$ and $\mathbb{R}((t))$ of formal power series, and the field of all germs of real meromorphic functions if $t \neq 0$, and on the other hand the field $\mathbb{Q}_p$ of p-adic numbers.

The model-theoretic properties are similar in the both considered situations (see for the p-adic case [1],[8]). The role of the Kochen operator

$$\gamma(x) = \frac{1}{p}\,\frac{x^p - x}{(x^p - x)^2 - 1}$$

from the theory of formally p-adic fields is played in the theory of formally t-adic fields by the square operator and by the operator

$$\delta(x) = (1 - t\,x^2)(1 - t\,x^4)$$

considered by Roquette. A variant of the operator δ was first considered by Jacob [9]:

$$R(x,y) = x^6 - 5\,t\,x^4y^2 - 5\,t\,x^2y^4 + t^2y^6.$$

§ 5. The Kochen ring

Let K be a field, t a fixed element of K, and u a subset of K.

Definition. An ordering P of K, is called a t-ordering over u if P is a t-ordering and $u \subset P$.

Definition. The field K is called formally t-adic over u if there is at least one t-ordering over u of K.

If the set u is empty we recover the concept of a formally t-adic field.

Thus K is formally t-adic over u iff K is formally real over the subset $\{t, 1-t\} \cup \delta(K) \cup u$, i.e. there is at least one ordering P of K such that $\{t, 1-t\} \cup \delta(K) \cup u \subset P$.

Here $\delta(K) = \{\delta(x) \mid x \in K\}$. In other words, K is formally t-adic over u iff -1 does not belong to the semiring $\mathcal{J}_u$ of K generated by the subset $\{t, 1-t\} \cup \delta(K) \cup K^2 \cup u$. Let $A = \mathbb{Z}[u, t]$ be the subring of K generated by $u \cup \{t\}$. It follows that the class of those field extensions F of A which are formally t-adic over u is axiomatizable in the first order language of fields extended with individual constants which are names for the elements of $u \cup \{t\}$. If F is a member of this class then every intermediate field between A and F is also formally t-adic over u. This class is non-empty iff the field of fractions of A is formally t-adic over u.

Denote by $X_{K/u}$ the set of all t-orderings over u of K. In the particular case when u is empty $X_{K/u}$ coincides with the set X_K of all t-orderings of K. The set $X_{K/u}$ is a closed subset of the space X_K.

Definition. A valuation v of K is called a t-valuation over u if t generates the maximal ideal m_v and the residue field K_v is formally real over the subset $\overline{\mathcal{J}_u \cap O_v} = \{\bar{a} \in K_v \mid a \in \mathcal{J}_u \cap O_v\}$.

For u empty we recover by (3.3) the concept of a t-valuation. If $t = 0$, a valuation v of K is a 0-valuation over u iff v is the trivial valuation and K is formally real over u.

If P is a t-ordering over u then the associated t-valuation v_P is a t-valuation over u.

If v is a t-valuation over u let us denote by $X^v_{K/u}$ the set of all t-orderings P over u, which are compatible with v, i.e. $1 + m_v \subset P$. Proceeding as in § 3, we obtain the following result:

(5.1) Theorem: i) Let v be a t-valuation over u of K.

Denote by Y^v_u the set of those orderings of K_v which contain the subset $J_u \cap O_v$, and by Z^v_u the group of characters of the factor group $v(K)/_{v(J^\circ_u)}$. Here $J^\circ_u = J_u \setminus \{0\}$. [illegible] $= J_u$ O .

Then there is a (non-canonical) bijective map from $X^v_{K/u}$ onto the cartesian product $Y^v_u \times Z^v_u$. Moreover this bijection is also a homeomorphism if we consider the canonical topology on Y^v_u and the topology induced on Z^v_u by the product topology on $\{\pm 1\}^{v(K)/_{v(J^\circ_u)}}$.

ii) The space $X_{K/u}$ of t-orderings over u is the disjoint union $\bigcup X^v_{K/u}$ where v ranges over the set of t-valuations over u of K.

iii) If $P \in X^v_{K/u}$ then $O_v = O(P) = \{x \in K / 1 - tx^2 \in P\}$.

iv) K is formally t-adic over u iff there exists at least one t-valuation over u of K.

<u>Proof.</u> Only i) needs some explanations. The other statements follow easily as in § 3.

Lev $\tilde{v} : K^\circ/_{J^\circ_u} \longrightarrow v(K)/_{v(J^\circ_u)}$ be the surjective morphism induced by the valuation v. Denote by $\mu : v(K)/_{v(J^\circ_u)} \longrightarrow K^\circ/_{J^\circ_u}$ a $\mathbb{F}_2$ - linear map subject to $\tilde{v} \circ \mu = 1$. We define a map $f_\mu : X^v_{K/u} \longrightarrow Y^v_u \times Z^v_u$ by putting : $f_\mu(P) = (\overline{P}, \sigma)$, where $\overline{P}$ is the ordering of K_v induced by P, and σ is the composite morphism

$$v(K)/_{v(J^\circ_u)} \xrightarrow{\mu} K^\circ/_{J^\circ_u} \longrightarrow K^\circ/_{P^\circ} \cong \{\pm 1\}.$$

The map f_μ has an inverse g_μ defined as follows. Let $(\overline{P}, \sigma) \in Y^v_u \times Z^v_u$, and $x \in K^\circ$. We have $x = yz$ with $y J^\circ_u = \mu(v(x) + v(J^\circ_u))$ and $z \in O^\circ_v$. We put $x \in g_\mu(\overline{P}, \sigma)$ iff $\bar{z}.\sigma(v(x) + v(J^\circ_u)) \in \overline{P}$. If $x = y'z'$ is another representation of the same type of x, we have $z.z'^{-1} \in O^\circ_v \cap J_u$ and hence $\bar{z}.\bar{z}'^{-1} \in \overline{P}^\circ$. Thus $g_\mu(\overline{P}, \sigma)$ is well defined. It follows easily that $g_\mu(\overline{P}, \sigma) \in X^v_{K/u}$, $f_\mu \circ g_\mu = 1$, and $g_\mu \circ f_\mu = 1$. Q.E.D.

Now let us consider the following subrings of K:

$$B_u = \mathbb{Q}[t]_{(t)}\left[\frac{1}{1+z}\,\middle|\, z\in J_u\ ,\ z \neq -1\right]$$

$$R_u = \left\{x\in K \,\middle|\, 1 - tx^2 \in J_u\right\} = \bigcap \mathcal{O}(P),$$

where P ranges over the set of all t-orderings over u of K (we use the fact that J_u is the intersection of all t-orderings over u of K).

Observe that for t = 0, $\mathbb{Q}[t]_{(t)} = \mathbb{Q}$ and $R_u = K$. If t = 0 and u is empty then B_u coincides with the Baer ring of $K/\mathbb{Q}$ (see for instance [4] Theorem 2.4).

By (5.1) it follows $R_u = \bigcap O_v$ where v ranges over the set of all t-valuations over u of K. By analogy with the theory of formally p-adic fields we call R_u the <u>Kochen ring of K over</u> u. For u empty we obtain the (absolute) <u>Kochen ring of</u> K, denoted by $R \doteq R(K)$.

(5.2) <u>Lemma</u>. B_u is a subring of R_u.

<u>Proof.</u> The result is trivial for t = 0, so we assume $t \neq 0$. The result is also trivial if K is not formally t-adic over u; i.e. $-1 \in J_u$ and hence $R_u = K$. Assume that K is formally t-adic over u and let P be a t-ordering over u of K. P induces on $\mathbb{Q}(t)$ the canonical t-ordering of $\mathbb{Q}(t)$, and hence $\mathbb{Q}[t]_{(t)} \subset \mathcal{O}(P) = \{x\in K \mid 1-tx^2 \in P\}$. Now let $z \in J_u$. We have to show that $\frac{1}{1+z} \in \mathcal{O}(P)$. Indeed, we have $1-t\left(\frac{1}{1+z}\right)^2 = \frac{z^2+2z+(1-t)}{(1+z)^2} \in P$. since $z,\ 1-t \in J_u \subset P$. We conclude that $B_u \subset R_u$. Q.E.D.

(5.3). <u>Lemma.</u> B_u is a Prüfer ring with K as its field of fractions.

<u>Proof.</u> It suffices to prove the lemma for the particular case when u is empty. Let $B = \mathbb{Q}[t]_{(t)}\left[\frac{1}{1+z}\,\middle|\, z\in J,\ z \neq -1\right]$ where J is the semiring of K generated by the subset $\{t, 1-t\} \cup \delta(K) \cup K^2$. J coincides with the intersection of all t-orderings of K.

If K is not formally t-adic then J = K and hence B = K. Assume that K is formally t-adic. First let us show that K is the field of fractions of B. If $z \in J$ then $\frac{1}{1+z} \in B$, $\frac{z}{1+z} = 1 - \frac{1}{1+z} \in B$, and hence $z \in Q(B)$. On the other hand $K = J - J \subset Q(B)$

since $z = (\frac{z+1}{2})^2 - (\frac{z-1}{2})^2 \in J-J$ for every $z \in K$. We conclude that $Q(B) = K$.

Now let us show that B is a Prüfer ring, i.e. for every maximal ideal m in B, the local ring B_m is a valuation ring of K. First observe that either $z \in B_m$ or $z^{-1} \in m B_m$ if $z \in J$. Indeed, if $z \in J \setminus B_m$ then $\frac{1}{1+z} \in m$ and $\frac{z}{1+z} = 1 - \frac{1}{1+z} \in B \setminus m$ and hence $z^{-1} \in m B_m$. Let us show that the integral closure B_m' of B_m in K is a valuation ring. Let m' be a maximal ideal in B_m' such that $B_m \cap m' = m B_m$. Given any $z \in K$ we have either $z^2 \in B_m$ or $z^{-2} \in B_m$ since $z^2 \in J$. Thus either $z \in B_m'$ or $z^{-1} \in B_m'$ and hence B_m' is a valuation ring of K. Let us show that $B_m = B_m'$. If $z \in B_m'$ and $z^2 \notin B_m$ then $z^{-2} \in m B_m \subset m'$, which is absurd. Hence $z^2 \in B_m$ if $z \in B_m'$. Observe that the characteristic of the residue field B_m'/m' is zero. Let $z \in B_m'$. Since $z^2 \in B_m$ and $(1+z)^2 \in B_m$, we conclude that $z = \frac{(1+z)^2 - z^2 - 1}{2} \in B_m$. Thus B_m is a valuation ring of K for each maximal ideal m of B and hence B is a Prüfer ring with K as its field of fractions. Q.E.D.

(5.4) <u>Theorem.</u> Assume $t \neq 0$. Then the following hold:

i) $R_u = B_u\left[\frac{z}{z^2-t} \,\middle|\, z \in K,\ z^2 \neq t\right]$.

ii) Let v be a valuation of K. Then v is a t-valuation over u iff $O_v \supset R_u$ and $t \in m_v$.

iii) Every overring D of R_u, in particular R_u itself, is a Prüfer ring with K as its field of fractions, and the ideal-class group C(D) of D is a 2-group.

<u>Proof.</u> i) Denote by M the ring $B_u\left[\frac{z}{z^2-t} \,\middle|\, z \in K,\ z^2 \neq t\right]$. If K is not formally t-adic over u, $J_u = K$ and hence $M = B_u = K$.

Assume that K is formally t-adic over u. By (5.2), $B_u \subset R_u$. On the other hand, $R_u = \bigcap O_v$ where v ranges over the set of all t-valuations over u of K. Let v be such a valuation. Since v(t) is the smallest positive element of v(K), it follows $\frac{z}{z^2-t} \in O_v$ for every $z \in K$. We conclude that $M \subset R_u$. To prove the equality $M = R_u$ it suffices to show that R_u is integral over M and M is integrally closed in K. The last condition is satisfied since M is an overing of B_u which is a Prüfer ring in K by (5.3), so it remains to show that R_u is integral over M.

First let us observe that $t R_u \subset B_u \subset M$. Indeed, let $0 \neq z \in R_u$. Then $1 - tz^2 \in J_u$ and hence $t^{-1} z^{-2} - 1 \in J_u$. It follows $tz^2 = \frac{1}{1 + (t^{-1}z^{-2}-1)} \in B_u$. Thus $tz^2 \in B_u$ for each $z \in R_u$. We conclude that $tz = t\left(\frac{z+1}{2}\right)^2 - t\left(\frac{z-1}{2}\right)^2 \in B_u$ for every $z \in R_u$, i.e. $t R_u \subset B_u$.

Now let us remark that M equals its ring of fractions with respect to the monoid $1+tM$. Indeed, let $z = (1+ty)^{-1}$ with $y \in M$. Since $y \in M \subset R_u$ it follows $y \in O_v$ and hence $z \in O_v^*$ for each t-valuation v over u of K. Thus $z \in R_u$ and hence $tz \in B_u \subset M$. Therefore $z = 1 - (tz)\, y \in M$ as contended.

Let us show that R_u coincides with the integral closure M' of M in K. This fact follows from the general theory of formally p-adic fields [3] Theorem 2.2. For the convenience of the reader we include here a proof. Since M' is the intersection of those valuation rings of K which contain M, we have $M' \subset R_u$. Let $z \in K \setminus M'$. Since K is formally t-adic over u, the ideal t M is proper. It follows that the ideal $b = tM[z^{-1}] + z^{-1}M[z^{-1}]$ in the ring $M[z^{-1}]$ is proper. Indeed, if $1 \in b$ then $-1 = t\sum_{i=0}^{n} a_i z^{-i} + z^{-1} \sum_{i=0}^{n} b_i z^{-i}$ with $a_i, b_i \in M$ and hence $(1 + t a_o)z^{n+1} + \sum_{i=0}^{n-1} (a_{i+1} + b_i) z^{n-i} + b_n = 0$.

Since $1 + t a_o$ is a unit in M we conclude that z is integral over M which is absurd. Therefore the ideal b is proper.

Then $z^{-1} \in b \subset c$ for some maximal ideal c of $M[z^{-1}]$. As the canonical morphism $M \longrightarrow M[z^{-1}]/_c$ is surjective, its kernel $c \cap M$ is a maximal ideal which contains t M.

According to the place extension theorem there is a valuation ring O_v of K subject to $M[z^{-1}] \subset O_v$ and $c = m_v \cap M[z^{-1}]$. Let us show that v is a t-valuation over u. Since $M \subset O_v$, and hence $\frac{z}{z^2-t} \in O_v$ for each $z \in K$, it follows that $v(t)$ is the smallest positive element in $v(K)$. It remains to show that the residue field K_v is formally real over the subset $\overline{J_u \cap O_v}$. Assuming the contrary it follows $-1 = z - ty$ for some $z \in J_u$, $y \in O_v$, and hence $v(1+z) > 0$ for some $z \in J_u$. We derived a contradiction since $B_u \subset M \subset O_v$ and hence $\frac{1}{1+z} \in O_v$, i.e. $v(1+z) \leq 0$. We conclude that K_v is formally real over $\overline{J_u \cap O_v}$, and hence v is a t-valuation over u of K. On the other hand $z \notin O_v$ since $z^{-1} \in b \subset c \subset m_v$. It follows $z \notin R_u$. Thus R_u is the integral closure of M in K.

ii) Let v be a valuation of K. If v is a t-valuation over u of K then clearly $t \in m_v$ and $R_u \subset O_v$. Conversely, assume that $t \in m_v$ and $R_u \subset O_v$. Since $v(t) > 0$ and $\frac{z}{z^2-t} \in R_u \subset O_v$ for each $z \in K$ it follows that $v(t)$ is the smallest positive element of $v(K)$. In particular $R_u \neq K$ and hence K is formally t-adic over u. To conclude that v is a t-valuation over u it remains to show K_v is formally real over the subset $\overline{J_u \cap O_v}$. Assuming the contrary it follows $-1 = z - ty$ for some $z \in J_u$, $y \in O_v$, and hence there is $z \in J_u$ such that $v(1+z) > 0$, i.e. $\frac{1}{1+z} \notin O_v$. On the other hand, since $z \in J_u$, we have $\frac{1}{1+z} \in B_u \subset R_u \subset O_v$, i.e. a contradiction.

iii) Since R_u is a Prüfer ring in K, every overring D of R_u is a Prüfer ring too. As every such ring D is an intersection of valuation rings having formally real residue fields (in particular the polinomial $X^2 + 1$ has no solutions in the corresponding residue fields) it follows by the Prüfer criterion for holomorphy rings ([17] Theorem 1) that the factor group C(D) of the finitely generated fractional ideals of D modulo the principal ones is a 2-group. Q.E.D.

(5.5) <u>Corollary.</u> Suppose that $t \neq 0$. Then there is a canonical bijective map from the space of t-valuations over u of K onto the space of prime ideals of the factor ring $R_u/_t R_u$.

<u>Proof.</u> Since R_u is a Prüfer ring in K, the map $p \mapsto (R_u)_p$ is a bijection from the space of prime ideals of R_u onto the space of those valuation rings of K which lie over R_u. This map induces a bijective map from the space of prime ideals

of R_u/tR_u onto the space of those valuation rings O_v of K which satisfy the conditions: $R_u \subset O_v$ and $t \in m_v$. By (5.4)ii), the later space coincides with the space of valuation rings of all t-valuations over u of K.

§ 6. The Riemann space of places

In the general context of this section we consider an arbitrary base field K, a fixed element $t \neq 0$ of K, and a fixed subset P of K. We assume that K is formally t-adic over P. In particular we consider the case when P is a fixed t-ordering of K. Also we consider the special situation when K is the field of rational functions $\mathbb{Q}(t)$ and P is empty. We don't consider here the case t = 0 of formally real fields. For the corresponding theory in the case t = 0 see for instance [4].

Consider a field extension F of K. The space of those places Q of F/K which satisfy the condition: the residue field extension F.Q/K is formally t-adic over P, is called the Riemann space of F/(K,P) and is denoted by S(F). In order to simplify the notation we shall use the symbol S instead of S(F). If F is formally t-adic over P the trivial place 1_F is a member of S, and hence S is non-empty. In fact we shall see that a necessary and sufficient condition for S to be non-empty is that F is formally t-adic over P.

Let Q be a place of F/K. We denote by J_P, respectively by $J_P(F.Q)$, the semiring in F generated by the subset $\{t, 1-t\} \cup F^2 \cup \delta(F) \cup P$, respectively the semiring in F.Q generated by $\{t, 1-t\} \cup (F.Q)^2 \cup \delta(F.Q) \cup P$. Let R_P, respectively $R_P(F.Q)$, be the Kochen ring of F, respectively of F.Q, over P. Since K and P are fixed in the following, we ommit the index P, writting J, J(F.Q), R, R(F.Q) instead of J_P, $J_P(F.Q)$, R_P and $R_P(F.Q)$. The objects J, J(F.Q), R and R(F.Q) must not be confounded with the corresponding absolute objects attached to F and F.Q in the special case $P = \emptyset$.

(6.1) Theorem. Let Q be a place in S. Then the following hold:

i) Q lies over the Kochen ring R of F over P, i.e. R is contained in the valuation ring $\mathcal{O}_Q$ of Q.

ii) F is formally t-adic over P.

iii) The Kochen ring $R(F.Q)$ of $F.Q$ over P equals the image $R.Q$ by Q of the Kochen ring R of F over P.

Proof. i) Since $Q \in S$, $F.Q$ is formally t-adic over P. The inclusion $R \subset \mathcal{O}_Q$ is trivial if $Q = 1_F$. Assume that Q is not the trivial place, i.e. $\mathcal{O}_Q \neq F$. We have to show that $v_Q(x) \geqslant 0$ for each $x \in F$ subject to : $1 - tx^2 \in J$. Assuming the contrary, there exists $x \in F$ such that $v_Q(x) < 0$ and $1 - tx^2 \in J$. Let us put $y = x^{-1}$. I follows $v_Q(y) > 0$ and $y^2 - t \in J$, and hence $y^2 - t$ can be represented in the form $y^2 - t = \sum_{i=1}^{n} y_i$, where

$$(1) \quad y_i = t^{\alpha_i}(1-t)^{\beta_i}\, u_{i1}\cdots u_{ij_i}\, z_i^2 \prod_{k=1}^{\ell_i}(1-tx_{ik}^2)(1-tx_{ik}^4)$$

with $\alpha_i, \beta_i \in \{0,1\}$, $u_{i1},\ldots,u_{ij_i} \in P\setminus\{0\}$, $z_i \in F\setminus\{0\}$, $x_{i1},\ldots,x_{i\ell_i} \in F$.

Since $v_Q(y^2-t) = 0$ it follows $\lambda = \min\{v_Q(y_i) \mid i = 1,\ldots,n\} \leqslant 0$. If $\lambda < 0$ then $\lambda \in 2\, v_Q(F)$; we use the fact that $v_Q(1 - tx^2) = 0$ if $v_Q(x) \geqslant 0$, since $F.Q$ is formally t-adic and hence $1 - t(x.Q)^2 \neq 0$. Let $w \in F$ and $i_0 \in \{1,\ldots,n\}$ be such that $v_Q(w^2) = -\lambda$ and $v_Q(y_{i0}) = \lambda$. Then $v_Q(y_i w^2) \geqslant 0$, $(y_i w^2)Q \in J(F.Q)$ for $i = 1,\ldots,n$, and $v_Q(y_{i0}w^2) = 0$ and hence $0 \neq -(y_{i0}w^2).Q = \sum_{i \neq i_0}(y_i w^2).Q \in -J(F.Q) \cap J(F.Q)$, contrary to the hypothesis that $F.Q$ is formally t-adic over P.

Therefore $\lambda = 0$ and $-t = \sum_{i=1}^{n} y_i.Q \in J(F.Q)$. Thus we derive again a contradiction and hence $R \subset \mathcal{O}_Q$.

ii) Let Q be a member of S. If $Q = 1_F$ then clearly F is formally t-adic over P. If $Q \neq 1_F$ it follows by i) that $R \subset \mathcal{O}_Q \neq F$ and we obtain the same conclusion.

iii) If $Q = 1_F$ the equality $R(F.Q) = R.Q$ is trivial. Suppose that $Q \neq 1_F$. First let us show that $R.Q \subset R(F.Q)$. Let $x \in R$, i.e. $1-tx^2 \in J$. If $v_Q(x) > 0$ then $x.Q = 0 \in R(F.Q)$. Assume that $v_Q(x) = 0$. Since $1-tx^2 \in J$, we have $1-tx^2 = \sum_{i=1}^{n} y_i$, where the y_i 's have the form (1). Since F.Q is formally t-adic it follows $v_Q(1-tx^2) = 0$. Proceeding as in i) it follows $\lambda = \min \{v_Q(y_i) \mid i = 1,\dots,n\} = 0$ and $1-t(xQ)^2 \in J(F.Q)$, i.e. $x.Q \in R(F.Q)$.

Now let us show that $R(F.Q) \subset R.Q$. By (5,4),

$R(F.Q) = \mathbb{Q}[t]_{(t)}\left[\frac{1}{1+z} \,\middle|\, z \in J(F.Q)\right]\left[\frac{z}{z^2-t} \,\middle|\, z \in F.Q\right]$. It remains to show that $\frac{1}{1+z} \in R.Q$ for each $z \in J(F.Q)$ and $\frac{z}{z^2-t} \in R.Q$ for each $z \in F.Q$. Let $z \in J(F.Q)$; then there exists $x \in O_Q$ such that $x.Q = z$ and x belongs to the semiring in $\mathcal{O}_Q$ generated by $\{t, 1-t\} \cup \mathcal{O}_Q^2 \cup \delta(\mathcal{O}_Q) \cup P$.

Thus $\frac{1}{1+z} = \left(\frac{1}{1+x}\right).Q \in R.Q$. Let $z \in F.Q$ and $x \in \mathcal{O}_Q$ be such that $x.Q = z$. Then $\frac{z}{z^2-t} = \left(\frac{x}{x^2-t}\right).Q \in R.Q$. We conclude that $R(F.Q) = R.Q$ as contended. Q.E.D.

The following result gives a characterization of the places Q belonging to S.

(6.2) <u>Theorem.</u> Suppose that F is formally t-adic over P and let Q be a place of F/K. Then a necessary and sufficient condition for Q to belong to S is that Q lies above R.

<u>Proof.</u> If $Q \in S$ then, by (6.1), Q lies above R. Conversely, assume that Q lies above R, i.e. $R \subset \mathcal{O}_Q$. We have to show that F.Q is formally t-adic over P. Assuming the contrary, we have $-1 \in J(F.Q)$, and hence there is $y \in J$ such that $v_Q(1+y) > 0$. Since F is formally t-adic over P, $1 + y \neq 0$. Thus we obtain the contradictory statements: $\frac{1}{1+y} \in R \subset \mathcal{O}_Q$ and $\frac{1}{1+y} \notin \mathcal{O}_Q$. Q.E.D.

<u>Remark.</u> The previous results are analogues of Theorems 3.5.a and 3.5.b from [10] concerning formally p-adic fields,

and also of Proposition 2.3 from [4] concerning formally real fields.

If x is an arbitrary subset of F, denote by S^x the subset of S containing those places $Q \in S$ which satisfy the condition $x \subset \mathcal{O}_Q$.

The following result offers a criterion for a field extension F of K to be formally t-adic over P.

(6.3) <u>Theorem</u>. The following statements are equivalent:

i) F is formally t-adic over P.

ii) For every subset x of F, the set S^x is non-empty.

iii) The space S is non-empty.

<u>Proof.</u> i) implies ii). If F is formally t-adic over P then the identity place 1_F is a member of S^x for every subset x of F. The implication ii) $\Rightarrow$ iii) is trivial, and the implication iii) $\Rightarrow$ i) follows by (6.1).

Q.E.D.

Let us denote by H = H(F) the holomorphy ring $\bigcap_{Q \in S} \mathcal{O}_Q$ of the Riemann space S. Since $R \subset H$ and R is a Prüfer ring it follows by (6.2) that we may identify the Riemann space S with the prime spectrum Spec (H) of the holomorphy ring H and consider on S the Zariski topology having as basis of open sets the family $\{Y_f\}_{f \in H}$ where $Y_f = \{Q \in S \mid f.Q \neq 0\}$. Moreover S has a natural structure of ringed space. The structural sheaf G is given by $G(Y_f) = H_{(f)} = \bigcap_{Q \in Y_f} \mathcal{O}_Q$. Equipped with the Zariski topology, the Riemann space S is quasi-compact. It is easy to see that the family $\{S^x\}_x$ where x ranges over the family of finite subsets of F is a basis of open sets for the Zariski topology on S and $G(S^x) = H^x = \bigcap_{Q \in S^x} \mathcal{O}_Q$ for each finite subset x of F.

(6.4) <u>Theorem.</u> Let x be an arbitrary subset of F. Then $H^x = \bigcap_{Q \in S^x} \mathcal{O}_Q = R.K[x]$. In particular for x empty we obtain $H = R.K = K\left[\frac{1}{1+z} \,\middle|\, z \in J,\ z \neq -1\right]\left[\frac{z}{z^2-t} \,\middle|\, z \in F,\ z^2 \neq t\right]$. H^x is a Prüfer ring with F as its field of fractions and its ideal class-group is a 2-group.

Proof. The statement follows easily from (5.4) and (6.2).

Remark. The previous result is an analogue of [10] Theorem 3.7 on formally p-adic fields and of [3] Proposition 4.1 on formally real fields.

Let u be an arbitrary subset of F. Denote by S_u the subset of S containing those places $Q \in S$ which satisfy the condition: the elements of u are holomorphic in Q, i.e. $u \subset O_Q$, and the residue field extension $F.Q|K$ is formally t-adic over $P \cup u.Q$. In particular, if u is empty, $S_u = S$. If x and u are subsets of F, denote by S^u_x the intersection $S^x \cap S_u$, and by $H^x_u = \bigcap_{Q \in S^x_u} \mathcal{O}_Q$ the holomorphy ring of S^x_u. First let us observe that for every subset x of F, S^x_u is non-empty if F is formally t-adic over $P \cup u$. Indeed, in this case, the trivial place 1_F is contained in S^x_u. The converse is not generally true. (For instance, let K be the field of rational functions $\mathbb{Q}(t)$ and P the canonical t-ordering of K. Let $F = {}^*K$ be an enlargement of K in Robinson'sense [15], and ${}^*P$ the corresponding internal t-ordering of ${}^*K$. Let $a \neq 0$ be an infinitely small element of F with respect to ${}^*P/P$, i.e. $b \pm a \in {}^*P$ for every $b \in P^\circ$. Denote by Q the place of F/K whose veluation ring is the ring $F_{fin} = \{x \in F \mid \bigvee_{b \in P^\circ} b \pm x \in P^\circ\}$ of finite elements of F with respect to ${}^*P/P$. Then $Q \in S_u$, where $u = \{a, -a\}$ but F is not formally t-adic over $P \cup u$). If the monoid generated by u is a subgroup of the multiplicative group F°, it follows easily that S_u is non-empty iff F is formally t-adic over $P \cup u$.

We end this section with a description of the holomorphy rings H^x_u for arbitrary subsets x and u. The particular case when u is empty was considered in (6.4).

(6.5.) Proposition. Suppose that S^x_u is non-empty. Then H^x_u is the smallest overring A of $H[x,u]$ subject to $1 + J_u(A) \subset A^\circ$, where $J_u(A)$ denotes the semiring generated by $\{t, 1-t\} \cup A^2 \cup \delta(A) \cup P \cup u$, and A° is the group of units in A.

Proof. First let us observe that the intersection $A = \bigcap_{i \in I} A_i$ of a family $\{A_i\}_{i \in I}$ of overrings of $H[x,u]$

subject to $1 + J_u(A_i) \subset A_i^\circ$ satisfies the condition $1+J_u(A)\subset A^\circ$ too.

Let A denote the smallest overring of $H[x,u]$ subject to $1 + J_u(A)\subset A^\circ$. Observe that $A\subset H_u^x$. Indeed, let $Q\in S_u^x$ and $J_u(\mathcal{O}_Q)$ be the semiring generated by $\{t,1-t\}\cup \mathcal{O}_Q^2\cup\delta(\mathcal{O}_Q)\cup P\cup u$. If $1 + J_u(\mathcal{O}_Q)\not\subset \mathcal{O}_Q^\circ$ then -1 belongs to the semiring $J_{uQ}(F.Q) = J_u(\mathcal{O}_Q).Q$ generated by $\{t,1-t\}\cup(F.Q)^2\cup\delta(F.Q)\cup P\cup u.Q$, i.e. $F.Q$ is not formally t-adic over $P\cup u.Q$, which is absurd.

On the other hand A is a Prüfer ring with F as its field of fractions and hence $A = \bigcap A_p$ where p ranges over the set of maximal ideals of A. For each maximal ideal p of A let Q_p be the place of F/K associated to the valuation ring A_p. If we show that $Q_p\in S_u^x$ for every maximal ideal p then $H_u^x =$

$$= \bigcap_{Q\in S_u^x} \mathcal{O}_Q \subset \bigcap_{p\in \mathrm{Max}(A)} \mathcal{O}_{Q_p} = A$$

and hence $A = H_u^x$. It remains to show that $Q_p\in S_u^x$ for $p\in \mathrm{Max}(A)$. We have $u\cup x\subset \mathcal{O}_{Q_p} = A_p$, $F.Q_p \cong A/p$, $J_{uQ_p}(F.Q_p) = J_u(A).Q_p$ and hence $-1\notin J_{u.Q_p}(F.Q_p)$ because $(1 + J_u(A))\cap p = \emptyset$. We conclude that $F.Q_p$ is formally t-adic over $P\cup u.Q_p$ and hence $Q_p\in S_u^x$.

Q.E.D.

The following result describes the holomorphy ring H_u^x as the inductive limit of certain overrings of $H[x,u]$.

(6.6) <u>Proposition.</u> Assume that S_u^x is non-empty. Then there exists a unique sequence $(A_n)_{n\in\mathbb{N}}$ of intermediate rings between $H[x,u]$ and H_u^x satisfysing the conditions:

i) $A_0 = H[x,u]$.

ii) $-1\notin J_u(A_n)$.

iii) A_{n+1} is the ring of fractions of A_n with respect to the monoid $1 + J_u(A_n)$.

At last we have $H_u^x = \bigcup_{n\in\mathbb{N}} A_n$.

<u>Proof.</u> First we have to show by induction that $-1\notin J_u(A_n)$ and $A_n\subset H_u^x$ for each $n\in \mathbb{N}$.

For $n = 0$, if $-1\in J_u(A_0)$ then $F.Q.$ is not formally t-adic over $P\cup u.Q$ for every $Q\in S_u^x \neq \emptyset$, which is absurd. We conclude that $-1 \notin J_u(A_o)$.

Suppose that $A_n \subset H_u^x$ and $-1 \notin J_u(A_n)$. We have to show that $A_{n+1} \subset H_u^x$ and $-1 \notin J_u(A_{n+1})$. Since $1 + J_u(A_n) \subset H_u^x \subset \mathcal{O}_Q$ and F.Q. is formally t-adic over $P \cup u$. Q for each $Q \in S_u^x$ it follows $1 + J_u(A_n) \subset \mathcal{O}_Q^{\bullet}$ for every $Q \in S_u^x$ and hence $A_{n+1} \subset H_u^x$. With the same argument it follows $1 + J_u(A_{n+1}) \subset (H_u^x)^{\bullet}$ and hence $-1 \notin J_u(A_{n+1})$.

Now we have to show that $H_u^x = A = \bigcup_{n \in \mathbb{N}} A_n$. By (6.5) it suffices to prove that A is minimal with the properties: $H[x,u] \subset A$ and $1 + J_u(A) \subset A^{\bullet}$. First let us show that A satisfies the later condition. Let $z \in J_u(A)$; we have to show that $1+z$ is invertible in A. By construction of A, there is $n \in \mathbb{N}$ such that $z \in J_u(A_n)$. Since $-1 \notin J_u(A_n)$ we conclude that $\frac{1}{1+z} \in A_{n+1} \subset A$. Now let C be an overring of $H[x,u]$ subject to $1 + J_u(C) \subset C^{\bullet}$. We have to show that $A_n \subset C$ for every $n \in \mathbb{N}$. For $n = 0$ we have nothing to show. Assume that $A_n \subset C$ for some $n \in \mathbb{N}$. We must show that $A_{n+1} \subset C$. Since $A_n \subset C$ it follows $1+J_u(A_n) \subset 1+ J_u(C)$ and hence $1 + J_u(A_n) \subset C^{\bullet}$. We conclude that $A_{n+1} \subset C$. Q.E.D.

Remark. The last two results are analogues of Propositions 4.1 and 4.2 from [4] concerning the formally real fields.

§ 7. The Nullstellensatz for holomorphy rings-a weak form

Let K, t and P be as in § 6. We suppose that K is formally t-adic over P. Let F be a field extension of K and, x and u arbitrary subsets of F. Our goal in this section is to give a week form of the Nullstellensatz for an arbitrary subring A of H_u^x which contains $H[x,u]$.

Definition. Griven a subset M of S_u^x, let $I_A(M)$ be the ideal of A consisting of those elements $z \in A$ which vanish on M, i.e. z.Q = 0 for each $Q \in M$. Given an ideal a of A let $S_u^x(a)$ be the set of common zeros $Q \in S_u^x$ of elements in a, i.e. z Q = 0 for each $z \in a$.

Definition. (Stengle [21]). Let C be a commutative ring, a an ideal in C, and Γ a semiring in C containing all squares in C. Then the Γ -radical of a is the subset:

$$r_\Gamma(a) = \left\{ z \in C \;\middle|\; z^{2m} + b \in a \text{ for some } m \geqslant 1,\ b \in \Gamma \right\}$$

An ideal in C is a Γ - radical ideal if it is own Γ- radical. According to Stengle [21] Proposition 2, $r_\Gamma(a)$ is a Γ -radical ideal, and equals the intersection of all prime Γ- radical ideals containing a.

If A is an intermediate ring bewtween $H[x,u]$ and H_u^x, we denote by $J_u(A)$ the semiring in A generated by $\{t, 1-t\} \cup A^2 \cup \delta(A) \cup P \cup u$. If a is an ideal in A we denote by $r_u(a) = r_{J_u(A)}(a)$ the $J_u(A)$ - radical of a. It follows $S_u^x(a) = S_u^x(r_u(a))$, and $r_u(a) \subset I_A(S_u^x(a))$.

(7.1). Proposition. Let A be a subring of H_u^x containing $H[x,u]$, and Q a place of F/K. Then $Q \in S_u^x$ iff Q lies above A and the center $m_Q \cap A$ of Q on A is a $J_u(A)$ - radical ideal.

Proof. If $Q \in S_u^x$ then $\mathcal{O}_Q \supset H_u^x \supset A$ and F.Q is formally t-adic over $P \cup u.Q$. Let $z \in A$ be such that $z^{2m} + b \in m_Q$ for some $m \geqslant 1$ and $b \in J_u(A)$. Then $(z.Q)^{2m} + b.Q = 0$ and hence $z.Q = 0$, i.e. $z \in m_Q \cap A$, since F.Q is formally t-adic over $P \cup u.Q$.

Conversely, we have only to show that F.Q is formally t-adic over $P \cup u.Q$. Since A is a Prüfer ring it follows $\mathcal{O}_Q = A_p$ where $p = m_Q \cap A$ and F.Q is isomorphic to the field of fractions of $A/_p$. If F.Q is not formally t-adic over $P \cup u.Q$, i.e. $-1 \in J_{u.Q}(F.Q)$, then $z^{2m} + b \in p$ for some $z \in A \setminus p$, $m \geqslant 1$ and $b \in J_u(A)$, and hence p is not a $J_u(A)$ - radical ideal, which contradicts the hypothesis. Q.E.D.

The following weak Nullstellensatz is an immediate consequence of (7.1) and of Stengle [21] Proposition 2.

(7.2) Proposition. Let A be a subring of H_u^x containing $H[x,u]$ and a an ideal in A. Then $I_A(S_u^x(a)) = r_u(a)$.

(7.3) Corollary. Let a be an ideal in $H^x = H[x]$. Then $I_{H^x}(S^x(a))$ equals the nilradical $\sqrt{a}$ of a. In particular

$$I_{H^x}(S^x) = \bigcap_{Q \in S^x} m_Q = 0 \text{ if F is formally t-adic over P.}$$

§ 8. The restricted Riemann space

In the general context of the rest of this paper we consider an arbitrary base field K, $t \neq 0,1$ a fixed element of K, and P a fixed t-ordering of K. Let v be the t-valuation associated to P, and $O_v = O(P) = \{x \in K \mid 1 - tx^2 \in P\}$ the corresponding valuation ring. We assume in addition that the value group v(K) is a $\mathbb{Z}$-group. By (3.3), the t-ordering P is completely determined by the induced ordering $\bar{P}$ of K_v. Denote by $\hat{K}$ the t-adic closure of K satisfying the condition: its unique t-ordering $\hat{P} = \hat{K}^2 \cup t\hat{K}^2$ extends P. By (4.6), $(\hat{K}, \hat{P})$ is uniquely determined up to an isomorphism of t-ordered fields over (K,P).

If F is a field extension of K we define the restricted Riemann space $\hat{S} = \hat{S}(F)$ of $F|(K,P)$ as the subspace of the Riemann space S = S(F) consisting of those places Q of $F|K$ which are rational over K, i.e. $K \subset F.Q \subset \hat{K}$. It is possible that $\hat{S}$ is empty though S is non-empty, i.e. F is formally t-adic over P. For instance, let $K = \mathbb{Q}(t)$, P the unique t-ordering of K, and $F = \mathbb{R}((t))$. Since F is t-adically closed, the unique t-ordering of F is $T = F^2 \cup tF^2$, $R = R(F) = \mathcal{O}(T) = \{x \in F \mid 1 - tx^2 \in T\} = \mathbb{R}[[t]]$, and $H = H(F) = K.R = F$.

Thus $S = S(F) = \{1_F\}$ and hence S is empty. However, if F is a finitely generated field extension of K, we shall show that the necessary and sufficient condition for F/K to admit a non-empty restructed Riemann space $\hat{S}$ is that F is formally t-adic over P.

<u>We shall assume in the rest of this paper that F/K is finitely generated.</u>

The Zariski topology on S induces a topology on $\hat{S}$. A basis of open sets for this induced topology is given by the sets $\hat{S}^x = \hat{S} \cap S^x = \{Q \in \hat{S} \mid x \subset \mathcal{O}_Q\}$ where x ranges over all finite subsets of F. There is also another topology on $\hat{S}$ induced by the ordering $\hat{P}$ of $\hat{K}$. This topology admits as basis the sets $\hat{S}_u = \{Q \in \hat{S} \mid u.Q \subset \hat{P}\}$ where u ranges over all finite subsets of F. We also consider sets of the form $\hat{S}_u^x = \hat{S}_u \cap \hat{S}^x$ where u and x are finite subsets of F. Amy such set will be called a <u>basic subset</u> of $\hat{S}$. The folowing result establishes a non-trivial relation between the sets S_u^x and $\hat{S}_u^x$.

(8.1) <u>Proposition.</u> Let u, x, u' and x' be finite subsets of F. If $\hat{S}^x_u \subset \hat{S}^{x'}_{u'}$ then $S^x_u \subset S^{x'}_{u'}$.

<u>Proof.</u> Suppose that $S^x_u \not\subset S^{x'}_{u'}$, i.e. either the set $\{Q \in S^x_u \mid u' \cup x' \not\subset \mathcal{O}_Q\}$ is non-empty or the set $\{Q \in S^x_u \mid u' \subset \mathcal{O}_Q$, $F.Q$ is not formally t-adic over $P \cup u'.Q\}$ is non-empty. Then the statement is a consequence of the following two lemmata.

(8.2) <u>Lemma.</u> Let u and x be finite subsets of F and z be an element of F. Then the following assertions are equivalent:

i) The set $\{Q \in S^x_u \mid z.Q = 0\}$ is non-empty.

ii) The set $\{Q \in \hat{S}^x_u \mid z.Q = 0\}$ is non-empty.

<u>Proof.</u> Only the implication i) $\Longrightarrow$ ii) requires a proof. Suppose that there is a place $U \in S^x_u$ such that $z.U = 0$. We have to show that the set $M_x = \{Q \in \hat{S}^x_u \mid z.Q = 0\}$ is non-empty. If $M_{x'}$ is non-empty for some finite subset x' of F containing x, then clearly M_x is a also non-empty. Hence we may enlarge x if convenient by adding finitely many elements of F. After a suitable enlargement we may assume that $F = K(x)$, $u \cup \{z\} \subset x \subset \mathcal{O}_U$ and, by Zariski's local uniformization theorem [23], $x.U$ is a simple point on the affine model V of F/K whose generic point is x. After a renumbering of the elements $x_1, \ldots, x_n$ of x we may assume that $u = \{x_1, \ldots, x_m\}$, $z = x_{m+1}$, and $m+1 \leqslant n$. Thus there exists a t-ordering T on $F.U$ which extends P and $x_i.U \in T$ for $i = 1, \ldots, m$. In addition the point $x.U \in V(F.U)$ is simple and $x_{m+1}.U = 0$.

We envisage the affine variety V in n-space as being defined by a finite system of polynomial equations over K. Let $f_1, \ldots, f_s \in K[X]$, where $X = (X_1, \ldots, X_n)$, be some polynomials defining V. The condition for a point to be simple on V is that at least one the minors of order n-dim (V) of the Jacobian matrix $(\frac{\partial f_i}{\partial X_j})$ does not vanish at that point. Let $h \in K[X]$ be a proper minor such that $h(x.U) \neq 0$. Thus the t-ordered field extension $(F.U, T)$ of (K, P) satisfies the following sentence in the language L_K of t-ordered fields extended with constants which are names for the elements of K:

$$\varphi := (\exists X) \bigwedge_{i=1}^{s} f_i(X) = 0 \wedge h(X) \neq 0 \wedge \bigwedge_{i=1}^{m} X_i \geqslant 0 \wedge X_{m+1} = 0$$

where $X = (X_1, \ldots X_n)$.

By (4.3), $(F.U,T)$ can be embedded into a model (L,T') of $\widetilde{W}$. Since φ is an existential sentence in L_K it follows that φ is also true on (L,T'). Let us consider the commutative diagram in the category of t-ordered fields:

$$\begin{array}{ccc} (K,P) & \longrightarrow & (F.U,T) \\ \downarrow & & \downarrow \\ (\hat{K},\hat{P}) & \xrightarrow{\ \lambda\ } & (L,T') \end{array}$$

where $(\hat{K}, \hat{P})$ is the t-adic closure of (K,P). By (4.3), λ is an elementary embedding and hence $(\hat{K}, \hat{P})$ satisfies φ. Thus there exists a point $b = (b_1, \ldots, b_n)$ of V, which is rational over $\hat{K}$, such that $h(b) \neq 0$ (therefore b is simple), $b_i \in \hat{P}$ for $i = 1, \ldots, m$, and $b_{m+1} = 0$. Since the point b is simple on V it follows by a well known result of the algebraic geometry (see for instance [10] Corollary A2) that the specialization $x \to b$ can be extended to a $\hat{K}$ - rational place Q of F/K. It follows $Q \in \hat{S}_u^x$ and $z.Q = 0$ and hence the set M_x is non-empty as contended.

Q.E.D.

(8.3) Lemma. Let x, u and u' be finite subsets of F. If there is a place $U \in S_u^{x \cup u'}$ such that $F.U$ is not formally t-adic over $P \cup u'.U$ then there is a place $Q \in \hat{S}_u^{x \cup u'}$ such that $u'.Q \not\subset \hat{P}$.

The proof of this lemma is similar with the proof of (8.2).

(8.4) Corollary. Let x and u be finite subsets of F. Then S_u^x is non-empty iff $\hat{S}_u^x$ is non-empty. In particular the restricted Riemann space $\hat{S}$ is dense in the Riemann space S with respect to the Zariski topology.

(8.5) Corollary. Let u be a finite subset of F such that the multiplicative monoid generated by u is a subgroup of F^*. Then the following assertions are equivalent:

i) F is formally t-adic over $P \cup u$.

ii) $\hat{S}_u^x$ is non-empty for each finite subset x of F.

iii) $\hat{S}_u$ is non-empty.

In particular, a necessary and sufficient condition for $\hat{S}$ to be non-empty is that F is formally t-adic over P.

(8.6). Corollary. Let u be a finite subset of F such that the multiplicative monoid generated by u is a subgroup of $F^\bullet$. Assume that F is formally t-adic over $P \cup u$ and F cannot be embedded in $\hat{K}$ over K; in particular we may assume that F is transcendental over K. Then the set $\hat{S}_u^x$ is infinite for every finite subset x of F.

Proof. Suppose that $\hat{S}_u^x$ is finite for some finite set $x = \{x_1, \ldots, x_n\}$. By (8.5) the set $\hat{S}_u^x$ is non-empty. Let $Q_1, \ldots, Q_e$ $(e \geqslant 1)$ be the elements of $\hat{S}_u^x$. Since F cannot be embedded in $\hat{K}$ over K, there exist $y_1, \ldots, y_e \in F$ such that $y_i Q_i = \infty$ for $i = 1, \ldots, e$. Let x' be the set $\{x_1, \ldots x_n, y_1, \ldots, y_e\}$. We conclude that $\hat{S}_u^{x'}$ is empty, contrary to (8.5) Q.E.D.

§ 9. The Nullstellensatz for holomorphy rings

Let us consider the same situation as in § 8. First we give a description of the holomorphy rings H_u^x in terms of holomorphy rings of basic subsets of the restricted Riemann space $\hat{S}$.

(9.1) Theorem. Let x and u be finite subsets of F. Then the holomorphy ring H_u^x of S_u^x equals the holomorphy ring of the basic subset $\hat{S}_u^x$ of $\hat{S}$, i.e. $H_u^x = \bigcap_{Q \in \hat{S}_u^x} \mathcal{O}_Q$.

Proof. We have to show that $\bigcap_{Q \in \hat{S}_u^x} \mathcal{O}_Q \subset H_u^x$. Let $z \in F \setminus H_u^x$, i.e. there is $Q \in S_u^x$ such that $z^{-1}.Q = 0$. By (8.2), there exists a place $U \in \hat{S}_u^x$ such that $z^{-1}.U = 0$ and hence $z \notin \bigcap_{Q \in \hat{S}_u^x} \mathcal{O}_Q$. Q.E.D.

Now let us consider an arbitrary subring A of H_u^x which contains $H[x,u]$. Given a subset M of $\hat{S}_u^x$, let $I_A(M)$ be the ideal of A consisting of those elements $z \in A$ which vanish on M, i.e. $z.Q = 0$ for each $Q \in M$. Given an ideal $\mathfrak{a}$ of A let $\hat{S}_u^x(\mathfrak{a})$ be the set of common zeros $Q \in \hat{S}_u^x$ of elements in $\mathfrak{a}$, i.e. $z\,Q = 0$ for each $z \in \mathfrak{a}$.

(9.2). Theorem. (Nullstellensatz - a strong form). Let A be a subring of H_u^x which contains $H[x,u]$, and $\mathfrak{a}$ a finitely generated ideal in A. Then $I_A(\hat{S}^x_u(a)) = r_u(a)$.

<u>Proof.</u> By (7.2), $r_u(a) = I_A(S^x_u(a))$. Since $\hat{S}^x_u(a) = S^x_u(a) \cap \hat{S}^x_u$ we have to show that $I_A(\hat{S}^x_u(a)) \subset I_A(S^x_u(a))$. Let $f \in A \setminus I_A(S^x_u(a))$, i.e. the set $\{Q \in S^x_u \mid z.Q = 0$ for every $z \in a$, and $f.Q \neq 0\}$ is non-empty. Since a is finitely generated we may use a similar argument as in the proof of (8.2) to conclude that the set $\{Q \in \hat{S}^x_u \mid z.Q = 0$ for every $z \in a$, and $f.Q \neq 0\}$ is non-empty too and hence $f \notin I_A(\hat{S}^x_u(a))$.

Q.E.D.

(9.3) <u>Corollary.</u> Let a be a finitely generated ideal of $H^x = H[x]$. Then $I_{H^x}(\hat{S}^x(a)) = \sqrt{a}$. In particular $I_{H^x}(\hat{S}^x) = \bigcap_{Q \in \hat{S}^x} m_Q = 0$

The preceeding results can be interpreted as statements about the radical ideal structure of the holomporphy ring H^x_u. More precisely we have:

(9.4) <u>Proposition.</u> Let A be an intermediate ring between $H[x,u]$ and H^x_u and a a finitely generated ideal in A. Then the $J_u(A)$ -radical $r_u(a)$ of a coincides with the intersection of those maximal ideals p of A which contain a and are $J_u(A)$ - radical ideals. In particular, H^x is a generalized Jacobson ring, i.e. for every finitely generated ideal a in H^x, the milradical of a equals the Jacobson radical of a.

<u>Proof.</u> Let A be a ring such that $H[x,u] \subset A \subset H^x_u$ and a be a finitely generated ideal in A. Denote by c the intersection of those maximal ideals p in A which contain a and are $J_u(A)$ -radical ideals. We have to show that $c \subset r_u(a)$. Observe that every place $Q \in S^x_u$ determines a maximal ideal in A, namely its center $m_Q \cap A$ on A. Indeed, Q induces a K-monomorphism from $A/_{m_Q \cap A}$ into $\hat{K}$. Since the extension $\hat{K}/K$ is algebraic, we conclude that $A/_{m_Q \cap A}$ is a field, i.e. $m_Q \cap A$ is a maximal ideal in A. In addition, by (7.1), $m_Q \cap A$ is a $J_u(A)$ - radical ideal. Therefore for each element $h \in c$, we have $h.Q = 0$ for every $Q \in \hat{S}^x_u(a)$. By (9.2), $h \in r_u(a)$ and hence $c \subset r_u(a)$. For the particular case $u = \emptyset$ we apply (9.3).

Q.E.D.

§ 10. The Nullstellensatz for the coordinate ring of a variety over a t-ordered field

Let K, t, P, $\hat{K}$, $\hat{P}$ be as in § 9. We consider the following situation:

V	an affine variety defined over K
$x = (x_1, \ldots, x_n)$	a generic point of V over K
$K[x]$	its coordinate ring; the elements in $K[x]$ are regarded as polynomial functions defined on V
$F = K(x)$	the field of rational functions on V over K
$u = (u_1, \ldots, u_m)$	a finite family of elements in $K[x]$
$V(\hat{K})$	the space of $\hat{K}$ - rational points of V
$V_u(\hat{K})$	the subset of $V(\hat{K})$ containing those points $a \in V(\hat{K})$ which satisfy the condition $u_i \in \hat{P} = \hat{K}^2 \cup t\hat{K}^2$ for $i = 1, \ldots, m$
$J_u(K[x])$	the semiring in $K[x]$ generated by the subset $\{t, 1-t\} \cup K[x]^2 \cup \delta(K[x]) \cup P \cup u$
a	an ideal in $K[x]$
$r_u(a)$	the $J_u(K[x])$ - radical of a
$V_u(\hat{K})(a)$	the subset of $V_u(\hat{K})$ containing those points $b \in V_u(\hat{K})$ which satisfy the condition $f(b) = 0$ for every $f \in a$.
$I(V_u(\hat{K})(a))$	the ideal in $K[x]$ consisting of those elements $f \in K[x]$ which satisfy the condition $f(b) = 0$ for every $b \in V_u(\hat{K})(a)$.

The main result of this section is the following.

(10.1) Theorem. If the variety V is nonsingular then $I(V_u(\hat{K})(a)) = r_u(a)$.

Clearly, the nonsingularity condition is satisfied if V is the full affine space. In this particular case we obtain Jacob's Nullstellensatz [9] Theorem 2.

Let us consider the restricted Riemann space $\hat{S}$ of $F/(K,P)$ and the basic subset $\hat{S}^x$. If $Q \in \hat{S}^x$ then $x.Q = (x_1Q, \ldots, x_nQ)$ is a specialization of x over $\hat{K}$ and hence $x.Q$

is a $\hat{K}$ - rational point of the variety V. If we attach to each $Q \in \hat{S}^x$ the point $x.Q \in V(\hat{K})$ we obtain the projection map $\hat{S}^x \longrightarrow V(\hat{K})$. If $Q \in \hat{S}^x_u$ then Q satisfies the conditions $u_i Q \in \hat{P}$ for $i = 1,\dots,m$. Considering $u_i = u_i(x)$ as polynomial expressions in $K[x]$ we observe that $u_i\, Q = u_i(x.Q)$ and hence $x.Q \in V_u(\hat{K})$. Conversely, if $x.Q \in V_u(\hat{K})$ it follows in the same way that $Q \in \hat{S}^x_u$. Thus we have:

(10.2) Lemma. $\hat{S}^x_u$ is the inverse image of $V_u(\hat{K})$ with respect to the projection map $\hat{S}^x \longrightarrow V(\hat{K})$.

According to [10] Corollary A.2, we also have:

(10.3) Lemma. The image of $\hat{S}^x_u$ with respect to the projection map contains at least all simple points in $V_u(\hat{K})$. In particular, if the variety V is nonsingular then the projection map $\hat{S}^x_u \longrightarrow V_u(\hat{K})$ is surjective.

As a consequence of this lemma we obtain the following criterion for $V_u\,(\hat{K})$ to contain a simple point. This criterion is of birational nature, reffering only to the function field F and not to the particular variety V.

(10.4) Theorem. Suppose that the multiplicative monoid generated by u is a subgroup of F°. Then the necessary and sufficient condition for $V_u(\hat{K})$ to contain a simple point is that the function field F is formally t-adic over $P \cup u$.

Proof. By (10.3), $V_u(\hat{K})$ contains a simple point iff $\hat{S}^x_u$ is non-empty. By (8.5), $\hat{S}^x_u$ is non-empty iff F is formally t-adic over $P \cup u$.

Q.E.D.

Proof of (10.1)

Since V is nonsingular it follows that $\hat{S}^x_u(aH^x_u)$ is the inverse image of $V_u\,(\hat{K})(a)$, and $V_u\,(\hat{K})(a)$ is the image of $\hat{S}^x_u\,(a\,H^x_u)$ with respect to the projection map $\hat{S}^x_u \longrightarrow V_u\,(\hat{K})$.

It follows that $I(V_u(\hat{K})(a) = K[x] \cap I_{H^x_u}(\hat{S}^x_u\,(a\,H^x_u))$.

By (9.2), $I_{H^x_u}\,(\hat{S}^x_u\,(a\,H^x_u))$ equals the $J_u\,(H^x_u)$ - radical $r_u\,(a\,H^x_u)$ of the ideal $a\,H^x_u$ in H^x_u and hence $I\,(V_u(\hat{K})(a)) = K[x] \cap r_u\,(a\,H^x_u)$. It remains to show that the contraction of the ideal $r_u(a\,H_u^x)$ on $K[x]$ is the $J_u\,(K[x])$ - radical $r_u(a)$ of the ideal a. The inclusion $r_u(a) \subset K[x] \cap r_u\,(a\,H_u^x)$ follows easily from definitions. Conversely, let $g \in K[x] \setminus r_u(a)$. We have to show that $g \notin r_u\,(a\,H_u^x)$. By [21] Proposition 2, $r_u(a)$ is the intersection of all prime $J_u\,(K[x])$ - radical ideals in $K[x]$ containing a. It follows that there is a prime $J_u(K[x])$ - radical ideal p containing a such that $g \notin p$. Let $L = K\,(x \bmod p)$ be the field of fractions of the factor ring $K[x]/_p$. Then the field extension L of K is formally t-adic over $P \cup u \bmod p$, and x mod p is an L - rational point of the variety V. Since by hypothesis V is nonsingular, the point x mod p is simple on V. By [10] Corollary A 2, the specialization $x \longrightarrow x \bmod p$ can be extended to an L-rational place Q of F/K. It follows that the center $m_Q \cap K[x]$ of the place Q on $K[x]$ is the prime ideal p, and hence $g \notin m_Q$. On the other hand, $Q \in S_u^x$ and the center $m_Q \cap H_u^x$ of Q on H_u^x is a prime ideal containing the ideal a H_u^x. Since by (7.1), $m_Q \cap H_u^x$ is a $J_u(H_u^x)$ - radical ideal, it follows by [21] Proposition 2 that $g \notin r_u(a\,H_u^x)$. Thus we succeeded to show that $r_u(a) = K[x] \cap r_u\,(a\,H_u^x)$, as contended. Q.E.D.

Remark. (10.1) is an analogue of [10] Theorem 1.2 for p-adically closed fields and of [21] Theorem 1 and [4] Theorem 1.1. for ordered fields.

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