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TOWARD THE UNIFICATION OF THE THEORY OF NONLINEAR SEMIGROUPS

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1. Introduction . The problem of the generation of nonlinear semigroups by a dissipative (possible multivalued) operator, is one of the most interesting problems of nonlinear analysis. The main motivation of this assertion is that nonlinear semi-groups are a convenient approach for the treatment of many partial differential equations arising in physics.

The mathematical contribution of this work is the proof of Martin's theorem via Kobayashi's theorem (section 2). This proof is important because is very simple and contributes to the unification of the theory of the generation of nonlinear semi-groups. Moreover, using this proof, the presentation of this theory becomes simpler and shorter with at least ten pages. This since the original proof of Martin (which takes about 15 pages) is very natural and concise, so that there is no any hope to be simplified and reduced in lenght.

2. Let X be a Banach space with norm $\|\cdot\|$ and $A, B \subset X \times X$. As usual we define

$$A\,x = \{ y \in X;\ [x,y] \in A \},\quad D(A) = \{ x \in X;\ A\,x \neq \phi \}$$

$$R(A) = \bigcup_{x \in D(A)} A\,x\ ,\quad A^{-1} = \{ [y,x];\ [x,y] \in A \}$$

$$\lambda A = \{ [x, \lambda y]\ ;\ [x,y] \in A \}\ ,\quad \lambda \in R$$

$$A + B = \{ [x, y + z]\ ;\ [x,y] \in A,\ [x,z] \in B \}.$$

One identifies the mappings with their graphs. A subset $B \subset X \times X$ is said to be dissipative if for each $\lambda > 0$ and $[x_i, y_i] \in A$, $i = 1,2$ we have

$$\|x_1 - x_2\| \leq \|(x_1 - \lambda y_1) - (x_2 - \lambda y_2)\|$$

The subset B is said to be ω- dissipative if $A - \omega I$ is dissipative (where I is the identity and $\omega \in R$).

A semi-group on a subset $D \subset X$ is a function S on $[0, \infty)$ which satisfies

$$S(t) \text{ maps } D \text{ into } D\ , \quad \forall\, t \geq 0 \tag{2.1}$$

$$S(t+\tau) = S(t)\, S(\tau)\ , \quad t, \tau \geq 0 \tag{2.2}$$

$$\lim_{t \downarrow 0} S(t)x = S(0)x = x\ , \quad \forall\, x \in D \tag{2.3}$$

If in addition

$$\| S(t)x - S(t)y \| \leq e^{\omega t}\, \| x - y \|\ , \quad \forall\, x,y \in D, t \geq 0 \tag{2.4}$$

then S is said to be a semi-group of type ω on D.

If $z \in X$, denote by $d[z,D]$ the distance from the point z to the subset D.

The semigroup S is said to be differentiable at $t_0 \in (0,\infty)$ if the derivative of the function $t \to S(t)x$ at t_0 exists (for each $x \in D$). For an arbitrary $T > 0$,denote by $\{t_i^n\}$ the partition $\Delta_n = \{ 0 = t_0^n < t_1^n \ldots t_{N_n-1}^n < T \leq t_{N_n} \}$ of the interval $[0,T]$ satisfying the condition

$$|\Delta_n| = \max_{1 \leq i \leq N_n} (t_i^n - t_{i-1}^n) \to 0 \quad \text{as} \quad n \to \infty \tag{2.5}$$

Let us assume that there are $\{x_i^n\} \subset D(A)$ and $\{p_i^n\}$ such that

$$\frac{x_i^n - x_{i-1}^n}{t_i^n - t_{i-1}^n} - p_i^n \in A\, x_i^n \quad , \ i = 1,2,\ldots,N_n \ , \ n \geq 1 \tag{2.6}$$

$$x_0^n \to x \quad \text{as } n \to \infty \quad , \quad x \in X$$

$$p_n = \sum_{i=1}^{N_n} \| p_i^n \| \ (t_i^n - t_{i-1}^n) \to 0 \quad \text{as } n \to \infty$$

The step function $u_n : [0,T] \to X$ defined by

$$u_n(t) = \begin{cases} x_0^n & ,t = 0 \\ x_i^n & ,t \in (t_{i-1}^n , t_i^n] \cap (0,T] \quad ,i = 1,\ldots,N_n \end{cases} \tag{2.7}$$

is said to be a DS- approximate solution of the Cauchy problem

$$(d/dt)u(t) \in Au(t) \quad , \ t \in (0,T) \ , \ u(0) = x_0 \tag{2.8}$$

We now are prepared to state the following

<u>Kobayashi's theorem</u> . <u>If</u> $A \subset X \times X$ <u>is</u> ω <u>-dissipative and satisfies</u>

$$\lim_{h \downarrow 0} \frac{1}{h}\, d\, [\, x, R(I - hA)] = 0 \ , \ x \in \overline{D(A)} \tag{2.9}$$

then A generates a semigroup S(t) of type ω on D. Precisely, for each $T > 0$ there is a DS - approximate solution u_n defined by (2.7) such that

$$S(t)x = \lim_{n\to\infty} u_n(t) \quad , \quad \forall t \in [0,T] , x \in \overline{D(A)} \qquad (2.10)$$

Any other DS - approximate solution u_n satisfies also (2.10)

The semigroup S associated with A via above theorem is said to be generated by A. The function $t \to S(t)x$ $(x \in \overline{D(A)})$ is the integral solution of (2.8)(see e.g. [2]). In some particular cases, if the semigroup S(t) is a.e. differentiable on $(0,+\infty)$, then $u(t) = S(t)x$ is just the strong solution of (2.8) ([1] , [3])

Recall also that in the particular case

$R(I - \lambda A) \supset \overline{D(A)}$, for all sufficiently small positive λ,

S(t) is given by the exponential formula of Crandall and Liggett[1]

$$S(t)x = \lim_{n\to\infty} (I - \frac{t}{n} A)^{-n} x \ , \ x \in \overline{D(A)} \ , \ t \geq 0$$

In the continuous case, esentially the basic result is given by

Martin's theorem [3] Let be a closed subset of X. If $A: D \to X$ is a continuous and dissipative operator satisfying (2.11), then the semigroup S generated by A is everywhere differentiable on $[0,\infty)$

Actually, in [3] Martin has obtained a result somewhat more general There, the nonautonomous case is treated. Finally, in [3] the condition (2.9) appears under the equivalent form

$$\liminf_{h\downarrow 0} \frac{1}{h} d[x + hA x ;D] = 0 \ , \quad x \in D \qquad (2.11)$$

A proof of the equivalence of (2.9) and (2.11) even in more general conditions, may be found in [5]

We now proceed to show how Martin's result can be easily derived from that of Kobayashi.

Proof of Martin's theorem via Kobayashi's theorem.

Let $x \in D$. Since A is continuous at x there are $M > 0$ and $R > 0$ such that

$$\|Au\| \leq M, \quad \forall u \in B(R) \qquad (2.12).$$

where B(R) is the ball of radious R about zero.

Let $T > 0$ with the property

$$T(M+1) < R \tag{2.13}$$

Fix an arbitrary natural number n and set $t_o^n = 0$, $x_o^n = x$. Following Martin [3] ,inductively define t_{i+1}^n and x_{i+1}^n as follows:

If $t_i^n = T$ set $t_{i+1}^n = T$ and if $0 < t_i^n < T$, define δ_i^n as the largest number $(0,\frac{1}{n}]$ satisfying

$$t_i^n + \delta_i^n \leq T \tag{2.14}$$

$$\| Au - Ax_i^n \| \leq \frac{1}{n} \quad , \forall u \in D , \quad \| u - x_i^n \| \leq (M+1)\delta_i^n \tag{2.15}$$

$$\frac{1}{\delta_i^n} d[x_i^n + \delta_i^n Ax_i^n ; D] \leq \frac{1}{2n} \tag{2.16}$$

In view of (2.11) , $\delta_i^n > 0$. Since there is no danger of confusion we drop the index n, writting $t_i^n = t_i$, $x_i^n = x_i$ and $\delta_i^n = \delta_i$. Define

$$t_{i+1} = t_i + \delta_i \tag{2.17}$$

According to (2.16), there is an element $x_{i+1} \in D$ such that

$$\frac{1}{\delta_i} \| x_i + \delta_i Ax_i - x_{i+1} \| \leq \frac{1}{n} \tag{2.18}$$

As in Pavel-Ursescu [5] (see also [4]) the following notation is useful

$$p_i = (x_{i+1} - x_i - \delta_i Ax_i)/\delta_i \ , \ i = 0,1,\ldots \tag{2.19}$$

Thus we have

$$x_{i+1} = x_i + (t_{i+1} - t_i)(Ax_i + p_i) \ , \| p_i \| \leq \frac{1}{n} \tag{2.20}$$

It is easy to verify thatt $x_i \in B(R)$, hence (2.20) and (2.15) give

$$\| Ax_{i+1} - Ax_i \| \leq \frac{1}{n} \tag{2.21}$$

It is also known that $\lim_{i\to\infty} t_i = T$ (see [3])

Let $T_1 = T_2$ and $t \in (0,T_1]$. There is an integer $i = i(n,t)$ such that $t \in (t_i, t_{i+1}]$ and another one $i_n = N_n$ such that $t_{i_n} < T_1 \leq t_{i_n}+1$.

Define $y_n : [0,T_1] \to X$ by

$$y_n(t) = x_i + (t - t_i)(Ax_i + p_i), t_i \leq t \leq t_{i+1}, i=0,1\ldots N_n \tag{2.22}$$

It is immediate that

$$\|y_n(t) - y_n(s)\| \le (M + 1)|t-s| \;,\; t,s \in [0,T_1]$$

The crucial point is now to prove the sequence (of polygonal lines) y_n is convergent. This is the difficult (and long) part of Martin's proof. To avoid it, let us observe that (2.20) yields

$$\frac{x_{i+1} - x_i}{t_{i+1} - t_i} - \varepsilon_i = Ax_{i+1} \;,\; i = 0,1,\ldots,N_n \tag{2.23}$$

with

$$\varepsilon_i = \varepsilon_i^n = p_i + Ax_i - Ax_{i+1} \tag{2.24}$$

By (2.17) and (2.18) we have $\|\varepsilon_i\| \le \frac{2}{n}$, therefore

$$\varepsilon^n = \sum_{i=1}^{\infty} \|\varepsilon_i^n\| \, (t_{i+1} - t_i) \le \frac{T}{n} \tag{2.25}$$

Consequently, the function u_n: $[0,T_1] \to D$ given by

$$u_n(t) = \begin{cases} x, t = 0 \\ x_{i+1}^n \;,\; t_i < t \le t_{i+1} \;,\; i = 0,1,\ldots,N_n \end{cases} \tag{2.26}$$

is A DS - approximate solution.

By Kobayashi's theorem, $\lim_{n\to\infty} u_n(t) = u(t) = S(t)x \in D$. On the other hand, it is easy to check that

$$\|y_n(t) - u_n(t)\| \le \frac{2(M + 1)}{n} \;,\; \forall t \in [0,T_1] \tag{2.27}$$

Indeed, if $t = 0$ then the left hand side of (2.27) is zero. If $t_i < t \le t_{i+1}$, then we have

$$y_n(t)-u_n(t) \;=\; x_i - x_{i+1} + (t-t_i)(Ax_i+P_i) \qquad (t_{i+1}-t_i)+(t-t_i)(M+1)$$

This implies (2.27) since $t - t_i \le t_{i+1} - t_i \le \frac{1}{n}$

From (2.27) it follows that

$$\lim_{n\to\infty} y_n(t) = \lim_{n\to\infty} u_n(t) = S(t)x \tag{2.28}$$

uniformly on $[0,T]$.

Finally, the fact that $t \to u(t) = S(t)x$ is differentiable on $[0, T_1]$ it follows from the integral equation

$$u(t) = x + \int_0^t Au(s)ds \;,\; t \in [0,T_1] \tag{2.29}$$

and from the continuity of A and of $t \to u(t)$. The equation (2.29) is obtained in Martin [3]. However, for the sake of simplicity we shall indicate here a proof of it, as in [5] on [4].

Define the step function

$$a_n(s) = \begin{cases} t_i \ , \ t_i \leq t < t_{i+1} \\ T_1 \ , \ t = T_1 \end{cases}$$

It is easy to check that y_n can be written under the form

$$y_n(t) = x + \int_0^t Ay_n(a_n(s))ds + g_n(t) \qquad (2.30)$$

where

$$g_n(t) = \sum_{j=0}^{i-1} (t_{j+1} - t_j)p_j + (t - t_1)p_i ; t_i \leq t \leq t_{i+1}$$

hence $\| g_n(t) \| \leq \frac{T_1}{n}$, $t \in [0,T_1]$.

Since $y_n(a_n(s)) \to u(s)$ as $n \to \infty$, uniformly on $[0,T_1]$, (2.30) yields (2.29).

By standard arguments, one prove that the local solution u of (2.29 can be extended to the whole $[0,+\infty)$ and therefore (2.29) holds for all $t \geq 0$. This implies the differentiability of $t \to S(t)x$ everywhere on $[0,+\infty)$.

Remark 1. Similarly to (the "tangent condition on the right") (2.11) let us consider the following "tangent condition on the left"

$$\liminf_{h \uparrow 0} \frac{1}{h} d[x + hAx;D] = 0, \ x \in D \qquad (2.31)$$

where $h \uparrow 0$ means $h \to 0$ with $h < 0$. For the existence of the solution $u: (-\infty,0] \to D$ to the Cauchy problem

$$u'(t) = Au(t) \ , u(0) = x, x \ D \ , \ t \leq 0 \qquad (2.32)$$

the condition (2.31) is necessary. If $A: D \to X$ is Lipschitz continuous on D, then (2.31) is also sufficient (hence necessary and sufficient) for the existence of the solution u(t) of (2.32).

Necessity Since $u(h) = u(h,x) \in D$, $h < 0$, we have

$$\frac{1}{-h}\, d[x + hA\,x\,;D] \le \frac{1}{-h}\|x + hAx - u(h)\| = \left\|\frac{u(h) - x}{h} - Ax\right\| \to 0 \text{ as } h \uparrow 0$$

which implies (2.31) with "$\lim\limits_{h\uparrow 0}$" instead of "lim inf".

• <u>Sufficiency</u> Set $A_1x = -Ax$, $x \in D$ and consider the problem

$$y'(s) = A_1(y(s)) \ , \ s \geqslant 0 \ y(o) = x, \ x \in D \qquad (2.33)$$

In view of (2.31), it follows that A_1 satisfies (2.11) and consequently the solution of (2.33) is just $y(s) = S(s)x$, $s \geqslant 0$. Now, the function $u = y(-t) = S(-t)x$, $t \leq 0$ is the solution to (2.32). It follows that if (2.11) holds and (2.31) is not true for $x_o \in D$, then $S(t)x_o$, $t \geqslant 0$ cannot be extended to the left of zero (such that $S(t)x_o \in D$ for $t < 0$). Moreover, if $A: D \to X$ is Lipschitz continuous on D, and both (2.11) and (2.31) are satisfied, then the semigroup S generated by A is defined on $R = (-\infty, +\infty)$ (i.e. is a group).

A simple example in this direction is the following one. Take $X = R$, $D = [0,1] \subset R$ and $Ax = -x + 1$. It is easy to see that (2.11) holds while (2.31) is not satisfied for $x = 0$. Consequently, by the above theory $S(t)o$ does not belong to $[0,1]$ for $t < 0$. Indeed, in this case $S(t)o = 1 - e^{-t} \in [0,1]$ for all $t \geqslant 0$ (but not for $t < 0$).

Remark 1 has been added after some conversations with D. Motreanu and G. Moroşanu.

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