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EXTENSION OF PLACES AND CONTRACTION PROPERTIES FOR FUNCTION FIELDS OVER p-ADICALLY CLOSED FIELDS

By Serban A. Basarab[x)]

Abstract: Let N/F be a field extension over a base field K. We investigate some situations when certain objects attached to F/K (places of F/K, the Kochen ring of F/K and the holomorphy ring of F/K with K p-adically closed) can be obtained by contraction from the corresponding objects attached to N/K. As applications we prove the existence of some bounds in the theory of fields and formally p-adic fields.

Contents

0. Introduction

Let K be a base field and N/F be a field extension over K. We are interested to investigate situations when the following questions admit affirmative answers.

I) Let P be a place of F/K which is rational over a field extension L of K. Can P be extended to a place Q of N/K which is rational over the same field L?

By Chevalley's place extension theorem this is always possible if L is algebraically closed. There exists a rich

x) The work was prepared while the author was a Humboldt fellow at the University Heidelberg.

mathematical literature concerning this problem and its various aspects. We shall show in Section 2 that the afore mentioned question has a partially affirmative answer if the base field K is an internal field in an enlargement in Robinson' sense, F is a finitely generated field extension of K, N is an internal field extension of K naturally attached to F/K as in Section 1, and P is a place of F/K which is rational over an internal field extension L of K.

II) Let K be a p-adically closed field (see for definitions Section 3) and N/F be a field extension over K. If N is formally p-adic then F is formally p-adic too. We are interested in situations when the converse is also true.

III) Consider the same data as in Problem II. Let $\widetilde{R}(F)$ and $\widetilde{R}(N)$ denote the corresponding Kochen rings of the fields F/K and N/K. Let $\widetilde{r}(F)$ and $\widetilde{r}(N)$ be the corresponding Kochen ideals in $\widetilde{R}(F)$ and $\widetilde{R}(N)$. By definition $\widetilde{R}(F) \subset \widetilde{R}(N) \cap F$ and $\widetilde{r}(F) \subset \widetilde{r}(N) \cap F$. We look for situations when the previous inclusions become equalities.

IV) With the same data as above let H(F) and H(N) denote the corresponding holomorphy rings of F/K and N/K. We are interested in situations where H(F) is the contraction of H(N), i.e. $H(F) = H(N) \cap F$.

We shall prove in Section 4 and 5 that the problems II - IV admit affirmative answers if K is an internal p-adically closed field in a suitable enlargement, F is finitely generated over K and N is an internal field extension of K naturally attached to F, as in Section 1.

As applications, we derive in Sections 6 and 7 some results concerning the existence of bounds in the theory of fields and formally p-adic fields.

In a first version of this paper, the stress was laid upon the existence of bounds in the theory of formally p-adic fields. Prof.Dr.Peter Roquette suggested to change the point of view and accentuate the structural aspects as in the formulations of problems I) - IV). It is a great pleasure for me to express here my warmest thanks for his advices and encouragement in the preparation of this work.

1. Finitely generated fied extensions over an internal field

Let us consider a mathematical structure M containing a non - empty family $\{K_i\}_{i\in I}$ of fields, the polynomial rings $K_i[X_1, \ldots X_n]$ for $i\in I$ and arbitrary n, the finitely generated field extensions of K_i for $i\in I$, the set $\mathbb{N}$ of natural numbers, e.t.c., and take an enlargement $^*M$, in Robinson's sense, of M.

Let K denote an internal field in the enlargement $^*M$, considered fixed in the following. The main aim of this section is to construct a functor with good properties from the category of finitely generated field extensions of K into the category of finitely *generated internal field extensions of K.

As a first step we construct a functor from the category of finitely generated K-algebras into the category of finitely *generated internal K-algebras.

Theorem 1.1. Every finitely generated K-algebra A can be embedded in a functorial manner into a finitely *generated internal K-algebra $\hat{A}$ such that the following conditions are satisfied:

1) $\hat{A}$ is *generated by A;

2) $\hat{A}$ is a faithfully flat A-module.

3) If $x = (x_1, \ldots x_n)$ is an arbitrary family of generators of the K-algebra A and a denotes the kernel of the canonical K - morpfism:
$K[X] = K[X_1, \ldots, X_n] \to A : X \mapsto x$, then the kernel of the canonical internal K - morphism $K^*[X] \to \hat{A} : X \mapsto x$ is a $K^*[X]$, i.e. $\hat{A} \cong K^*[X]/_{aK^*[X]} \cong A \otimes_{K[X]} K^*[X]$. In particular $\hat{A}$ is uniquely determined up to an internal isomorhism of K-algebras.

4) A has no nilpotent elements iff $\hat{A}$ has no nilpotent elements.

5) A is an integral domain iff $\hat{A}$ is an integral domain.

Proof. Let C denote the category whose objects are the K-algebras of the standard type $K[X]/_a$, where $X = (X_1, \dots, X_n)$, $n \in \mathbb{N}$, and a is an ideal in $K[X]$, and whose morphisms are the morphisms of K-algebras. Let $\hat{C}$ denote the internal category whose objects are the internal K-algebras of the type $K^*[X]/_{a'}$, where $X = (X_1, \dots, X_n)$, $n \in {}^*\mathbb{N}$, $K^*[X]$ is the internal ring of internal polynomials in X with coefficients in K, a' is an internal ideal in $K^*[X]$, and whose morphisms are the internal morphisms of K-algebras.

Let us show that C can be identified with a non-full subcategory of $\hat{C}$. If $A = K[X]/_a$ is an arbitrary object in C, let $\hat{A}$ denote the internal K-algebra $K^*[X]/_{a'}$, where a' is internal ideal in $K^*[X]$ generated by a. Since $K[X] = K[X_1, \dots, X_n]$ is Noetherian, a is finitely generated, and hence a' is the ideal $a K^*[X]$ generated by a and $\hat{A} \cong A \otimes_{K[X]} K^*[X]$. If $A = K[X]/_a$, $B = K[Y]/_b$, $X = (X_1, \dots, X_n)$, $Y = (Y_1, \dots, Y_m)$, and $f: A \to B$ is a morphism in C then f induces a morphism $\hat{f}: \hat{A} \to \hat{B}$ in $\hat{C}$ and we have the commutative diagram of K-algebras:

$$\begin{array}{ccc} \hat{A} & \xrightarrow{\ \hat{f}\ } & \hat{B} \\ {\scriptstyle i_A}\uparrow & & \uparrow{\scriptstyle i_B} \\ A & \xrightarrow{\ f\ } & B \end{array}$$

where i_A and i_B are canonic. Since $K^*[X]$ is a faithfully flat $K[X]$ - module [7] 2.3., it follows by [4] I, § 3,3, that $\hat{A}$ is a faith - fully flat A - module, in particular i_A and i_B are injective and hence the map $C(A, B) \to \hat{C}(\hat{A}, \hat{B}): f \mapsto \hat{f}$ is injective. It is immediate that the maps $A \mapsto \hat{A}$ and $f \mapsto \hat{f}$ define a functor $\hat{}: C \to \hat{C}$. Since for arbitrary objects A and B in C, the map $C(A, B) \to \hat{C}(\hat{A}, \hat{B}): f \mapsto \hat{f}$ is injective we conclude that the functor $\hat{}: C \to \hat{C}$ is an embedding. By enlargement principle [19] the image of C is a non-full subcategory of $\hat{C}$.

Since $\hat{}: C \to \hat{C}$ is a functor it follows that if A and B are arbitrary objects in C and $f: A \to B$ is an isomorphism in C, then $\hat{f}: \hat{A} \to \hat{B}$ is an isomorphism in $\hat{C}$. So, if A is a finitely generated K-algebra, we can choose an arbitrary family $x = (x_1, \ldots, x_n)$ of generators of A over K and define $\hat{A}$ to be the factor K-algebra $K^*[X]/_{aK^*[X]}$, where a is the kernel of the canonical K - morphism $K[X] \to A: X \mapsto x$. $\hat{A}$ is a faithfully flat A-module and is uniquely determined up to an internal isomorphism of K-algebras. Thus the statements 1) - 3) are proved. It remains to prove 4) and 5).

Let $A = K[X]/_a$, $X = (X_1, \ldots, X_n)$. By [7] 2.7.,

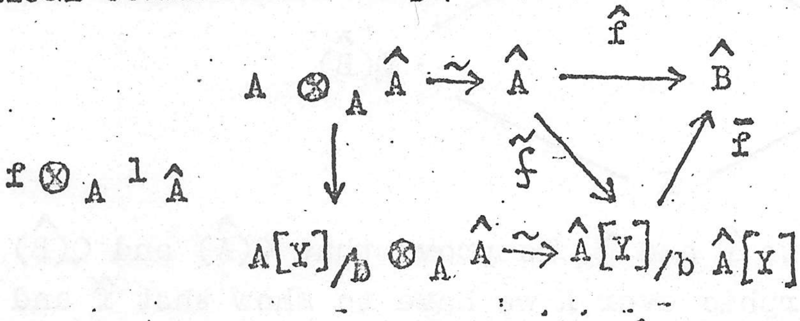
$$\sqrt{aK^*[X]} = {}^*\!\!\sqrt{aK^*[X]} = \{P \in K^*[X] \mid (\exists \omega \in {}^*N) P^{\omega} \in aK^*[X]\} = \sqrt{aK^*[X]}$$

It follows that a is a radical ideal in $K[X]$ iff $aK^*[X]$ is a radical ideal in $K^*[X]$, therefore A has no nilpotent elements iff $\hat{A}$ has no nilpotent elements, as contended.

By [7] 2.6., a is prime in $K[X]$ iff $aK^*[X]$ is prime in $K^*[X]$, hence A is an integral domain iff $\hat{A}$ is an integral domain. Q.E.D.

<u>Proposition 1.2.</u> Let $f: A \to B$ be a morphism between finitely generated K - algebras and assume that A is an integral domain. Let $\hat{f}: \hat{A} \to \hat{B}$ denote the corresponding internal morphism of K-algebras. Then f is injective iff $\hat{f}$ is injective.

<u>Proof.</u> Let $y = (y_1, \ldots, y_m)$ be a family of generators of the A - algebra B and let b denote the kernel of the canonical A-morphism $A[Y] \to B: Y \mapsto y$. Then $B \cong A[Y]/b$ and it is easy to see that $\hat{B} \cong \hat{A}^*[Y]/_{b\hat{A}^*[Y]}$. Consider the canonical commutative diagram of $\hat{A}$ - algebras:

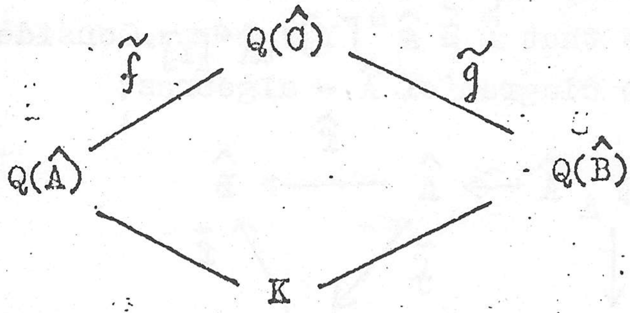


Since, by Theorem 1.1., $\hat{A}$ is a faithfully flat A-module, it follows that f is injective iff $\tilde{f}$ is injective. We have to shown that the injectivity of $\tilde{f}$ implies the injectivity of $\hat{f}$. The opposite implication is trivial.

Assume that $\hat{f}$ is not injective. Since A is an integral domain it follows by Theorem 1.1. that $\hat{A}$ is an integral domain. Let F denote the field of quotients of $\hat{A}$. In particular F is an internal field. As $\hat{f}$ is not injective it follows that $b\,F^{*}[Y] = F^{*}[Y]$. Since F is an internal field, $F^{*}[Y]$ is a faithfully flat $F[Y]$ - module, and hence $bF[Y] = b\,F^{*}[Y] \cap F[Y] = F[Y]$. We conclude that $\tilde{f}$ is not injective. Q.E.D.

Corollary. Let A and B be finitely generated K-algebras which are integral domains and whose fields of quotients Q(A) and Q(B) are isomorphic over K. Then the corresponding internal K-algebras $\hat{A}$ and $\hat{B}$ are integral domains and the isomorphism between Q(A) and Q(B) can be lifted to an internal K-isomorphism between the fields of quotients $Q(\hat{A})$ and $Q(\hat{B})$.

Proof. We may identify the K-algebras A and B with finitely generated subalgebras in their common field of quotients F. Thus $A = K[x_1, \ldots, x_n]$, $B = K[y_1, \ldots, y_m]$ for $x_1, \ldots, x_n, y_1, \ldots, y_m \in F$, and $Q(A) = Q(B) = F$. Let $C = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be the subalgebra generated by $A \cup B$. We have $Q(C) = F$. The inclusions $f: A \to C$ and $g: B \to C$ induce the corresponding internal morphisms of K - algebras $\hat{f}: \hat{A} \to \hat{C}$ and $\hat{g}: \hat{B} \to \hat{C}$. By Theorem 1.1., $\hat{A}$, $\hat{B}$ and $\hat{C}$ are integral domains, and by Proposition 1.2., $\hat{f}$ and $\hat{g}$ are injective, and hence we have the commutative diagram of internal field extensions:



with $\hat{f}$ and $\hat{g}$ lifted to $\tilde{f}$ and $\tilde{g}$. To prove that $Q(\hat{A})$ and $Q(\hat{B})$ are internally isomorphic over K we have to show that $\tilde{f}$ and $\tilde{g}$ are isomorphisms. Since $Q(A) = Q(C) = F$ it follows that $Q(\hat{A})$ and $Q(\hat{C})$ are *generated by the same subfield F. As the image by $\tilde{f}$ of $Q(\hat{A})$ is an internal subfield of $Q(\hat{C})$ extending F we conclude that $\tilde{f}$ is an isomorphism. With the same argument it follows that $\tilde{g}$ is an isomorphism too. Q.E.D.

Thus we obtained an analogue of Theorem 1.1. for finitely generated field extensions of the internal field K.

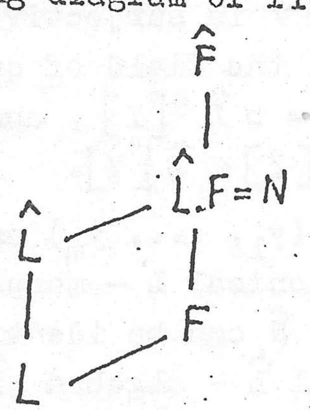
Theorem 1.3. Each finitely generated field extension F of the internal field K can be embedded in a functorial way into a finitely *generated internal field extension $\hat{F}$ of K such that the following conditions are satisfied:

1) $\hat{F}$ is *generated by F.

2) If $x = (x_1, \ldots, x_n)$ is an arbitrary family of generators of F/K and a denotes the kernel of the canonical K-morphism $K[X] \to F$: $X \mapsto x$, then the kernel of the canonical internal K-morphism $K^*[X] \to \hat{F}$: $X \mapsto x$ is a $K^*[X]$, i.e. there is a canonical internal K-isomorphism from the field of quotients of the factor ring $K^*[X] / aK^*[X]$ onto $\hat{F}$.

Thus $\hat{F}$ is uniquely determined up to an internal K-isomorphism of fields. The map $F \mapsto \hat{F}$ induces an embedding of the category of finitely generated field extensions of K into a non-full external subcategory of the internal category of *finitely* generated internal field extensions of K.

Now let us investigate the properties of the field extension $\hat{F}/F$. Let F/L be a field extension over K where L and F are finitely generated over K. The extension F/L induces naturally an internal field extension $\hat{F}/\hat{L}$ over K. Thus we have the following diagram of field extensions of K:



On the other hand, since $\hat{L}$ is an internal field in the same enlargement $^*M$ and $N = \hat{L}.F$ is finitely generated over $\hat{L}$, we can embedd as above the field extension $N/\hat{L}$ into a corresponding internal field extension $\hat{N}/\hat{L}$.

Proposition 1.4. a). $\hat{F}$ and $\hat{N}$ are internally isomorphic over $\hat{L}$.

b) $\hat{L}$ and F are linearly disjoint over L.

<u>Proof</u>. a) Let us show that $\hat{F}$ satisfies the necessary conditions from Theorem 1.3. to be identified with the internal field extension $\hat{N}$ of $\hat{L}$ attached to the finitely generated field extension N/L. Since $\hat{F}$ is *generated by F and $F \subset N$ it follows that $\hat{F}$ is *generated by N. As the construction of $\hat{N}$ does not depend on the choice of the generators of $N/\hat{L}$, it suffices to show that $\hat{F}$ satisfies condition 2) from Theorem 1.3. for a particular family of generators of $N/\hat{L}$. Assume that $F = L(y)$, where $y = (y_1, \ldots, y_m) \in F^m$ and hence $N = \hat{L}(y)$. We have to show that the kernel of the canonical internal $\hat{L}$ - morphism $\rho : \hat{L}^*[Y] \to \hat{F}$: $Y \mapsto y$ is generated by polynomials in $\hat{L}[Y]$. Let $L = K(x)$, where $x = (x_1, \ldots, x_n) \in L^n$. Let $A = K[x]$, and $\hat{A} = K^*[x]$ be the corresponding internal K - algebra attached to A. By definition, $\hat{L}$ is the field of quotients of $\hat{A}$. Consider the commutative diagram of internal K-algebras

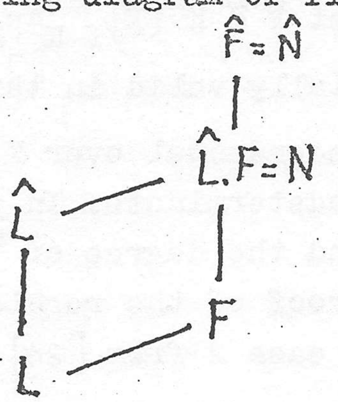
$$\begin{array}{ccccc} \hat{A}^*[Y] & \xrightarrow{\ i\ } & \hat{L}^*[Y] & \xrightarrow{\ \rho\ } & \hat{F} \\ \uparrow \nu & & & \nearrow \mu & \\ K^*[X,Y] & & & & \end{array}$$

where $\mu(X) = x$, $\mu(Y) = y$, $\nu(X) = x$, $\nu(Y) = Y$, and i is the canonical inclusion. Since, by definition of $\hat{F}$, $\operatorname{Ker}\mu = bK^*[X,Y]$ for some ideal b in $K[X, Y]$, and ν is surjective, it follows that $\operatorname{Ker}(\rho \circ i) = b\hat{A}^*[Y]$. As $\hat{L}$ is the field of quotients of $\hat{A}$, we have $\operatorname{Ker}\rho = \operatorname{Ker}(\rho \circ i)$. $\hat{L}^*[Y] = b\hat{L}^*[Y]$, and hence $\operatorname{Ker}\rho$ is generated by polynomials in $L[Y] \subset \hat{L}[Y]$.

b) Let $F = L(y)$ whit $y = (y_1, \ldots, y_m)$ and let $B = L[y]$. Denote by b the kernel of the canonical L - morphism $L[Y] \to F$: $Y \mapsto y$. As we showed at a), $\hat{F} = \hat{N}$ can be identified with the field of quotients of the internal $\hat{L}$ - algebra $\hat{L}^*[Y]/_{b\hat{L}^*[Y]}$, therefore, by definition of $\hat{N}$, $N = \hat{L}$. F can be identified with the field of quotients of the $\hat{L}$ - algebra $C = \hat{L}[Y]/_{b.\hat{L}[Y]} \cong B \otimes_L \hat{L}$. Since $C = \hat{L}[y]$ is the subring of N generated by $B \cup \hat{L}$ and $C \cong B \otimes_L \hat{L}$ it follows that the L - algebras B and $\hat{L}$ are linearly disjoint over L. As F is the field of quotients of B we conclude that F and $\hat{L}$ are linearly disjoint over L. Q.E.D.

Theorem 1.5. Let F be a finitely generated field extension of the internal field K and $\hat{F}$ be the coresponding internal field extension of K attached to F. Then the extension $\hat{F}/F$ is regular and the degree of transcendency of F/K equals the degree of *transcendency of $\hat{F}/K$.

Proof. We use induction with respect to the number n of generators of F/K, in order to reduce the proof to the case where F/K is generated by a single element. This reduction step is carried out as follows: Let L be an intermediate field between K and F and assume that F/L is generated by a single element, whereas L/K is generated by $n-1$ elements. Let us consider the following diagram of field extensions of K:



By Proposition 1.4., the internal field extension $\hat{N}$ of the internal field $\hat{L}$, attached to the finitely generated field extension $N/\hat{L}$ can be identified with $\hat{F}$. By induction $\hat{L}$ is regular over L and the degree of transcendency of L/K equals the degree of *transcendency of $\hat{L}/K$. By Proposition 1.4., F and $\hat{L}$ are linearly disjoint over L and hence the regularity of $\hat{L}/L$ implies the regularity of N/F. On the other hand, since linearly disjointness implies algebraic independence, it follows that the degrees of trancendency of F/L and $N/\hat{L}$ are equal.

Now, if we assume the theorem proved for one-generator extensions over an internal field, then we apply this to $N/\hat{L}$ and conclude that $\hat{N} = \hat{F}$ is regular over N and the degree of transcendency of $N/\hat{L}$ equals the degree of *transcendency of $\hat{F}/\hat{L}$. Since a tower of regular extensions is again regular, it follows that $\hat{F}/F$ is regular. On the other hand we have the equalities:

$$\operatorname{trdeg}(F/K) = \operatorname{trdeg}(L/K) + \operatorname{trdeg}(F/L) = {}^{*}\operatorname{trdeg}(\hat{L}/K) + \operatorname{trdeg}(N/\hat{L}) = {}^{*}\operatorname{trdeg}(\hat{L}/K) + {}^{*}\operatorname{trdeg}(\hat{F}/\hat{L}) = {}^{*}\operatorname{trdeg}(\hat{F}/K)$$

Thus it suffices to prove Theorem 1.5. under the additional hypothesis that $F = K(x)$ is generated by a single element x.

We distinquish two cases:

Case 1: x is algebraic over K. Let $f \in K[X]$ denote the irreducible polynomial of x over K. Then $F = K[x] \cong K[X]/_{fK[X]}$. The polynomial $f \in K[X]$ remains irreducible in $K^{*}[X]$. Indeed, if f admits a decomposition $f = g.h$ with g and in $K^{*}[X]$, then $\deg(g) \leqslant \deg(f)$ and $\deg(h) \leqslant \deg(f)$, hence g and h are contained in $K[X]$. We conclude that $\hat{F} \cong K^{*}[X]/_{f K^{*}[X]} \cong K[X]/_{fK[X]} \cong F$. Thus Theorem 1.5. is trivially valid in this case.

Case 2: x is transcendental over K. Then $F \cong K(X)$ and $\hat{F} \cong K^{*}(X)$ where X is an indeterminate. In particular the degree of transcendency of F/K and the degree of *transcendency of $\hat{F}/K$ are equal to 1. The proof of the regularity of $\hat{F}/F$ follows the line of proof for the case 2 from [24] Theorem 3.5. Q.E.D.

Remark. The regularity of the extension $\hat{F}/F$ in the particular case $F = K(X)$, with $X = (X_1, \ldots, X_n)$ and $n \in {}^{*}\mathbb{N}$, is announced in [7] 2.5. without proof.

2. Extension of places

Let K be an internal field as in Section 1. We show in this section that Problem I from Introduction has a partially affirmative answer if F is a finitely generated field extension of K and N is the internal field extension $\hat{F}$ of K attached to F as in Section 1.

Theorem 2.1. Let F be a finitely generated field extension of the internal field K and let $\hat{F}$ denote the internal field extension of K attached to F. Let P be an arbitrary place of F/K which is rational over an internal field extension L of K, i.e. the residue field FP is a subfield of L/K. Let $x = (x_1, \ldots, x_n)$ be a family of generators of F/K such that $K[x]$ is a subring of the valuation ring O_P of P. Assume that xP is a simple point on the affine model V of F/K whose generic point is x. Then there exists an internal place Q of $\hat{F}/K$ which is rational over L, i.e. $\hat{F}.Q \subset L$, and Q coincides with P on the subring $K[x]$.

Proof. Let a denote the kernel of the canonical K-morphism $K[X] \to F$: $X \mapsto x$. Then $K[x] \cong K[X]/_a$ is the coordinate ring of the affine variety V and $F = K(x)$ is the field of rational functions on V over K. Let $r_1, \ldots, r_s \in K[X]$ be some generators of the ideal a. We envisage V as being defined by the system of polynomial equations over K:

(2.1) $r_j(X_1, \ldots, X_n) = o$ for $j = 1, \ldots, s$

The internal extension $\hat{F}$ of K is, by definition, the field of quotients of the internal integral domain $K^*[x] \cong$ $= K^*[X]/_{aK^*[X]}$. Let $\hat{V}$ denote the internal affine variety over K whose *generic point is x. Then $K^*[x]$ is the internal coordinate ring of $\hat{V}$ and $\hat{F} = K^*(x)$ is the internal field of internal rational functions on $\hat{V}$ over K. We envisage $\hat{V}$ as being defined by the same system (2.1) of finitely many polynomial equations over K. Since, by Theorem 1.5., the degree of transcendency of F/K and the degree of *transcendency of $\hat{F}/K$ are equal, it follows that the dimension of V and the internal dimension of $\hat{V}$ are equal.

Now, the condition for a point to be simple on V can be expressed by saying that at least one of the minors of order n-dim (V) of the Jacobian matrix $\left(\frac{\partial r_j}{\partial X_i}\right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq s}}$ does not vanish at that point. Since V and $\hat{V}$ are defined by the same system (2.1) of polynomial equations over K and dim (V) = *dim($\hat{V}$), we conclude that the condition for an internal point to be simple on $\hat{V}$ coincides with the corresponding condition for V.

As, by hypothesis, xP is simple on V and xP is rational over the internal field L it follows that xP is a simple point on $\hat{V}$. By enlargement principle [19] and by [10] Corollary A.2, the internal specialization $x \longrightarrow xP$ can be extended to an internal place Q of $\hat{F}/K$ such that $\hat{F}.Q = K^*(xP) \subset L$. By construction Q coincides with P on $K[x]$ as contended. Q.E.D.

Corollary. With the same data as in Theorem 2.1., assume that the place P is rational over K, i.e. FP = K, and the point xP is simple on V. Then there is an internal place Q of $\hat{F}/K$ such that Q is rational over K, i.e. $\hat{F}.Q = K$, and coincides with P on K[x].

Theorem 2.1. can be improved if we assume that the finitely generated field extension F/K satisfies <u>Zariski's local uniformization theorem</u>: For each model V of F/K in the sense of algebraic geometry and for each place P of F/K which lies over the coordinate ring K[x] of V over K, there is a model W of F/K such that P lies over the coordinate ring K[y] of W over K, K [x] is a subring of K[y], and the center yP of P on W is a simple point.

It is known that Zariski's local uniformization theorem does hold over an arbitrary field K of characteristic zero [28] and also for function fields F/K of dimension ≤ 3 without any restriction concerning the characteristic. We shall prove in Section 6 the existence of some bounds related to Zariski's theorem.

<u>Theorem 2.2.</u> Let F be a finitely generated field extension of the internal field K such that F/K satisfies Zariski's local uniformization theorem. Let P be a place of F/K which is rational over an internal field extension L of K, and A=K [x] be a finitely generated subalgebra contained in the valuation rind $\mathcal{O}_P$ of P. Then there is an internal place Q of $\hat{F}$/K which is rational over L and coincides with P on A.

<u>Corollary</u>. With the some data and hypotheses as in Theorem 2.2., assume that P is rational over K, i.e. F.P.=K. Then there is an internal place Q of $\hat{F}$/K which is rational over K and coincides with P on A.

3. <u>Background from the theory of formally p-adic fields</u>

The theory of formally p-adic fields was initiated by Kochen in [11], in a complete analogy to the classical theory of formally real fields. Important developments of this theory were achieved by Roquette in [20]- [23] and, together with Jarden in [10], where a Nullstellensatz over p-adically closed fields is proved. Amajor role in the proof of the results of [10] is played by the model theory of Henselian valued fields initiated by the wellknown series of papers of Ax and Kochen [1], and Ershov [8].

The possibility to extend the framework of the theory of formally p-adic fields is suggested in [11] as well as in

[10]. An extension of the theory was fulfilled by Transier [26], [27]. A different framework is developed by the author in [2], where the principal results of [10] are proved in a more general context and some examples and applications are discussed.

Now let us give the main definitions and results from [2] which are useful in the rest of this work.

Let k be a fixed field equipped with a non-trivial place p_o and the corresponding valuation v_o. Let L, L_o and $\tilde{L}$ denote the following first order languages:

L the language of fields extended with individual contants naming the elements of the residue field k_{v_o}.

L_o the language of ordered groups extended with individual constants naming the elements of the value group $v_o(k)$.

$\tilde{L}$ the language of valued fields extended with individual constants naming the elements of k.

Assume that there are given a theory T in the language L, and a theory T_o in the language L_o subject to the conditions:

a) The models of T are field extensions of k_{v_o}, and k_{v_o} is a model of T.

b) T is $\forall$ - axiomatizable, i.e. if ℓ is an arbitrary model of T then every intermediate field between k_{v_o} and ℓ is a model of T too.

a') The models of T_o are Abelian linearly ordered groups extending $v_o(k)$, and $v_o(k)$ is a model of T_o.

b') T_o is $\forall$ - axiomatizable, i.e. if H is an arbitrary model of T_o then each intermediate ordered group between $v_o(k)$ and $\hat{H}$ is a model of T_o too.

Let W denote the theory in the language $\tilde{L}$ having as models the valued fields (K, v) subject to the conditions: v extends v_o, the residue field K_v is a model of T, and the value group v(K) is a model of T_o. Thus, the theory W is $\forall$ - axiomatizable and the base field (k, v_o) is a prime model of W.

Let Λ and Γ be arbitrary sets of rational functions with coefficients in k in countably many indeterminates. We say that the ordered pair (Λ, Γ) satisfies condition (c_1) if for every valued field (K, v) extending (k, v_o) the following statements are equivalent:

1) (K, v) is a model of W.

2) a) For every $f(z_1, \ldots, z_n) \in \Lambda$, and for every $(a_1, \ldots, a_n) \in K^n$, $v(f(a_1, \ldots, a_n)) \geqslant o$ if f is defined in $(a_1, \ldots, a_n)$, and

b) for every $f(z_1, \ldots, z_n) \in \Gamma$, and for every $(a_1, \ldots, a_n) \in K^n$, $v(f(a_1, \ldots, a_n)) > o$ if f is defined in $(a_1, \ldots, a_n)$.

We assume in the following that there is at least one ordered pair (Λ, Γ) satisfying (c_1). In other words, W admits a system of axioms of the special form afore described.

Now let us consider an arbitrary model (K, v) of W, and let p be the corresponding place of K. Let F denote an aritraty field extension of K and let u and u' be arbitrary subsets of F.

<u>Definition</u> F is called formally p-adic if there exists a valuation w of F such that (F, w) is a model of W extending (K, v). F is called <u>formally p-adic over</u> (u, u') if the exists a valuation w of F such that (F, w) is a model of W, $u \subset \mathcal{O}_w$ and $u' \subset m_w$.

If F is formally p-adic over (u, u') then each intermediate field betwen $K(u, u')$ and F is formally p-adic over (u, u'). In particular, if F is formally p-adic then every intermediate field betwen K and F is formally p-adic too.

<u>The Kochen ring</u> $\tilde{R}_{u,u'}(F)$ <u>of</u> F <u>over</u> (u,u') is defined as the intersection of the valuation rings $\mathcal{O}_w$ where w ranges over the set of valuations of F extending v such that (F,w) is a model of W, $u \subset \mathcal{O}_w$ and $u' \subset m_w$. F is formally p-adic over (u, u') iff $\tilde{R}_{u,u'}(F) \neq F$.

<u>The Kochen ideal</u> $\tilde{r}_{u,u'}(F)$ of $\tilde{R}_{u,u'}(F)$ is defined as the intersection of the maximal ideals m_w of the corresponding valuation rings $\mathcal{O}_w$ afore mentioned.

In particular, for $u \subset \mathcal{O}_v$ and $u' \subset m_v$, we obtain the (absolute) Kochen ring $\tilde{R}(F)$ of F and the Kochen ideal $\tilde{r}(F)$ in

$\tilde{R}(F)$.

<u>The Riemann space</u> S(F) of F/K is defined as the set of all places P of F subject to the following conditions:

1) P is trivial on K; im particular the residue field FP is an extension of K.

2) The residue field F.P. is formally p-adic.

If u, u' and x are arbitrary subsets of F we denote by $S^x_{u,u'}(F)$ the subset of S(F) containing all places P satisfying the conditions:

1) The elements of u, u' and x are holomorphic in P, i.e. $u \cup u' \cup x \subset \mathcal{O}_P$.

2) The residue field F.P is formally p-adic over (u.P, u'.P).

<u>The</u> (absolute) <u>holomorphy ring</u> H(F) of F is defined as the holomorphy ring of the set S(F), i.e. the intersection of all valuation rings $\mathcal{O}_P$ of F for $P \in S(F)$.

If u, u' and x are arbitrary subsets of F, <u>the holomorphy ring</u> $H^x_{u,u'}(F)$ <u>of</u> F <u>over</u> (u, u', x) is defined as the holomorphy ring of the set $S^x_{u,u'}(F)$, i.e. the intersection of all valuation rings $\mathcal{O}_P$ of F where P ranges over the set $S^x_{u,u'}(F)$.

We assume in the rest of this paper that the following model theoretic condition is satisfied:

(c_4) The theory W is companionable, i.e. there exists a theory $\tilde{W}$ in the language $\tilde{L}$ such that the following conditions are satisfied:

1) $\tilde{W}$ extends W.

2) $\tilde{W}$ is model-complete.

3) Each model of W can be embedded in some model of $\tilde{W}$.

The theory $\tilde{W}$, supposed to exist, is uniquely determined by the conditions 1) - 3) ([5] III Theorem 18), and is called <u>the model-companion of</u> W.

It follows that the models of $\tilde{W}$ are algebaically complete, i.e. without proper algebraic immediate extensions, and hence, they are Henselian valued fields.

A model (K, v, p) of W is called p-adically closed if (K, v) is a model of $\widetilde{W}$.

Now we mention the principal results from [2] which are useful in Sections 4, 5 and 7.

A first result givesa description of the Kochen ring in terms of integral definite functions. In u, u' and x are arbitrary subsets of F we denote by $\widetilde{S}^x_{u,u'}(F)$ the subset of $S^x_{u,u'}(F)$ containing the places P of F/K satisfying the conditions: $FP = K$, $u \cup u' \cup x \subset \mathcal{O}_p$, $uP \subset \mathcal{O}_v$ and $u'P \subset m_v$. An element $z \in F$ is called <u>integral definite on</u> $\widetilde{S}^x_{u,u'}(F)$ if $zP \in \mathcal{O}_v$ for each $P \in \widetilde{S}^x_{u,u'}(F)$, and <u>strictly integral definite on</u> $\widetilde{S}^x_{u,u'}(F)$ if $zP \in m_v$ for each $P \in \widetilde{S}^x_{u,u'}(F)$. Denote by $\widetilde{I}^x_{u,u'}(F)$ the ring of integral definite functions on $\widetilde{S}^x_{u,u'}(F)$, and by $\widetilde{i}^x_{u,u'}(F)$ the ideal of strictly integral definite functions on $\widetilde{S}^x_{u,u'}(F)$. Then we have the following result:

<u>Theorem 3.1</u>. Assume that (c_4) is satisfied and let (K, v, p) be p-adically closed. Let F be a finitely generated field extension of K and u and u' be finite subsets of F. Then $\widetilde{R}_{u,u'}(F) = \widetilde{I}^x_{u,u'}(F)$ and $\widetilde{r}^x_{u,u'}(F) = \widetilde{i}^x_{u,u'}(F)$ for each finite subset x of F.

For proof see [2] Theorem 7.1. For formally p-adic fields in Kochen-Roquette sense see [11] Theorem 2 and [10] Theorem 7.2.

A second result is a Nullstellensatz for Kochen rings. With the same data as in Theorem 3.1 let us consider the representation μ of the factor ring $A(F) = \widetilde{R}_{u,u'}(F) / \widetilde{r}_{u,u'}(F)$ into the ring J of functions on $\widetilde{S}^x_{u,u'}(F)$ with values in the residue field K_v: if $z \in \widetilde{R}_{u,u'}(F)$ then $zP \in \mathcal{O}_v$ for each $P \in \widetilde{S}^x_{u,u'}(F)$, and z determines a function from $\widetilde{S}^x_{u,u'}(F)$ to K_v. By Theorem 3.1, the function induced by z is the zero map iff $z \in \widetilde{r}_{u,u'}(F)$, so we have a canonical representation $\mu : A(F) \to J$. Now, if a is an ideal in A(F), let $\widetilde{S}^x_a$ denote the subset of $\widetilde{S}^x_{u,u'}(F)$ containing the places P subject to $\mu(z)(P)=0$ for each $z \in a$, and j_a denote the ideal in J containing the functions vanishing on $\widetilde{S}^x_a$. Then we have.

<u>Theorem 3.2.</u> Assume that the hypotheses of Theorem 3.1

are satisfied, and in addition, the following condition is verified:

(c_2) There exists a non-constant monic polynominal $g \in k_{v_0}[X]$ such that $T \vdash (\forall x) g(x) \neq 0$, i.e. g has no root in any model of T.

Let a be a finitely generated ideal in A(F). Then the inverse image $\mu^{-1}(j_a)$ of the ideal j_a equals the radical $\sqrt{a}$ of a.

For proof see [2] Theorem 7.3.

Another result we need is the following one giving a description of the holomorphy rings.

Theorem 3.3. Let (K, v, p) be a p-adically closed field and F be a finitely generated field extension of K. Assume that (c_2) is satisfied and F/K satisfies Zariski's local uniformization theorem. Let u, u' and x be finite subsets of F. Then $H^x_{u,u'}(F) = \widetilde{R}_{u,u'}(F).\mathbb{Z}[x] = \widetilde{H}^x_{u,u'}(F)$, where $\widetilde{H}^x_{u,u'}(F) = \{z \in F \mid zP \neq \infty \text{ for each } P \in \widetilde{S}^x_{u,u'}(F)\}$ is the ring of holomorphic functions on $\widetilde{S}^x_{u,u'}(F)$.

For proof see [2] Theorem 7.5. For formally p-adic fields in Kochen-Roquette sense see [10] Theorem 2.1.

The following result is a Nullstellensatz for holomorphy rings. For the formally p-adic fields in Kochen-Roquette sense see [11] Theorem 3 and [10] Theorem 2.2.

Theorem 3.4. Assume the same data and hypotheses as in Theorem 3.3. Let a be a finitely generated ideal in $H^x_{u,u'}(F)$. Then the radical $\sqrt{a}$ of a coincides with the set of the elements $h \in H^x_{u,u'}(F)$ vanishing at all common zeros $P \in \widetilde{S}^x_{u,u'}(F)$ of the ideal a.

For proof see [2] Theorem 7.6.

4. Contraction properties for Kochen rings

Let W and $\widetilde{W}$ be some theories as considered in Section 3. Consider a mathematical structure M containing a non-empty family $\{(K_i, v_i)\}_{i \in I}$ of models of $\widetilde{W}$, the polynomial rings $K_i[X_1, \dots, X_n]$ for $i \in I$ and arbitrary $n \in \mathbb{N}$, the finitely generated field extensions of K_i for $i \in I$, the set $\mathbb{N}$ of natural numbers, e.t.c., and take au enlargement $^*M$ of M.

By principles of permanence [19] there exist internal valued fields in the enlargement $^*M$ which are models of $\widetilde{W}$. Let (K, v) be such a field and let p denote the corresponding place on K. In other words (K, v, p) is an internal p-adically closed field.

Let F be an arbitrary finitely generated field extension of K and let $\hat{F} \supset F$ be the internal field extension of K attached to F as in Section 1. We show in this section that Problems II and III from Introduction as well as other related questions admit an affirmative answer for the extension $\hat{F}/F$ over the internal p-adically closed field (K, v, p).

Let $u = (u_1, \ldots, u_m)$ and $u' = (u'_1, \ldots, u'_{m'})$ be arbitrary finite families of elements in F. Denote by $\widetilde{R}_{u,u'}(F)$, respectively $\widetilde{R}_{u,u'}(\hat{F})$, the Kochen ring of F/K, respectively of $\hat{F}/K$, over (u,u'). Let $\widetilde{r}_{u,u'}(F)$, respectively $\widetilde{r}_{u,u'}(\hat{F})$, denote the Kochen ideal in $\widetilde{R}_{u,u'}(F)$, respectively in $\widetilde{R}_{u,u'}(\hat{F})$. By definition we have the inclusions: $\widetilde{R}_{u,u'}(F) \subset \widetilde{R}_{u,u'}(\hat{F}) \cap F$ and $\widetilde{r}_{u,u'}(F) \subset \widetilde{r}_{u,u'}(\hat{F}) \cap F$. We shall show that the opposite inclusions hold too.

Let $\hat{R}_{u,u'}(\hat{F})$, respectively $\hat{r}_{u,u'}(\hat{F})$, denote the internal Kochen ring of $\hat{F}/K$ over (u,u'), respectively the internal Kochen ideal in $\hat{R}_{u,u'}(\hat{F})$. $\hat{R}_{u,u'}(\hat{F})$ is an internal subring of $\hat{F}$ and coincides with the intersection of all internal valuation rings Q_w where w ranges over the internal set of internal valuations of $\hat{F}$ such that $(\hat{F}, w)$ is a model of W extending (K, v), $u \subset Q_w$ and $u' \subset m_w$. The internal ideal $\hat{r}_{u,u'}(\hat{F})$ in $\hat{R}_{u,u'}(\hat{F})$ equals the intersection of the maximal ideals m_w of the corresponding internal valuation rings Q_w afore mentioned. Thus we have the inclusions: $\widetilde{R}_{u,u'}(\hat{F}) \subset \hat{R}_{u,u'}(\hat{F})$ and $\widetilde{r}_{u,u'}(\hat{F}) \subset \hat{r}_{u,u'}(\hat{F})$.

<u>Theorem 4.1.</u> The following equalities hold:

$$\widetilde{R}_{u,u'}(F) = \widetilde{R}_{u,u'}(\hat{F}) \cap F = \hat{R}_{u,u'}(\hat{F}) \cap F$$

and

$$\tilde{r}_{u,u'}(F) = \tilde{r}_{u,u'}(\hat{F}) \cap F = \hat{r}_{u,u'}(\hat{F}) \cap F.$$

Proof. It suffices to prove that $\hat{R}_{u,u'}(\hat{F}) \cap F \subset \tilde{R}_{u,u'}(F)$ and $\hat{r}_{u,u'}(\hat{F}) \cap F \subset \tilde{r}_{u,u'}(F)$.

Let us show that the first inclusion holds. Let h be an arbitrary element in $\hat{R}_{u,u'}(\hat{F}) \cap F$. We have to show that h belongs to $\tilde{R}_{u,u'}(F)$.

By Theorem 3.1., the Kochen ring $\tilde{R}_{u,u'}(F)$ of F/K over (u,u') equals the ring $\tilde{I}^x_{u,u'}(F)$, defined in Section 3, where x is an arbitrary finite family of elements in F.

We choose in a convenient way a finite family x as follows. First let $y = (y_1, \ldots, y_k)$ be a family of generators of F/K such that h and the elements of the families u and u' appear among the elements of y. After suitable renumbering of the y_i we may assume that $u_i = y_i$ for $i = 1, \ldots, m$, $u'_i = y_{i+m}$ for $i = 1, \ldots, m'$, $h = y_{m+m'+1}$ and $k \geqslant m + m' + 1$. Let V denote the affine variety over K whose generic point is y. Thus K[y] is the coordinate ring of V over K and F = K(y) is the field of rational functions on V over K. We envisage V as being defined by a system of polynomial equations over K:

(4.1) $\qquad r_j(Y_1, \ldots Y_k) = o$ for $j = 1, \ldots, s$;

The point y is simple on V since y is generic on V over K. The condition for the point y to be simple on V can be expressed by saying that at least one of the minors of order k-dim (V) of the Jacobian matrix $\left(\frac{\partial r_j}{\partial Y_i}\right)_{\substack{1 \leq i \leq k \\ 1 \leq j \leq s}}$ does not vanish at y. Let $\Gamma \in K[Y]$ be some proper minor of the Jacobian matrix such that $\Gamma(y) \neq o$. Let $x = (x_1, \ldots, x_n)$ be a finite family of elements in F such that $\Gamma(y)^{-1}$ and the elements of the family y appear among the elements of x. Thus we may assume that $x_i = y_i$ for $i = 1, \ldots, k$, $x_{k+1} = \Gamma(y)^{-1}$ and $n \geqslant k+1$. We have to show that $hP \in Q_V$ for each place P belonging to the set $\tilde{S}^x_{u,u'}(F)$ defined in Section 3.

Let P be an arbitrary member of $\tilde{S}^x_{u,u'}(F)$. As $F.P = K$, $yP \in K^k$ and $\Gamma(yP) = \Gamma(y)P \neq o$, we conclude that yP is a simple point on V, rational over K. By Corollary to Theorem 2.1., there exists an internal place Q of $\hat{F}/K$ such that $\hat{F}.Q = K$ and Q coincides with P on $K[y]$. Since $u \cup u' \cup \{h\} \subset y$, it follows that $u_i Q = u_i P \in O_V$ for $i = 1, \ldots, m$, $u'_i Q = u'_i P \in m_V$ for $i = 1, \ldots, m'$, and $hQ = hP \neq \infty$. Thus Q belongs to the internal set $\hat{S}_{u,u'}(\hat{F})$ containing the internal places T of $\hat{F}/K$ subject to conditions: $\hat{F}.T = K$, $uT \subset O_Q$ and $u'T \subset m_Q$.

According to enlargement principle [19] and to Theorem 3.1., the interbal Kochen ring $\hat{R}_{u,u'}(\hat{F})$ of $\hat{F}/K$ over (u, u') coincides with the internal ring $\hat{I}_{u,u'}(\hat{F})$ of integral functions of $\hat{F}$ on the internal set of places $\hat{S}_{u,u'}(\hat{F})$ afore defined. Since, by hypothesis, $h \in \hat{R}_{u,u'}(\hat{F})$ and, by construction, $Q \in \hat{S}_{u,u'}(\hat{F})$, we conclude that $h.P = h.Q \in O_V$. Thus we proved that hP is contained in O_V for every place $P \in \tilde{S}^x_{u,u'}(F)$, i.e. $h \in \tilde{I}^x_{u,u'}(F)$. We conclude, by Theorem 3.1, that h belongs to $\tilde{R}_{u,u'}(F)$ as contended.

The inclusion $\hat{r}_{u,u'}(\hat{F}) \cap F \subset \tilde{r}_{u,u'}(F)$ follows in a similar way. Q.E.D.

Taking $u \subset O_V$ and $u' \subset m_V$, we obtain:

<u>Corollary to Theorem 4.1.</u> The following equalities hold:

$$\tilde{R}(F) = \tilde{R}(\hat{F}) \cap F = \hat{R}(\hat{F}) \cap F,$$

and

$$\tilde{r}(F) = \tilde{r}(\hat{F}) \cap F = \hat{r}(\hat{F}) \cap F.$$

Thus we proved that Problem III from Introduction admits a positive answer for the extension $\hat{F}/F$. The answer to Problem II from Introduction for the extension $\hat{F}/F$ is an immediate consequence of Theorem 4.1.

<u>Theorem 4.2.</u> F is formally p-adic over (u,u') iff $\hat{F}$ is formally p-adic over (u, u') iff $\hat{F}$ is *formally p-adic over (u,u').

<u>Proff.</u> The implications $\Leftarrow$ are trivial . Assume that F is formally p-adic over (u,u'). Then $\tilde{R}_{u,u'}(F) \neq F$, and, by Theorem 4.1, $\hat{R}_{u,u'}(\hat{F}) \neq \hat{F}$, i.e. $\hat{F}$ is *formally p-adic over (u,u')

Q.E.D.

Corollary to Theorem 4.2. F is formally p-adic iff $\hat{F}$ is formally p-adic iff $\hat{F}$ is *formally p-adic.

Noe let us investigate the relation between the radical ideal structure of the factor rings $A(F) = \tilde{R}_{u,u'}(F)/\tilde{r}_{u,u'}(F)$, $A(\hat{F}) = \tilde{R}_{u,u'}(\hat{F})/\tilde{r}_{u,u'}(\hat{F})$ and $\hat{A}(\hat{F}) = \hat{R}_{u,u'}(\hat{F})/\hat{r}_{u,u'}(\hat{F})$. The inclusions $\tilde{R}_{u,u'}(F) \subset \tilde{R}_{u,u'}(\hat{F}) \subset \hat{R}_{u,u'}(\hat{F})$ and $\tilde{r}_{u,u'}(F) \subset \tilde{r}_{u,u'}(\hat{F}) \subset \hat{r}_{u,u'}(\hat{F})$ induce the ring morphisms from the commutative diagram:

$$\begin{array}{ccc} A(\hat{F}) & \xrightarrow{\ \ell\ } & \hat{A}(\hat{F}) \\ i \uparrow & \nearrow j & \\ A(F) & & \end{array}$$

By Theorem 4.1, the morphisms i and j are injective. Thus we can identify A(F) with a common subring in $A(\hat{F})$ and $\hat{A}(\hat{F})$. Let a be am arbitrary ideal in A(F). Then we have the inclusions:

$$\sqrt{a} \subset \sqrt{aA(\hat{F})} \cap A(F) \subset \sqrt{a\hat{A}(\hat{F})} \cap A(F)$$

We want to know if the opposite inclusions hold too. Since the rings A(F), $A(\hat{F})$ and $\hat{A}(\hat{F})$ have not nilpotent elements, the affore considered inclusions become equalities in the particular case a = o.

Now let us assume that the theory W satisfies condition (c_2) from Section 3, and let a be a finitely generated ideal in A(F). Then we have:

Theorem 4.3. With the data and hypotheses as above, the following equalities are satisfied:

$$\sqrt{a} = \sqrt{aA(\hat{F})} \cap A(F) = \sqrt{a\hat{A}(\hat{F})} \cap A(F) = \sqrt[*]{a\hat{A}(\hat{F})} \cap A(F).$$

Remark. Since $\hat{A}(\hat{F})$ is an internal ring and a is finitely generated, the internal ideal *generated by a in $\hat{A}(\hat{F})$ coincides with the ideal $a\hat{A}(\hat{F})$ generated by a, and

$$\sqrt[*]{a\hat{A}(\hat{F})} = \{x \in \hat{A}(\hat{F}) \mid x^{\omega} \in a\hat{A}(\hat{F}) \text{ for some } \omega \in {}^*\mathbb{N}\}.$$

Proof. It suffices to show that $\sqrt[*]{a\hat{A}(\hat{F})} \cap A(F) \subset \sqrt{a}$. Let h be an arbitrary element in $\sqrt[*]{a\hat{A}(\hat{F})} \cap A(F)$. We have to show that $h \in \sqrt{a}$ '.

By Theorem 3.2., $\sqrt{a} = \mu^{-1}(j_a) = \{z \in A(F) \mid \mu(z)(P) = o$ for each $P \in \widetilde{S}^x_a\}$, where x is an arbitrary finite family of elements in F. Let $h' \in \widetilde{R}_{u,u'}(F)$ be such that $h' \bmod \widetilde{r}_{u,u'}(F) = h$. We have to show that $h'P \in m_v$ for each $P \in \widetilde{S}^x_a$.

We choose in a convenient way a finite family x as follows Let $y = (y_1, \ldots, y_k)$ be a family of generators of F/K such that h', the elements of the families u and u', and some representatives $t_1, \ldots, t_e$ in $\widetilde{R}_{u,u'}(F)$ for the generators of the ideal a appear aamong the elements of y. Let V denote the affine variety over $\mathbb{K}$ whose generic point is y. We proceed as in the proof of Theorem 4.1. Since y is generic, y is simple on V. Let $\Gamma \in K[Y]$ be some proper minor, of the Jacobian matrix such that $\Gamma(y) \neq o$. Let x be a finite family of elements in F such that $\Gamma(y)^{-1}$ and the elements of the family y appear among the elements of x. We have to show that $h'P \in m_v$ for each $P \in \widetilde{S}^x_a$.

Let P be an arbitrary member of $\widetilde{S}^x_a$. It follows that yP is a simple point on V and gence, by Corollary to Theorem 2.1., there is an internal place Q of $\hat{F}/K$ such that $\hat{F}.Q. = \mathbb{K}$ and Q coincides with P on $K[y]$. It follows that Q belongs to the internal set $\hat{S}_{a\hat{A}(\hat{F})}$ containing the internal places T of $\hat{F}/K$ subject to the conditions: $\hat{F}.T = K$, $uT \subset O_v$ and $zT \in m_v$ for each $z \in \hat{R}_{u,u'}(\hat{F})$ such that $z \bmod \hat{r}_{u,u'}(\hat{F}) \in a\hat{A}(\hat{F})$.

By enlargement principle and by Theorem 3.2., the internal ideal $\sqrt[*]{a\hat{A}(\hat{F})}$ coincides with the set of those elements $z \in \hat{A}(\hat{F})$ which satisfy the condition: $z'T \in m_v$ for each representative z' of z in $\hat{R}_{u,u'}(\hat{F})$ and for each $T \in \hat{S}_{a\hat{A}(\hat{F})}$. Since, by hypothesis, $h \in \sqrt[*]{a\hat{A}(\hat{F})}$, and, by construction, $Q \in \hat{S}_{a\hat{A}(\hat{F})}$, we conclude that $h'P = h'Q \in m_v$. So we proved that $h'P \in m_v$ for each $P \in \widetilde{S}^x_a$, and hence, by Theorem 3.2, $h \in \sqrt{a}$, as contended Q.E.D.

5. Contraction properties for holomorphy rings

Let us consider the framework from Section 4, and let (K, v, p) be an internal p-adically closed field, F be a finitely generated field extension of K, and $\hat{F} \supset F$ be the internal field extension of K attached to F as in Section 1. The *aim* of this section is to show that Problem IV from Introduction and another related question can be solved for the case of an extension $\hat{F}/F$ as above.

Let $u = (u_1, \ldots, u_m)$, $u' = (u'_1, \ldots, u'_{m'})$ and $x = (x_1, \ldots, x_n)$ be arbitrary finite families of elements in F. Denote by $H^x_{u,u'}(F)$, respectively by $H^x_{u,u'}(\hat{F})$, the holomorphy ring of F/K, respectively of $\hat{F}/K$, over (u,u',x). By definition, $H^x_{u,u'}(F) \subset H^x_{u,u'}(F) \cap F$. We show that the opposite inclusion holds too if certain additional conditions are satisfied.

Let $\hat{H}^x_{u,u'}(\hat{F})$ denote the internal holomorphy ring of $\hat{F}/K$ over (u, u', x). $\hat{H}^x_{u,u'}(\hat{F})$ coincides with the intersection of the valuation rings O_P of internal places P of $\hat{F}/K$ subject to: $u \cup u' \cup x \subset O_P$ and $\hat{F}.P$ is *formally p-adic over $(uP, u'P)$. Thus we have the inclusion $H^x_{u,u'}(\hat{F}) \subset \hat{H}^x_{u,u'}(\hat{F})$.

<u>Theorem 5.1</u>. Assume that (c_2) is satisfied, and F/K satisfies Zariski's local uniformization theorem. Then

$$H^x_{u,u'}(F) = H^x_{u,u'}(\hat{F}) \cap F = \hat{H}^x_{u,u'}(\hat{F}) \cap F$$

<u>Proof</u>. It suffices to prove that $\hat{H}^x_{u,u'}(\hat{F}) \cap F \subset H^x_{u,u'}(F)$. Let h be an arbitrary element in $\hat{H}^x_{u,u'}(\hat{F}) \cap F$. We have to show that $h \in H^x_{u,u'}(F)$. By Theorem 3.3., the holomorphy ring $H^x_{u,u'}(F)$ of F/K over (u, u', x) equals the ring $\tilde{H}^x_{u,u'}(F)$ of holomorphic functions on the set $\tilde{S}^x_{u,u'}(F)$. Let P be an arbitrary member of $\tilde{S}^x_{u,u'}(F)$. We have to show that $hP \neq \infty$. Assume the contrary, i.e. $hP = \infty$. Then $h \neq 0$ and $h^{-1}P = 0$. By Corollary to Theorem 2.2, there is an internal place Q of $\hat{F}/K$ which is rational over K, i.e. $\hat{F}.Q = K$, and coincides with P on $K[u, u', x, h^{-1}]$. Thus Q belongs to the internal set $\hat{S}^x_{u,u'}(\hat{F})$ containing the internal places T of $\hat{F}/K$ subject to conditions: $\hat{F}.T = K$, $u \cup u' \cup x \subset O_T$, $uT \subset O_v$ and $u'T \subset m_w$. In addition we have $h^{-1}Q = h^{-1}P = 0$. By enlargement principle and by Theorem 3.3., the internal holomorphy ring $\hat{H}^x_{u,u'}(\hat{F})$ of $\hat{F}/K$ over (u,u',x) coincides with the ring of holomorphix functions on the internal set $\hat{S}^x_{u,u'}(\hat{F})$. Since, by hypothesis, $h \in \hat{H}^x_{u,u'}(\hat{F})$ and, by construction, $Q \in \hat{S}^x_{u,u'}(\hat{F})$, it follows that $h.Q \neq \infty$, and hence $h^{-1}Q \neq 0$ which gives a con-

tradiction Q.E.D.

Taking $u \subset O_v$, $u' \subset \mathfrak{m}_v$ and $x \subset K$, we obtain:

Corollary to Theorem 5.1. Suppose that (c_2) is satisfied, and F/K satisfies Zariski's local uniformization theorem. Then $H(F) = H(\hat{F}) \cap F = \hat{H}(\hat{F}) \cap F$.

Thus we proved that the Problem IV from Introduction is affirmatively solved in the case of the extension $\hat{F}/F$.

Now let us study the relation between the radical ideal structure of the rings $H^x_{u,u'}(F)$, $H^x_{u,u'}(\hat{F})$ and $\hat{H}^x_{u,u'}(\hat{F})$. If a is an ideal in $H^x_{u,u'}(F)$ then we have the inclusions:

$\sqrt{a} \subset \sqrt{aH^x_{u,u'}(\hat{F})} \cap F \subset \sqrt{a\hat{H}^x_{u,u'}(\hat{F})} \cap F$. We are interested to know if the opposite inclusions hold too.

Theorem 5.2. Assume that (c_2) is satisfied, and F/K satisfies Zariski's local uniformization theorem. Then

$$\sqrt{a} = \sqrt{aH^x_{u,u'}(\hat{F})} \cap F = \sqrt{a\,\hat{H}^x_{u,u'}(\hat{F})} \cap F = \sqrt[*]{a\,\hat{H}^x_{u,u'}(\hat{F})} \cap F,$$

where a is an arbitrary finitely generated ideal in $H^x_{u,u'}(F)$.

Proof. It suffices to prove that $\sqrt[*]{a\,\hat{H}^x_{u,u'}(\hat{F})} \cap F \subset \sqrt{a}$. Let h be an arbitrary element in $\sqrt[*]{a\,\hat{H}^x_{u,u'}(\hat{F})} \cap F$. We have to show that $h \in \sqrt{a}$.

First let us observe that $h \in H^x_{u,u'}(F)$ by Theorem 5.1. Then, by Theorem 4.4, we have to show that h vanishes at all common zeros $P \in \tilde{S}^x_{u,u'}(F)$ of the ideal a. Let $P \in \tilde{S}^x_{u,u'}(F)$ be such that $zP = o$ for each $z \in a$. We have to show that $hP = o$. Since $h \in H^x_{u,u'}(F)$, it follows that $hP \neq \infty$. By Corollary to Theorem 2.2, there exists an internal place Q of $\hat{F}/K$ such that $\hat{F}.Q = K$ and Q coincides with P on $K\left[u, u', x, t, h\right]$, where $t = (t_1, \ldots, t_s)$ is a family of generators of a. Thus Q belongs to the internal set $\hat{S}^x_{u,u'}(\hat{F})$ containing the internal places T of $\hat{F}/K$ satisfying the conditions: $\hat{F}.T = K$, $u \cup u' \cup x \subset O_T$, $uT \subset O_v$ and $u'T \subset m_v$. As $t_iQ = t_iP = o$ for $i = 1, \ldots, s$, we conclude that Q is a zero of the ideal $a\hat{H}^x_{u,u'}(\hat{F})$, i.e. $zQ = o$ for each $z \in a\,\hat{H}^x_{u,u'}(\hat{F})$.

By enlargement principle and by Theorem 3.4., the ideal $\sqrt[*]{a\hat{H}^x_{u,u'}(\hat{F})}$ coincides with the set of the elements in $\hat{H}^x_{u,u'}(\hat{F})$ vanishing at all common zeros $T \in \hat{S}^x_{u,u'}(\hat{F})$ of the ideal $a\hat{H}^x_{u,u'}(\hat{F})$. Since, by hypothesis, $h \in \sqrt[*]{a\hat{H}^x_{u,u'}(\hat{F})}$, and $Q \in \hat{S}^x_{u,u'}(\hat{F})$ is a zero of the ideal $a\hat{H}^x_{u,u'}(\hat{F})$, we conclude that $hP = hQ = o$, as contended. Q.E.D.

6. Bounds over arbitrary base fields

As A.Robinson asserts in [18], "there are in ideal theory and algebraic geometry many cases where bounds are known to exist, or are believed to exist, and where the existence of such bounds ... corresponds to the extraction of a finite from an infinite desjunction..." Many such old results were proved by J.König [12] and G.Herrmann [9] in a constructive manner using ideas of Kronecker, M.Noether and Hentzelt. Among recent works in this constructive style we mention [16], [25] and [15].

A.Robinson showed that the existance of such bounds can be established by "a method which relies only on the elements of ideal theory, coupled with an argument from non-standard analysis", more generally from model theory. "This may be contrasted with some lenghty computations involved in the methods of König and Herrmann" [18]. A systematic model theoretic approach of the existence of bounds in the theory of polynomial ideals is achieved by L. van den Dries in [6], [7].

In this section we derive the existence of some bounds from the non-standard theoretic results proved in Section 1 and 2. We begin with a consequence of Proposition 1.2.

Proposition 6.1. Given natural numbers n, m and d, there is $\alpha = \alpha(n, m, d) \in \mathbb{N}$ such that for every field K, for every prime ideal a in $K[X] = K[X_1, \ldots, X_n]$ of degree $\leqslant d$, for every ideal b in $K[X, Y] = K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ of degree $\leqslant d$, if $a \subsetneq b \cap K[X]$ then there is a polynomial $P \in (b \cap K[X]) \setminus a$ of degree $\leqslant \alpha$.

Remark. By the degree deg (a) of an ideal a in a polynomial ring $K[X_1, \ldots, X_n]$ we understand the smallest natural number ℓ for which there exists a family $r = (r_1, \ldots, r_s)$ of generators of a, such that $\sum_{i=1}^{s} \deg(r_i) = \ell$, where deg (r_i) is the total degree of r_i.

<u>Proof</u>. Suppose n, m, d given and α does not exist. So for each $\ell \in \mathbb{N}$ there exist a field K_ℓ, a prime ideal a_ℓ in $K[X]$ of degree $\leq d$, an ideal b_ℓ in $K[X,Y]$ of degree $\leq d$, such that $a_\ell \subsetneq b_\ell \cap K_\ell[X]$ and each polynomial $P \in (b_\ell \cap K_\ell[X]) \setminus a_\ell$ is of degree $\geq \ell$. Consider a structure containing all fields K_ℓ, polynomial rings $K_\ell[X, Y]$, $\mathbb{N}$, e.t.c. and take an enlargement of this structure. By enlargement principle there exist an internal field K in this enlargement, an internal prime ideal a' in $K^*[X]$ of degree $\leq d$, an internal ideal b' in $K^*[X, Y]$ of degree $\leq d$, and an infinite natural number ω such that $a' \subsetneq b' \cap K^*[X]$ and each internal polynomial $P \in (b' \cap K^*[X]) \setminus a'$ is of degree $\geq \omega$. Since the internal ideals a' and b' are of finite degree we conclude that $a' = aK^*[X]$ and $b' = bK^*[X,Y]$, where a is an ideal in $K[X]$ and b is an ideal in $K[X, Y]$. As, by [7] 2.3., $K^*[X]$ is a faithfully flat $K[X]$ - module, it follows that $a = a' \cap K[X]$ and $b = b' \cap K[X, Y]$. In particular a is prime and $a \subset b$. Thus we obtain a morphism $f: K[X]/a \to K[X, Y]/b$ of K-algebras. Let $\hat{f}: K^*[X]/a' \to K^*[X, Y]/b'$ denote the internal K-algebras morphism induced by f. Since, by hypothesis, $a' \subsetneq b' \cap K^*[X]$ it follows that $\hat{f}$ is not injective, and hence, by Proposition 1.2., f is not injective, i.e. $a \subsetneq b \cap K[X]$. Let $P \in (b \cap K[X]) \setminus a \subset \subset (b' \cap K^*[X]) \setminus a'$. Thus we obtained a standard polynomial $P \in (b' \cap K^*[X]) \setminus a'$, which gives a contradiction Q.E.D.

<u>Remark</u>. A standard proof of the previous result in the particular case a = o is given by D.Popescu [14] Corollary 4.3.

In the same way we can prove the following consequence of Proposition 1.4.

<u>Proposition 6.2</u>. Given natural numbers n, m and d, there is $\beta = \beta(n, m, d) \in \mathbb{N}$ subject to: for every field K, for every field extension F of K, for every family $x = (x_1, \ldots, x_n)$ of elements in F such that $F = K(x)$ and the kernel of the canonical K-morphism $K[X] \to F: X \mapsto x$ is of degree $\leq d$, for

every family $y = (y_1, \ldots, y_k)$ in F, such that $y_i = \frac{y_i'}{y_i''}$ where $y_i', y_i'' \in K[x]$ of degree $\leqslant d$ for $i = 1, \ldots, k$, for each family $z = (z_1, \ldots, z_m)$ of linearly dependent elements in F over $L = K(y)$ such that $z_i = \frac{z_i'}{z_i''}$ where $z_i', z_i'' \in K[x]$ of degree $\leqslant d$ for $i = 1, \ldots, m$, there exists a family $u = (u_1, \ldots, u_m)$ of elements in L, not all zero, such that $\sum_{i=1}^{m} u_i z_i = 0$ and $u_i = \frac{u_i'}{u_i''}$, where $u_i', u_i'' \in K[x]$ of degree $\leqslant \beta$ for $i = 1, \ldots, m$.

The following results are consequences of Theorem 1.5.

Proposition 6.3. Given natural numbers n and d, there is $\gamma = \gamma(n,d) \in \mathbb{N}$ withe the property: for each field K, for each field extension F of K, for each family $x = (x_1, \ldots, x_n)$ such that $F = K(x)$ and the kernel of the canonical K-morphism $K[X] \to F: X \mapsto x$ is of degree $\leqslant d$, if $a_1, \ldots, a_d, z$ are arbitrary elements in F such that $z^d + a_1 z^{d-1} + \ldots + a_d = 0$ and $a_i = a_i'/a_i''$ where $a_i', a_i'' \in K[x]$ of degree $\leqslant d$ for $i = 1, \ldots, d$, then $z = z'/z''$ with $z', z'' \in K[x]$ of degree $\leqslant \gamma$.

Proposition 6.4. Given natural numbers n, m, ℓ and d, there is $\delta = \delta(n,m,\ell,d) \in \mathbb{N}$ subject to: for each field K of positive characteristic $p \leq \ell$, for each field extension F of K, for each family $x = (x_1, \ldots, x_n)$ such that $F = K(x)$, and the kernel of the canonical K-morphism $K[X] \to F: X \mapsto x$ is of degree $\leqslant d$, if $y_1, \ldots, y_m$ are arbitrary elements in F such that $y_i = y_i'/y_i''$ where $y_i', y_i'' \in K[x]$ of degree $\leqslant d$ and $y_1^{p^{-d}}, \ldots, y_m^{p^{-d}} \in F^{p^{-d}}$ are linearly dependent over F, then there exist $u_1, \ldots, u_m \in F$, not all zero, such that $\sum_{i=1}^{m} u_i^{p^d} y_i = 0$ and $u_i = u_i'/u_i''$ where $u_i', u_i'' \in K[x]$ of degree $\leqslant \delta$ for $i = 1, \ldots, m$.

We end this section with some remarks concerning Zariski's local uniformization theorem.

Definition. Let n, d and m be natural numbers. We say that a field K satisfies Zariski's local uniformization theorem with respect to the ordered triple (n, d, m) if for each finitely generated field extension F of K, for each place P of F/K, if $x = (x_1, \ldots, x_n)$ is a family of generators of F/K such that $K[x]$ is contained in the valuation ring O_P of P and the kernel of the canonical K-morphism $K[X] \to F$: $X \mapsto x$ is of degree $\leq d$, then there exists a family $y = (y_1, \ldots, y_\ell)$ of generators of F/K such that $\ell \leq m$, $K[x] \subset K[y] \subset O_P$, $y_i = y_i'/y_i''$ where $y_i', y_i'' \in K[x]$ of degree $\leq m$ for $i = 1, \ldots, \ell$, and yP is a simple point on the model of F/K whose generic point is y.

Definition. Let n and d be natural numbers. We say that a field K satisfies Zariski's local uniformization theorem with respect to the ordered pair (n, d) if for each finitely generated field extension F of K, for each place P of F/K, if $x = (x_1, \ldots, x_n)$ is a family of generators of F/K such that $K[x]$ is contained in the valuation ring O_P and the kernel of the canonical K - morphism $K[X] \to F$: $X \mapsto x$ is of degree $\leq d$, then there is a family $y = (y_1, \ldots, y_\ell)$ of generators of F/K such that $K[x] \subset K[y] \subset O_P$ and y P is a simple point on the model of F/K whose generic point is y.

If K is of characteristic zero then K satisfies Zariski's local uniformizations theorem with respect to every ordered pair (n, d). Moreover we have the following stringer result.

Theorem 6.5. Given natural numbers n and d, there is $\mathcal{E} = \mathcal{E}(n, d) \in \mathbb{N}$ such that every field of characteristic $p \geq \mathcal{E}$ (the case of characteristic zero in included) satisfies Zariski's local uniformization theorem with respect to the triple $(n, d, \mathcal{E})$

Proof. Suppose n and d given and $\mathcal{E}$ does not exist. By enlargement principle, there exist the following objects in a suitable enlargement:

an infinite natural number ω;

an internal field K of internal characteristic $p \geq \omega$; in particular the external characteristic of K is zero;

an internal field extension N of K;

an internal place Q of N/K;

a family $x = (x_1, \ldots, x_n)$ of *generators of N/K, i.e. $N = K^*(x)$, such that the following conditions are satisfied:

$K^*[x] \subset O_Q$

the kernel a' of the canonical internal K-morphism $K[X] \to N: X \mapsto x$ is of degree $\leq d$; in particular $a' = aK^*[X]$ for some ideal a in $K[X]$, and

for each internal family $z = (z_1, \ldots, z_m)$ of * generators of N/K, where $m \leq \omega$, if $K^*[x] \subset K^*[z] \subset O_Q$ and $z_i = z_i'/z_i''$ with $z_i', z_i'' \in K^*[x]$ of degree $\leq \omega$ for $i = 1, \ldots, m$, then zQ is not * simple on the internal model of N/K whose *generic point is z.

Since $K^*[X]$ is a faithfully flat $K[X]$ - module and $a' = aK^*[X]$ it follows that $a = a' \cap K[X]$ and hence we can identify N with the internal field extension $\hat{F}$ of K attached by Theorem 1.3. to the field extension $F = K(x)$ of K.

Let P denote the restriction of the internal place Q on F. Thus P is a place of F/K which is rational over the internal field extension $\hat{F}Q$ of K, and $K[x]$ is contained in the valuation ring O_P of P. Since K is of characteristic zero we conclude by Zariski's local uniformization theorem [28] that there is a finite family $y = (y_1, \ldots, y_m)$ of generaters of F/K, such that $K[x] \subset K[y] \subset O_P$ and yP is simple on the affine model V of F/K whise generic point is y.

As $\hat{F}$ is *generated by $F = K(y)$ it follows that $\hat{F}$ is generated over K by the finite family y, i.e. $\hat{F} = K^*(y)$. Let $\hat{V}$ denote the internal affine model of $\hat{F}$/K whose *generic point is y. As we have shown in the proof of Theorem 2.1. we can envisage the internal affine variety $\hat{V}$ as being defined by the same system of polynomial equations over K as the external variety V. Since $yQ = yP$ is an internal point on $\hat{V}$ which is simple on V it follows, by the same argument as in the proof of Theorem 2.1., that yQ is * simple on $\hat{V}$. Thus we obtained a finite family $y = (y_1, \ldots, y_m)$ of * generators of $\hat{F}$/K such that $K^*[x] \subset K^*[y] \subset O_Q$ and yQ is * simple on $\hat{V}$, which contradicts the statement (6.1) Q.E.D.

Corollary to Theorem 6.5. Let n and d be natural numbers. Then the set A(n, d) of primes p, for which there exists a field of characteristic p which does not satisfy Zariski's local uniformization theorem with respect to the pair (n, d), is finite.

It is known [22] that the sets A(n, d) are empty for $n \leq 3$ and arbitrary d.

7. Bounds over p-adically closed fields

It is known [13], [17] that there exist bounds in the theory of formally real fields. The situation is similar for the theory of formally p-adic fields as we shall show in this section, where we shall derive the existence of some bounds from the non-standard theoretic results proved in Section 4 and 5.

Let W be a theory as considered in Section 3 and let (Λ, Γ) be a fixed ordered pair satisfying (c_1). Let (K, v, p) denote a model of W and let F be a field extension of K, and u and u' be arbitrary subsets of F. The Kochen ring $\tilde{R}_{u,u'}(F)$ of F/K over (u, u') can be described as the integral closure of a certain ring of fractions, as follows. Denote by B the subring of F generated over the valuation ring O_v by the union of the sets u,u' and f(F) for each $f \in \Lambda \cup \Gamma$, and by b the ideal in B generated by the maximal ideal m_v and the union of the sets u' and f(F) for each $f \in \Gamma$. Let N be the multiplicative submonoid $(1 + b) \setminus \{o\}$ in B, and $R_{u,u'}(F)$ be the ring of fraction of B with respect to the monoid N. If F is formally p-adic over (u,u') then, by [2], Theorem 2.2., $\tilde{R}_{u,u'}(F)$ coincides with the integral closure of $R_{u,u'}(F)$ in F.

Definition. Assume that F/K is finitely generated and let $x = (x_1, \ldots, x_n)$ be a family of generators of F/K. Let $u = (u_1, \ldots, u_m)$ and $u' = (u'_1, \ldots, u'_{m'})$ be finite families of elements in F. Suppose that F is formally p-adic over (u,u'). Let $z \in \tilde{R}_{u,u'}(F)$.

The complexity of z with respect to the family of generators x is the smallest natural number ℓ for which there exist:

I) some polynomials $H_i \in O_v[U; U'; T; T']$ of degree $\leq \ell$, where $U = (U_1, \ldots, U_m)$, $U' = (U'_1, \ldots, U'_{m'})$, $T = (T_1, \ldots, T_\ell)$,

$T' = (T'_1, \ldots, T'_\ell)$, $i = 1, \ldots, \ell$;

II) some polynomials $G_{io} \in m_v[U; U'; T; T']$ of degree $\leq \ell$, for $i = 1, \ldots, \ell$;

III) some polynomials $G_{ij} \in O_v[U; U'; T; T']$ of degree $\leq \ell$, for $i = 1, \ldots, \ell$, $j = 1, \ldots, m'$;

IV) some polynomials $G'_{ik} \in O_v[U; U'; T; T']$ of degree $\leq \ell$, for $i = 1, \ldots, \ell$; $j = 1, \ldots, m'$;

V) a family f of rational functions $f_i(X_1, \ldots, X_\ell) \in \Lambda$ for $i = 1, \ldots, \ell$, such that the numerators and the denominators are of degree $\leq \ell$;

VI) a family g of rational functions $g_i(X_1, \ldots, X_\ell)$ for $i = 1, \ldots, \ell$, such that the numerators and the denominators are of degree $\leq \ell$, and

VII) a family $a = (a_1, \ldots, a_\ell)$ of elements in F, with $a_i = a'_i/a''_i$, where $a'_i, a''_i \in K[x]$ of degree $\leq \ell$, for $i = 1, \ldots, \ell$, such that z satisfies the equation $z^\ell + A_1 z^{\ell-1} + \ldots + A_\ell = 0$, where

$$A_i = \frac{H_i(u;u';f(a);g(a))}{1 + G_{io}(u;u';f(a);g(a)) + \sum_{j=1}^{m'} u'_j \cdot G_{ij}(u;u';f(a);g(a)) + \sum_{k=1}^{\ell} g_k(a) G'_{ik}(u;u';f(a);g(a))}$$

for $i = 1, \ldots, \ell$. By [2] Theorem 2.2. such ℓ always exists.

The following result follows by enlargement principle and by Theorem 4.1.

Theorem 7.1. Given natural numbers n, m, m' and d, there is $\mu = \mu(n, m, m', d) \in \mathbb{N}$ subject to: "Let (K, v, p) be an arbitrary p-adically closed field, F be a finitely generated field extension of K, $x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_m)$, $u' = (u'_1, \ldots, u'_{m'})$ be arbitrary families of elements in F such that the following conditions are satisfied:

a) $F = K(x)$;

b) the kernel of the canonical K-morphism $K[X] \to F$:

$X \mapsto x$ is of degree $\leqslant d$;

c) $u_i = g_i/g_i'$ where $g_i, g_i' \in K[x]$ of degree $\leqslant d$ for $i = 1, \ldots, m$;

d) $u_i' = h_i/h_i'$ where $h_i, h_i' \in K[x]$ of degree $\leqslant d$ for $i = 1, \ldots, m'$;

e) F is formally p-adic over (u, u').

Let $z \in \widetilde{R}_{u,u'}(F)$ be such that $z = z'/z''$, where $z', z'' \in K[x]$ of degree $\leqslant d$. Then the complexity of z with respect to the family x is $\leqslant \mu$.

Remarks. I) The previous theorem is an analogue in this general p-adic context of some results from the theory of formally real fields [17] Theorem 8.5.22, [3] Theorem 2.6.

II) It would be interesting to try to give a constructive proof of the previous theorem for concrete theories W and $\widetilde{W}$, for instance in the case of the theory of formally p-adic fields in Kochen-Roquette' sense, and obtain more precise informations about the function μ whose existence was proved by a non-standard method.

III) With deep arguments, Pfister [13] proved the following beautiful theorem: "Let K be a real closed field and let $f \in F = K(X_1, \ldots, X_n)$ be positive definite. Then f is a sum of 2^n squares in F". Observe that the bound on the number of squares does not depend on the degrees of the numerator and the denominator of f. It would be interesting to know if a similar result holds at least in some particular cases of the general theory considered here.

The following result is a stronger form of Theorem 3.2. (Nullstellensatz for Kochen rings).

Theorem 7.2. Assume that the theory W satisfies (c_4) and (c_2). Given natural numbers n, m, m', s and d, there is $\nu = \nu(n, m, m', s, d) \in \mathbb{N}$ subject to: "Let (K, v, p) be an arbitrary p-adically closed field, F be a finitely generated field extension of K, $x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_m)$, $u' = (u_1', \ldots, u_{m'}')$ be arbitrary families of elements in F such that the conditions a) - e) from Theorem 7.1 are satisfied. Let $r_1, \ldots, r_s$ and z be elements in $\widetilde{R}_{u,u'}(F)$ which admit representations $r_i = r_i'/r_i''$, $z = z'/z''$ with $r_i', r_i'', z', z'' \in K[x]$ of degree $\leqslant d$.

Then the following assertions are equivalent:

I) $zP \in m_v$ for every place $P \in \widetilde{S}_{u,u'}(F)$ such that $r_i P \in m_v$ for $i = 1, \ldots, s$;

II) there exist $\ell \leqslant \nu$, $t_i \in \widetilde{R}_{u,u'}(F)$ for $i = 1, \ldots, s$, and $t_o \in \widetilde{r}_{u,u'}(F)$ with complexities with respect to x bounded by ν such that

$$z^{\ell} = t_o + \sum_{i=1}^{s} r_i t_i \text{.''}$$

The proof is immediate by enlargement principle and by Theorem 4.3.

The following results is a consequence of Theorem 5.1.

<u>Theorem 7.3.</u> Assume that W satisfies (c_4) and (c_2), and the prime model (k, v, p_o) of W is of characteristic zero. Given natural numbers n, m, m', s and d, there is $\rho = \rho(n, m, m', s, d) \in \mathbb{N}$ subject to: "Let (K,v,p) be an arbitrary p-adically closed field, F be a finitely generated field extension of K, $x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_m)$, $u' = (u'_1, \ldots, u'_{m'})$, $y = (y_1, \ldots, y_s)$ be arbitrary families of elements in F such that the conditions a) - e) from Theorem 7.1. are satisfied and, in addition,

f) $y_i = y'_i/y''_i$ where $y'_i, y''_i \in K[x]$ of degree $\leqslant d$ for $i = 1, \ldots, s$.

Let $z \in H^{y}_{u,u'}(F) = \widetilde{R}_{u,u'}(F).K[y]$ be such that $z = z'/z''$ where $z', z'' \in K[x]$ of degree $\leqslant$ d. Then there exists a polynomial $H \in K[Y, T]$ of degree $\leqslant \rho$ where $Y = (Y_1, \ldots, Y_s)$, $T = (T_1, \ldots T_{\rho})$, and a family $a = (a_1, \ldots, a_{\rho})$ of elements in $\widetilde{R}_{u,u'}(F)$ whose complexity with respect to x is $\leqslant \rho$, such that $z = H(y, a)$".

We end the paper with a consequence of Theorem 5,2, a stronger form of Theorem 3.4. (Nullstellensatz for holomorphy rings) for the case of characteristic zero.

Theorem 7.4. Assume the same hypotheses as in Theorem 7.3. Given natural numbers n, m, m', s, ℓ and d, there is $\eta = \eta(n, m, m', s, \ell, d) \in \mathbb{N}$ subject to: "Let (K, v, p) be an arbitrary p-adically closed field, F be a finitely generated field extension of K, $x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_m)$, $u' = (u'_1, \ldots, u'_{m'})$, $y = (y_1, \ldots, y_s)$ be arbitrary families of elements in F such that the conditions a) - f) from Theorem 7.3. are satisfied. Let $r_1, \ldots, r_\ell$ and z be elements in $H^y_{u,u'}(F)$ which admit representations $r_i = r'_i/r''_i$, $z = z'/z''$ with $r'_i, r''_i, z', z'' \in K[x]$ of degree $\leqslant d$. Then the necessary and sufficient condition for z to vanish at all common zeros $P \in \tilde{S}^y_{u,u'}(F)$ of $r_1, \ldots, r_\ell$, is that some power z^k, where $k \leqslant \eta$, admits a representation of the form $z^k = \sum_{i=1}^{\ell} r_i t_i$, where $t_i = H_i(y, a)$ for some polynomials $H_i \in K[Y, T]$ of degree $\leqslant \eta$, with $Y = (Y_1, \ldots, Y_s)$, $T = (T_1, \ldots, T_\eta)$ and $a = (a_1, \ldots, a_\eta)$ is a family of elements in $\tilde{R}_{u,u'}(F)$ whose complexity with respect to x is $\leqslant \eta$ ".

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