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BOUNDARY CONTROL PROBLEMS WITH NONLINEAR STATE EQUATION

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Abstract. First order necessary conditions of optimality for boundary control problems governed by parabolic equations with nonlinear boundary value conditions are obtained. Some implications in controllability theory of these systems are derived.

1. INTRODUCTION

We are concerned here with first order necessary conditions of optimality for convex control problems governed by nonlinear boundary-value problems of the form

$$
(1.1) \quad
\begin{aligned}
& y_t + Ay = 0 && \text{in } Q = \Omega \times]0,T[\\
& \frac{\partial y}{\partial \nu} + \beta_i(y) \ni B_i u_i + f_i && \text{in } \Sigma_i = \Gamma_i \times]0,T[\,;\ i = 1,2 \\
& y(x,0) = y_0(x) && \text{in } \Omega .
\end{aligned}
$$

Here Ω is a bounded and open subset of the Euclidean space R^N, A is a second order elliptic and symetric operator on Ω and β_i are maximal monotone graphs (in general multivalued) in $R \times R$. The controls u_i are taken from the Hilbert spaces U_i and B_i are linear continuous operators from U_i to $L^2(\Sigma_i)$, $i = 1,2$. The functions y_0 and f_i are fixed in $L^2(\Omega)$ and $L^2(\Sigma_i)$, $i = 1,2$, respectively.

The boundary Γ of Ω consists of two disjoint parts Γ_1 and Γ_2, i.e., $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

To make what follows more meaningfull, let me briefly describe some physical problems from which this system originate.We refer the reader to [7] and [10] for further examples and complete references.

1° Newton's colling law. In this case β_i are continuous and monotonically increasing functions.

2° The Stefan - Boltzman heat radiation law. The functions β_i , i = 1,2 , are of the following form

$$(1.2) \qquad \beta_i(r) = \begin{cases} a_i(r-c)^4 & \text{if } r \geqslant c \\ 0 & \text{if } r < c \end{cases}$$

where $a_i > 0$; i = 1,2.

3° The natural convection

$$(1.3) \qquad \beta_i(r) = \begin{cases} a\, r^{5/4} & \text{if } r \geqslant 0 \\ 0 & \text{if } r < 0 \end{cases} \qquad ; a > 0; i = 1,2 .$$

4° The enzyme diffusion (The Michaelis-Menten law)

In this case $\Gamma = \Gamma_1$ and $\beta = \beta_1$ is given by

$$(1.4) \qquad \beta(r) = \begin{cases} \dfrac{r}{r+m} & \text{for } r > 0 \\ -\infty & \text{for } r = 0 \\ \phi & \text{for } r < 0. \end{cases}$$

5° The termostat-control ([7], p.21). The process is described by (1.1) where (we shall take $\Gamma_1 = \Gamma$ and $\beta_1 = \beta$)

$$(1.5) \qquad \beta(r) = \begin{cases} \alpha_1(r-\theta_1) & \text{if } -\infty < r < \theta_1 \\ 0 & \text{if } \theta_1 \leq r \leq \theta_2 \\ \alpha_2(r-\theta_2) & \text{if } \theta_2 < r < \infty \end{cases}$$

where α_1 and α_2 are positive numbers.

6° The Signorini-problem ([7]). The graph β($\Gamma = \Gamma_1$ and $\beta_1 = \beta$) is given by

$$(1.6) \qquad \beta(r) = \begin{cases} 0 & \text{if} \quad r > 0 \\ ]-\infty, 0] & \text{if} \quad r \quad 0 \\ \phi & \text{if} \quad r < 0 \end{cases}$$

The contents of this paper are outlined below.In the Section 2 we shall study existence and approximation of solutions for the boundary control system (1.1).In Section 3 are given the main results,Theorems 1 and 2 which are concerned with necessary conditions for optimality in a control problem with convex cost criterion governed by (1.1) in two typ ical cases: β_i locally Lipschitzian functions and the Signorini problem (1.6).

The proofs are delivered in sections 5,6.The main idea of our approach consists to approximating the control problem by a family of smooth problems for which the optimality equations are immediate and to tend to limit in the approximating equations.In Section 4 is studied the convergence of this approximating control process.In Section 7 some applications of the present theory to controllability of nonlinear systems of the form (1.1) are given.

The results as well as the approach used here are similar to those from the author works [1] , [2] , [3] .For comparison with other literature on necessary conditions for boundary control problems the works [14] , [16] are most closely related to present paper.In particular Theorem 1 includes and refines those of [14] .

The following notation will be used in the sequel.Given a real Banach space E , and $[0,T]$ a real interval we shall denote by $L^p(0,T;E)$, $1 \leq p \leq \infty$ the space of all p-integrable E-valued functions on $[0,T]$ and by $C([0,T];E)$ the Banach space of all continuous functions from $[0,T]$ to E. By $C_w([0,T];E)$ we shall denote the space of all functions continuous from $[0,T]$ to the space E endowed with weak topology.

Given a lower semicontinuous convex function $\varphi : E \longrightarrow \overline{R} =]-\infty,+\infty]$ we shall denote by $\partial\varphi(x) \in E'$ (the dual space of E) the set of all <u>subgradients</u> of φ at x, i.e.,

$$(1.7) \quad \partial\varphi(x) = \{x^* \in E';\ \varphi(x) \leq \varphi(y) + (x^*, x-y) \quad \text{for all } y \in E\}.$$

If φ is Gâteaux differentiable at x then $\partial\varphi(x)$ consists of a single element, namely the gradient $\nabla\varphi(x)$ of φ at x. The mapping $\partial\varphi : E \longrightarrow E'$ is called the <u>subdifferential</u> of φ. If β is a locally Lipschitzian function on real axis R, the <u>generalized gradient</u> $\partial\beta$ (in the sense of Clarke [6]) of β is defined by

$$(1.8) \quad \partial\beta(r) = \text{conv}\{y \in R;\ y = \lim_{r_n \to r} \beta'(r_n)\},\ r \in R$$

where $\nabla\beta = \beta'$ denotes the ordinary derivate of β. For other concepts and results in convex analysis relevant to this paper we refer the reader to [4], [5], [8], [13].

Let k, r, s be real numbers. We shall denote by $H^k(\Omega)$, $H^k(\Gamma)$, $H^{r,s}(Q)$ and $H^{r,s}(\Sigma)$ the usual Sobolev spaces on Ω, Γ, Q and Σ, respectively (see, e.g. [11], p.14). By $L^2(\Omega)$, $L^2(\Gamma)$, $L^2(Q)$ and $L^2(\Sigma)$ we shall denote the corresponding spaces of square integrable functions. Finally we shall denote by W(Q) the space of all functions $y \in L^2(0,T;H^1(\Omega))$ such that $\frac{d}{dt}y \in L^2(0,T;(H^1(\Omega))')$. Here $(H^1(\Omega))'$ is the dual space of $H^1(\Omega)$ and $\frac{dy}{dt}$ denotes the derivative of y(t) in the sense of $(H^1(\Omega))'$-valued distributions on $]0,T[$. W(Q) is a Banach space with the natural norm

(1.9) $\|y\|^2_{W(Q)} = (\|y\|^2_{L^2(0,T;H^1(\Omega))} + \|\frac{dy}{dt}\|^2_{L^2(0,T;(H^1(\Omega))')})$

and it is well known that $W(Q) \subset C([0,T];L^2(\Omega))$, algebrically and topologically.

2. THE BOUNDARY CONTROL SYSTEM

Let Ω be a bounded and open subset of R^N with a sufficiently smooth boundary Γ. We shall assume that Γ consists of two smooth and disjoint parts Γ_1 and Γ_2 where meas $\Gamma_1 > 0$ (except the case $N = 1$ and $\Omega =]a,b[$ when $\Gamma_1 = \{a\}$ or $\Gamma_1 = \{a\} \cup \{b\}$).

Let A be a second order differential operators on Ω of the form

$$Ay = - \sum_{i,j=1}^{N} (a_{ij}(x) y_{x_i})_{x_j} + a(x)y$$

where $a_{ij} \in C^1(\bar{\Omega})$, $a \in L^{\infty}(\Omega)$, $a_{ij} = a_{ji}$ for all i,j and for some $\omega > 0$,

$$\sum_{i,j=1}^{N} a_{ij} \xi_i \xi_j \geq |\xi|^2 \quad \text{a.e. on } \Omega, \quad \xi \in R^n.$$

(Here y_{x_i} denotes the partial derivative of y with respect to x_i).

For $y_0 \in L^2(\Omega)$ and $v_i \in L^2(\Sigma_i)$, $i = 1,2$ consider the system

$$\text{(2.1)} \quad \begin{aligned} & y_t + A_y = 0 && \text{in } Q \\ & \frac{\partial y}{\partial \nu} + \beta_i(y) \ni v_i && \text{in } \Sigma_i;\ i = 1,2 \\ & y(x,0) = y_0(x) && x \in \Omega \end{aligned}$$

where y_t stands for partial derivative $\partial y / \partial t$ while

$\frac{\partial y}{\partial \nu}$ is the outward normal derivative associated with A.

Here $\beta_i, i = 1,2$ are two maximal monotone graphs in $R \times R$ which satisfy the conditions

$$(2.2) \qquad \beta_i(0) \ni 0 \qquad i = 1,2 .$$

Let us now give a precise meaning to system (2.1)

DEFINITION 1 . A function $y \in W(Q)$ is a solution to (2.1) if there exist the functions $w_i \in L^2(\Sigma_i), i = 1,2$ such that

$$(2.3) \quad w_i(\sigma,t) \in \beta_i(y(\sigma,t)) \quad \text{a.e. } (\sigma,t) \in \Sigma_i; i = 1,2$$

and

$$(2.4) \quad - \int_Q y \varphi_t dxdt + \int_0^T a(y,\varphi)dt + \sum_{i=1}^{2} \int_{\Sigma_i} (w_i - v_i)\varphi \, d\sigma dt = \int_\Omega y_0(x)\varphi(x,0)dx$$

for all $\varphi \in W(Q)$ such that $\varphi(x,T) = 0$.
Here $a: H^1(\Omega) \times H^1(\Omega) \to R$ is the bilinear functional

$$(2.5) \quad a(y,z) = \sum_{i,j=1}^{N} \int_\Omega (a_{ij} y_{x_i} z_{x_j} + ayz)dx; \; y,z \in H^1(\Omega).$$

Condition (2.4) can be equivalently defined as

$$(2.6) \quad \frac{d}{dt}(y(t),\psi) + a(y(t),\psi) + \sum_{i=1}^{2} \int_{\Gamma_i} (w_i - v_i)\psi \, d\sigma = 0, \qquad \text{a.e. } t \in]0,T[$$

$$y(o) = y_0$$

for all $\psi \in H^1(\Omega)$.
Here (.,.) is the usual inner product in $L^2(\Omega)$ and will be also used to denote the pairing between $H^1(\Omega)$ and $(H^1(\Omega))'$.

Let ρ be a C_0^∞ - function on R satisfying: $\rho(r) > 0$ for $r \in]-1,1[$, $\rho(r) = 0$ for $|r| > 1$, $\rho(r) = \rho(-r)$ for all $r \in R$ and $\int_{-\infty}^{\infty} \rho(r)dr = 1$. We define for $\varepsilon > 0$,

$$(2.7) \quad \beta_i^\varepsilon(r) = \int_{-\infty}^{\infty} \beta_{i\varepsilon}(r - \varepsilon\theta)\rho(\theta)d\theta \ ; \ i = 1,2, \ r \in R$$

where

$$(2.8) \quad \beta_{i\varepsilon}(r) = \varepsilon^{-1}(r - (1 + \varepsilon\beta_i)^{-1}r) \ , \quad i = 1,2.$$

It should be recalled that β_i^ε are monotonically increasing infinitely differentiable functions. Moreover, β_i^ε is Lipschitzian with Lipschitz constant ε^{-1} and in a certain sense which will be cleared below it approximates β_i for $\varepsilon \to 0$.

For each $\varepsilon > 0$ consider the approximating system

$$(2.9) \quad \begin{aligned} & y_t + Ay = 0 && \text{in } Q \\ & \frac{\partial y}{\partial \nu} + \beta_i^\varepsilon(y) = v_i && \text{in } \Sigma_i \ ; \ i = 1,2 \\ & y(.,0) = y_0 && \text{in } \Omega. \end{aligned}$$

Let $\mathcal{A}_\varepsilon : H^1(\Omega) \to (H^1(\Omega))'$ be the operator defined by

$$(2.10) \quad (\mathcal{A}_\varepsilon y, \psi) = a(y,\psi) + \sum_{i=1}^{2} \int_{\Gamma_i} \beta_i^\varepsilon(y)\psi \, d\sigma \ ; y, \psi \in H^1(\Omega)$$

and let $f \in L^2(0,T;(H^1(\Omega))')$ be given by

$$(2.11) \quad (f(t),\psi) = \sum_{i=1}^{2} \int_{\Gamma_i} v_i \psi \, d\sigma \ , \quad \psi \in H^1(\Omega).$$

Then in the sense of Definition 1 (see (2.6)) equation (2.9) can be written as

$$\frac{dy}{dt} + \mathcal{A}_\varepsilon y = f \qquad t \in [0,T]$$

(2.12)

$$y(o) = y_o \; .$$

Since $\mathcal{A}_\varepsilon$ is continuous monotone, coercive and sublinear from $H^1(\Omega)$ to $(H^1(\Omega))'$, according to a standard existence result due to Lions (see for instance [4] p.64) equation (2.12)(and therefore (2.9)) has a unique solution $y_\varepsilon \in W(Q)$.

Let $j_i: R \longrightarrow R$ $i = 1,2$ be two convex and lower semicontinuous functions such that $\partial j_i = \beta_i$ (it is well known that such functions alwayis exist)

PROPOSITION 1 . Let $y_o \in L^2(\Omega)$ and $v_i \in L^2(\Sigma_i)$ be given such that $j_i(y_o) \in L^1(\Omega)$, $i = 1,2$ .Then system (2.1) has a unique solution $y \in W(Q)$ .Furthermore, one has for $\varepsilon \to 0$

$$(2.13) \quad y_\varepsilon \longrightarrow y \text{ strongly in } C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)$$

and weakly in $W(Q)$.

There exists $C > 0$ independent of ε and v_i such that

$$(2.14) \quad \|y\|_{W(Q)} + \sum_{i=1}^{2} \|\beta_i(y)\|_{L^2(\Sigma_i)} \le C\Big(\sum_{i=1}^{2} \|v_i\|_{L^2(\Sigma_i)} + 1\Big)$$

(If β_i are multivalued we mean by $\beta_i(y)$ whe single valued section w_i which occurs in (2.3))

Proof. We take the inner product of equation (2.12)(where $y = y_\varepsilon$) by y_ε and integrate over $[0,t]$ .By (2.10) and (2.11) it follows that

$$(2.15) \quad \|y_\varepsilon(t)\|_{L^2(\Omega)} + \int_0^t \|y_\varepsilon(s)\|^2_{H^1(\Omega)} ds \le C(\|u_1\|^2_{L^2(\Sigma_1)} +$$

$$+ \|u_2\|^2_{L^2(\Sigma_2)} + 1) , \quad t \in [0,T]$$

where C is independent of ε.

Next, we take the inner product of (2.12) with $\beta_i^\varepsilon(y_\varepsilon)$. Inasmuch as $a(\psi, \beta_i^\varepsilon(\psi)) \geq 0$, for all $\psi \in H^1(\Omega)$, we find after some calculations,

$$(2.16) \quad \int_\Omega j_i^\varepsilon(y_\varepsilon)dx + \sum_{j=1}^{2} \int_{\Sigma_j} (\beta_j^\varepsilon(y_\varepsilon) - v_j)\beta_i^\varepsilon(y_\varepsilon)d\sigma\, dt \leq$$

$$\leq \int_\Omega j_i^\varepsilon(y)dx, \text{ for } i = 1,2$$

where

$$j_i^\varepsilon(r) = \int_0^r \beta_i^\varepsilon(s)ds \quad ; \quad i = 1,2.$$

Along with assumption (2.2), (2.16) yields

$$\sum_{i=1}^{2} \|\beta_i^\varepsilon(y_\varepsilon)\|^2_{L^2(\Sigma_i)} \leq C(\sum_{i=1}^{2} \|v_i\|^2_{L^2(\Sigma_i)} + 1)$$

and by (2.12) and (2.15) we see that

$$(2.17) \quad \|y_\varepsilon\|^2_{W(Q)} + \sum_{i=1}^{2} \|\beta_i^\varepsilon(y_\varepsilon)\|^2_{L^2(\Sigma_i)} \leq C(1 + \sum_{i=1}^{2} \|v_i\|^2_{L^2(\Sigma_i)})$$

where C is independent of ε.

Now using once again equation (2.12) for $\varepsilon, \lambda > 0$ we get

$$\|y_\varepsilon(t) - y_\lambda(t)\|^2_{L^2(\Omega)} + \|y_\varepsilon - y_\lambda\|^2_{L^2(0,T;H^1(\Omega))} +$$

$$+ C \sum_{i=1}^{2} \int_{\Sigma_i} (\beta_i^\varepsilon(y_\varepsilon) - \beta_i^\lambda(y_\lambda))(y_\varepsilon - y_\lambda)d\sigma\, dt \leq 0$$

Taking in account (2.7), (2.8), (2.17) and the monotonicity of β_i the latter implies by a standard procedure, that

$$(2.18) \quad \|y_\varepsilon - y_\lambda\|^2_{C([0,T];L^2(\Omega))} + \|y_\varepsilon - y_\lambda\|^2_{L^2(0,T;H^1(\Omega))} \leq C(\varepsilon + \lambda)$$

Hence $y = \lim_{\varepsilon \to 0} y_\varepsilon$ exists in the strong topology of $L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$. In particular, this implies that

$$y_\varepsilon \longrightarrow y \text{ strongly in } L^2(0,T;H^{1/2}(\Gamma) \subset L^2(\Sigma)$$

and by (2.17) we may assume that

$$(2.19) \qquad \beta_i^\varepsilon(y_\varepsilon) \longrightarrow w_i \quad \text{weakly in } L^2(\Sigma_i),\ i = 1,2$$

According to Definition 1, to prove that y is a solution to (2.1) it suffices to show that

$$(2.20) \qquad w_i \in \beta_i(y) \qquad \text{a.e. on } \Sigma_i\ ;\ i = 1,2\ .$$

To this purpose, we set

$$z_\varepsilon^i = \beta_{i\varepsilon}(y_\varepsilon - \varepsilon\theta)\ .$$

By (2.7) and (2.19) it follows that, on same subsequence $\varepsilon \to 0$ we have

$$(2.21) \qquad z_\varepsilon^i \longrightarrow z^i \quad \text{weakly in } L^2(\Sigma_i \times]-1,1[\ )\ ;\ i = 1,2.$$

On the other hand, since $z_\varepsilon^i \in \beta_i((1 + \varepsilon\beta_i)^{-1}(y_\varepsilon - \varepsilon\theta))$ and by (2.19) $(1 + \varepsilon\beta_i)^{-1}(y_\varepsilon - \varepsilon\theta)$ is strongly convergent to y in $L^2(\Sigma_i \times]-1,1[\ )$ we may infer that

$$z^i(\sigma,t,\theta) \in \beta_i(y(\sigma,t)) \qquad \text{a.e. on } \Sigma_i \times]-1,1[.$$

Along with (2.7) and (2.21) the latter implies (2.20) as claimed. The uniqueness of y is immediate from Definition 1.

To obtain estimates (2.13) and (2.14) we let λ tend to zero in (2.18) and $\varepsilon \to 0$ in (2.17).

Let us denote by $K: L^2(\Sigma_1) \times L^2(\Sigma_1) \longrightarrow W(Q)$ the operator defined by $y = K(v_1,v_2)$ where y is the solution to (2.1)

By K_ε we shall denote the corresponding operator associated with equation (2.9).

PROPOSITION 2 .Under conditions of Proposition 1 the operator K is weakly continuous from $L^2(\Sigma_1)\times L^2(\Sigma_1)$ to $W(Q)$ and compact from $L^2(\Sigma_1)\times L^2(\Sigma_1)$ to $L^2(Q)$.Furthermore,if for $\varepsilon\to 0$ the sequence $\{(v_1^\varepsilon, v_2^\varepsilon)\}$ is weakly convergent in $L^2(\Sigma_1)\times L^2(\Sigma_2)$ to (v_1,v_2) then on some subsequence,again denoted ε , one has

(2.22) $\quad K_\varepsilon(v_1^\varepsilon, v_2^\varepsilon) \longrightarrow K(v_1,v_2)$ weakly in $W(Q)$ and strongly in $L^2(Q)$.

If $(v_1^\varepsilon, v_2^\varepsilon) \longrightarrow (v_1,v_2)$ strongly in $L^2(\Sigma_1)\times L^2(\Sigma_2)$ then

(2.23) $K_\varepsilon(v_1,v_2) \longrightarrow K(v_1,v_2)$ strongly in $C([0,T];L^2(\Omega))$ $L^2(0,T;H^1(\Omega))$.

Proof. Let $\{(v_1^n,v_2^n)\}$ a sequence of $L^2(\Sigma_1)\times L^2(\Sigma_1)$ weakly convergent to (v_1,v_2).By estimate (2.14) it follows that $\{y_n = K(v_1^n, v_2^n)\}$ is weakly compact in $W(Q)$.

Hence on some subsequence again denoted y_n ,we have

(2,24) $\quad y_n \longrightarrow y$ weakly in $W(Q)$ and strongly in $L^2(Q)$.

As a matter of fact,since $\{y_n\}$ is bounded in $L^2(0,T;H^1(\Omega))$ and $\left\{\frac{dy_n}{dt}\right\}$ in $L^2(0,T;(H^1(\Omega))')$ according to a well-known compacity theorem, $\{y_n\}$ is a precompact subset of some $L^2(0,T;H^\delta(\Omega))$ where $1/2<\delta<1$.

Thus by the trace theorem we may conclude that $\{y_n\}$ is precompact in $L^2(\Sigma)$.Hence without no loss of generality we may assume that

(2.25) $\quad y_n \longrightarrow y$ strongly in $L^2(\Sigma)$.

Selecting further subsequence it follows by (2.14) that

(2.26) $\quad \beta_i(y_n) \longrightarrow w_i$ weakly in $L^2(\Sigma_i)$; $i = 1,2$

Since β_i are maximal monotone it follows by (2.25) and (2.26) that $w_i \in \beta_i(y)$ a.e. on $\Sigma_i, i = 1,2$. Along with (2.24) this implies that $y = K(v_1,v_2)$ as claimed.

Now let $\{(v_1^\varepsilon, v_2^\varepsilon)\}$ be such that for $\varepsilon \to 0$

$$(2.27) \qquad v_i^\varepsilon \longrightarrow v_i \quad \text{weakly in } L^2(\Sigma_i) \; ; \; i = 1,2.$$

Then in virtue of estimate (2.17) we may assume that

$$(2.28) \qquad \tilde{y}_\varepsilon = K_\varepsilon (v_1^\varepsilon, v_2^\varepsilon) \longrightarrow z \quad \text{weakly in } W(Q) \text{ and strongly in } L^2(0,T;H^\delta(\Omega)); 1/2 < \delta < 1$$

and

$$(2.29) \qquad \beta_i^\varepsilon(\tilde{y}_\varepsilon) \longrightarrow \tilde{w}_i \quad \text{weakly in } L^2(\Sigma_i); i = 1,2$$

Since the sequence of traces of $\{K_\varepsilon (v_1^\varepsilon, v_2^\varepsilon)\}$ converges strongly in $L^2(\Sigma)$ to the trace of z, arguing as in the proof of Proposition 1 we may infer by (2.29) that $\tilde{w}_i \in \beta_i(z)$ a.e. on $\Sigma_i; i = 1,2$. Hence z is a solution to (2.1) corresponding to v_1, v_2 and therefore $z = K(v_1, v_z)$.

If $(v_1^\varepsilon, v_2^\varepsilon) \longrightarrow (v_1, v_2)$ strongly in $L^2(\Sigma_1) \times L^2(\Sigma_2)$ then arguing as in the Proof of Proposition 1 we deduce (2.23). This completes the proof of Proposition 2.

REMARK It must be emphasized that more general systems of the form

$$y_t + Ay = F \qquad \text{in } Q$$

$$(2.29) \qquad \frac{\partial y}{\partial \nu} + \beta_i(y) \ni v_i^o \qquad \text{in } \Sigma_i; i = 1,2$$

$$y(o) = y_o \qquad \text{in } \Omega$$

where $F \in L^2(Q)$ and $v_i^o \in L^2(\Sigma_i)$ can be put into the form (2.1) where $v_i = v_i^o - \frac{\partial z}{\partial \nu}$ and $z \in H^{2,1}(Q)$ is the solution to

$$(2.30)\quad \begin{aligned} z_t + Az &= F && \text{in } Q \\ z &= 0 && \text{in } \Sigma \\ z(o) &= 0 && \text{in } \Omega . \end{aligned}$$

Since Γ_i are smooth parts of Γ and $\frac{\partial z}{\partial \nu} \in L^2(\Sigma)$ it follows that the restrictions of $\frac{\partial z}{\partial \nu}$ to Γ_i belong to $L^2(\Sigma_i)$ and therefore $v_i \in L^2(\Sigma_i)$, $i = 1,2$. Thus Propositions 1 and 2 are applicable and therefore their conclusions remain true for general systems (2.29).

3. THE MAIN RESULTS

We shall study the following control problem:

Minimize

$$(3.1)\quad \frac{1}{2}\int_Q h(x,t)\,|y(x,t) - y_d(x,t)|^2 dxdt + \psi_1(u_1) + \psi_2(u_2) + \varphi(y(T))$$

on the class of all $u_i \in U_i$, $i = 1,2$ and $y \in W(Q)$ subject to state system (1.1)

We shall assume that the following conditions are satisfied.

1^o U_i, $i = 1,2$ are real Hilbert spaces with norms $\|\cdot\|_i$ and inner products denoted $\langle .,. \rangle_i$.

2^o The functions $\psi_i : U_i \to \bar{R} =]-\infty,+\infty]$, $i = 1,2$, are convex, lower semicontinuous and $\not\equiv +\infty$.

3^o The function $\varphi : L^2(\Omega) \to R$ is convex and continuous on $L^2(\Omega)$.

4^o $h \in L^\infty(Q)$ and $y_d \in L^2(Q)$ are given ; $h \geq o$ a.e. on Q.

As regards the control system (1.1) we shall assume that

5^o A is the elliptic symmetric operator presented in Section 2 and β_i, $i = 1,2$ are two maximal monotone graph in $R \times R$ which satisfy condition (2.2).

6° $B_i: U_i \to L^2(\Sigma_i)$, $i = 1,2$ are linear continuous operators.

7° $f_i \in L^2(\Sigma_i)$, $i = 1,2$ and $y_0 \in L^2(\Omega)$ satisfies the assumptions of Proposition 1.

The solution to (1.1) is meant in the sense of Definition 1 and according to Proposition 1,under our assumptions,for every pair $(u_1,u_2) \in U_1 \times U_2$ the control system (1.1) has a unique solution $y \in W(Q)$.

We shall say that the state $y^* \in W(Q)$ and the controls $u_i^* \in U_i$ $i = 1,2$ are optimal in problem (3.1) if the infimum of functional (3.1) is attained for $y = y^*$ and $u_i = u_i^*$.

The first optimality result is given in the case in which β_i are single valued and satisfy the following condition

8° The functions $\beta_i, i = 1,2$ are monotonically increasing and locally Lipschitzian on real axis R.Moreover,there exists $C > 0$ such that

$$(3.2) \quad \beta_i'(r)r \le C(|\beta_i(r)| + r^2 + 1), \text{ a.e. } r \in R;\ i = 1,2.$$

THEOREM 1 Let $y^* \in W(Q)$ and $(u_1^*,u_2^*) \in U_1 \times U_2$ be optimal in problem (3.1).Assume that conditions 1°~ 8° are satisfied.Then there exists $p \in C_w([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ with $\frac{\partial p}{\partial \nu} \in L^1(\Sigma)$ which satisfies along with y^* and u_1^*, u_2^* the system

$$(3.3) \quad p_t - Ap = h(y^* - y_d) \quad \text{in } Q$$

$$(3.4) \quad \frac{\partial p}{\partial \nu} + \partial\beta_i(y^*)p \ni 0 \quad \text{in } \Sigma_i; i = 1,2$$

$$(3.5) \quad p(T) + \partial\varphi(y^*(T)) \ni 0 \quad \text{in } L^2(\Omega)$$

$$(3.6) \quad B_i^* p_i \in \partial\psi_i(u_i^*) \quad , \quad i = 1,2.$$

Here we have denoted by $B_i^* : L^2(\Sigma_i) \to U_i$ the adjoint of the operator B_i and by $p_i \in L^2(\Sigma_i)$ the restriction of p to Σ_i.We have also denoted by $\partial\psi_i$ and $\partial\varphi$ the subdifferentials of ψ_i, φ and by $\partial\beta_i$ the generalized gradient of β_i(see (1.8)).

The boundary value problem (3.3)~(3.5) must be interpreted in the following weak sense,

$$(3.7) \quad \int_Q p\gamma_t dxdt + \int_0^T a(p,\gamma)dt + \int_{\Sigma_1} \mu_1 \gamma \, d\sigma dt + \int_{\Sigma_2} \mu_2 \gamma \, d\sigma dt + \int_Q h(y^* - y_d)dxdt + \int_\Omega \xi \gamma(x,T)dx = 0$$

for all $\gamma \in W(Q)$ satisfying: $\gamma(x,o) = 0$, a.e. $x \in \Omega$ .Here the functions $\mu_i \in L^2(\Sigma_i)$, i = 1,2 and $\xi \in L^2(\Omega)$ satisfy the equation

$$(3.8) \quad \mu_i(\sigma,t) \in \partial\beta_i(y^*(\sigma,t)) \qquad \text{a.e.}(\sigma,t) \in \Sigma_i; i = 1,2$$

$$(3.9) \quad \xi(x) + \partial\varphi(y^*(\cdot,T))(x) \ni 0 , \quad \text{a.e. } x \in \Omega .$$

It should be emphasized that Theorem 1 covers the main part of the physical problems presented in Introduction.For instance in the case $\Gamma_1 = \Gamma$ and $\beta_1 = \beta$ given by (1.5) (the thermostat-control problem) equation (3.4) becomes

$$\frac{\partial p}{\partial \nu} = \begin{cases} -\alpha_1 p & \text{if} \quad y^* < \theta_1 \\ -[0,\alpha_1] & \text{if} \quad y^* = \theta_1 \\ 0 & \text{if} \quad \theta_1 < y^* < \theta_2 \\ -[0,\alpha_2] & \text{if} \quad y^* = \theta_2 \\ -\alpha_2 p & \text{if} \quad y^* > \theta_2 . \end{cases} \quad \text{in } \Sigma .$$

Now we shall consider the particular case of problem (3.1) where $\Gamma_1 = \Gamma$ and $\beta_1 = \beta$ is given by (1.6).In this case (1.1) reduces to the unilateral problem (see e.g. [7])

$$y_t + Ay = 0 \quad \text{in } Q$$

(3.10) $$y\left(\frac{\partial y}{\partial \nu} - B_1 u_1 - f_1\right) = 0, y \geq 0, \frac{\partial y}{\partial \nu} - B_1 u_1 - f_1 \geq 0 \text{ in } \Sigma$$

$$y(o) = y_o .$$

We shall assume that all conditions $1^o \sim 7^o$ are satisfied (for $i = 1$) and notice that in virtue of 7^o we assume that $y_o(x) \geq 0$, a.e. $x \in \Omega$.

Under these assumptions we shall prove the following optimality theorem

THEOREM 2 Let $y^* \in W(Q)$ and $u_1^* \in U_1$ be optimal in problem (3.1) with state system (3.10). Then there exists $p \in C_w([0,T]; L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ with $\frac{\partial p}{\partial \nu} \in M(\Sigma)$, which satisfies along with y^* and u_1^* the system

(3.11) $$p_t - Ap = h(y^* - y_d) \quad \text{in } Q$$

(3.12) $$\left(\frac{\partial p}{\partial \nu}\right)_a = 0 \quad \text{a.e.} \quad \text{in } \{(\sigma,t) \in \Sigma ; y^*(\sigma,t) > 0\}$$

(3.13) $$p = 0 \quad \text{a.e.} \quad \{(\sigma,t) \in \Sigma ; y^*(\sigma,t) = 0\} \cap \{(\sigma,t) \in \Sigma ; B_1 u_1^* - \frac{\partial y^*}{\partial \nu} - f_1 > 0\}.$$

(3.14) $$p(T) + \partial \varphi(y^*(T)) \ni 0 \quad \text{in } L^2(\Omega)$$

(3.15) $$B_1(\gamma_o p) \in \partial \psi_1(u_1^*) .$$

Here $\left(\frac{\partial p}{\partial \nu}\right)_a$ denotes the absolutely continuous part of the measure $\frac{\partial p}{\partial \nu} \in M(\Sigma)$ and $M(\Sigma)$ in the space of all bounded Radon measures on $\bar{\Sigma}$. In (3.15) we have denoted by $\gamma_o p \in L^2(\Sigma)$ the trace of p at Σ.

Postponing the proofs of these theorems for Sections 5 and 6 we shall discuss now a particular case of Theorem 1. We shall consider the following special case of problem (3.1)

(3.16) $h \equiv 0$; $m(\Gamma_1) > 0$

(3.17) $U_i = L^2(0,T;\Sigma_i)$, $B_i \equiv I$(identity operator), $i = 1,2.$

(3.18) $\psi_i(u_i) = \int_{\Sigma_i} g_i(\sigma, u_i(\sigma,t)) d\sigma dt$, $u_i \in U_i$; $i = 1,2.$

where $g_1 : \Gamma_1 \times R \to \bar{R}$ is defined by

(3.19) $g_1(\sigma, r) = 0$ if $|r| \leq \rho$

$+\infty$ otherwise

and $g_2 : \Gamma_2 \times R \longrightarrow \bar{R}$ is normal convex integrand on $\Gamma_2 \times R$ (see [12]).

In other words, we consider the following control problem:

Minimize

(3.20) $\int_{\Sigma_2} g_2(\sigma, u_2(\sigma,t)) d\sigma dt + \varphi(y(T))$

on all $y \in W(Q)$ and $(u_1,u_2) \in L^2(\Sigma_1) \times L^2(\Sigma_2)$ subject to (1.1) (where U_i and B_i satisfy (3.17)) and to control constraint

(3.21) $|u_1(\sigma,t)| \leq \rho$ a.e. $(\sigma,t) \in \Sigma_1$.

We shall assume that Γ and the coefficients of A are analytic and

(3.22) $0 \in \partial\varphi(y^*(T))$.

COROLAR 1. <u>Let</u> y^* <u>and</u> u_1^*, u_2^* <u>be optimal in problem</u> (3.20). <u>Then</u> u_1^* <u>is a bang-bang control on</u> Σ_1, i.e.,

(3.23) $|u_1^*(\sigma,t)| = \rho$ a.e. $(\sigma,t) \in \Sigma_1$.

<u>Proof</u>. Since Theorem 1 is applicable in the present situation it follows by (3.19) and equation (3.6) for $i = 1$,

(3.24) $u_1^*(\sigma,t) \in \rho \operatorname{sgn} p(\sigma,t)$ a.e. $(\sigma,t) \in \Sigma_1$

where $\operatorname{sgn} r = r/|r|$ for $r \neq 0$ and $\operatorname{sgn} 0 = [-1,1]$.

Let now

$$\Sigma_0 = \{(\sigma,t)\in \Sigma_1;\ p(\sigma,t) = 0\}.$$

By (3.4) we see that $\frac{\partial p}{\partial \nu} = 0$ a.e. on Σ_0. Then by a well-known argument involving Dirichlet series we see that $m(\Sigma_0) = 0$ unless $p \equiv 0$ (m denotes the Lebesgue measure). Since by (3.5) and (3.22) $p \not\equiv 0$ we may infer that $p \neq 0$ a.e. on Σ_1. Then by (3.24) we deduce (3.23) there by completing the proof.

4. THE APPROXIMATING CONTROL PROCESS

Let $y^* \in W(Q)$ and $(u_1^*,u_2^*) \in L^2(\Sigma_1) \times L^2(\Sigma_1)$ be optimal elements in problem (3.1).

For $\varepsilon > 0$ consider the following optimal control problem: Minimize

$$(4.1)\quad \frac{1}{2}\int_Q h\,|y^* - y_d|^2\,dxdt + \sum_{i=1}^{2}\left(\psi_{i\varepsilon}(u_i) + \frac{1}{2}\|u^* - u\|_i^2\right) + \varphi_\varepsilon(y(T))$$

over all $y \in W(Q)$ and $u_i \in U_i, i = 1,2$ subject to state system

$$(4.2)\quad \begin{aligned} &y_t + Ay = 0 && \text{in } Q\\ &\frac{\partial y}{\partial \nu} + \beta_i^\varepsilon(y) = B_i u_i + f_i && \text{in } \Sigma_i;\ i = 1,2\\ &y(o) = y_o. \end{aligned}$$

Here $\psi_{i\varepsilon} : U_i \to R$, $i = 1,2$ and $\varphi_\varepsilon : L^2(\Omega) \to R$ are the convex functions defined by (see e.g. [4] p.107)

$$(4.3)\quad \psi_{i\varepsilon}(u) = \inf\{\|u - v\|_i^2/2\varepsilon + \psi_i(v);\ v \in U_i\},\ i = 1,2$$

$$(4.4)\quad \varphi_\varepsilon(y) = \inf\{\|y - z\|^2_{L^2(\Omega)}/2\varepsilon + \varphi(\ );\ z \in L^2(\Omega)\}.$$

Let now

$$(4.5)\qquad F_\varepsilon(u_1,u_2) = \frac{1}{2}\int_Q h\,|K_\varepsilon(B_1u_1+f_1,B_2u_2+f_2) - y_d|^2\,dxdt + \sum_{i=1}^{2}\left(\psi_{i\varepsilon}(u_i) + \frac{1}{2}\|u_i^* - u_i\|_i^2\right) + \varphi_\varepsilon(K_\varepsilon(B_1u_1+f_1,B_2u_2+f_2)(T))$$

and

$$(4.6)\qquad F(u_1,u_2) = \frac{1}{2}\int_Q h\,|K(B_1u_1+f_1,B_2u_2+f_2) - y_d|^2\,dxdt + \sum_{i=1}^{2}\psi_{i\varepsilon}(u_i) + \varphi(K(B_1u_1+f_1,B_2u_2+f_2)(T))$$

where $K_\varepsilon : L^2(\Sigma_1)\times L^2(\Sigma_2)\longrightarrow W(Q)$ and $K: L^2(\Sigma_1)\times L^2(\Sigma_2)\longrightarrow W(Q)$ have been defined in Section 2.

In terms of F_ε problem (4.1) may be written as

$$(4.1)'\qquad \min\left\{F_\varepsilon(u_1,u_2);\ u_1\in L^2(\Sigma_1),\ u_2\in L^2(\Sigma_2)\right\}$$

while by (3.1) we have

$$(4.7)\qquad F(u_1^*,u_2^*) = \min\left\{F(u_1,u_2);u_1\in L^2(\Sigma_1),u_2\in L^2(\Sigma_2)\right\}.$$

Since the functions $\psi_{i\varepsilon}$ and φ_ε are weakly lower semicontinuous and by Proposition 2 the operator K_ε is weakly con - tinuous, we may infer that the functional F_ε is weakly lower semicontinuous on $L^2(\Sigma_1)\times L^2(\Sigma_2)$. Hence problem (4.1) (equivalently (4.1)') has at least one solution $(y_\varepsilon, u_{1\varepsilon}, u_{2\varepsilon})\in W(Q)\times L^2(\Sigma_1)\times L^2(\Sigma_2)$. On the other hand, since the functions $\psi_{i\varepsilon}$, $i = 1,2$ and φ_ε are Fréchet differentiable on U_i and $L^2(\Omega)$, respectively (see e.g. [4], p.107) it follows by a standard device the existence of some function $p_\varepsilon\in W(Q)$ satisfying along with y_ε and u_i^* the following system (the Euler-Lagrange system associated with problem (4.1))

$$(p_\varepsilon)_t - Ap_\varepsilon = h(y_\varepsilon - y_d) \qquad \text{in } Q$$

$$(4.8) \qquad \frac{\partial p_\varepsilon}{\partial \nu} + (\beta_i^\varepsilon)'(y_\varepsilon)p_\varepsilon = 0 \qquad \text{in } \Sigma_1; i = 1,2$$

$$p_\varepsilon(T) + \partial\varphi_\varepsilon(y_\varepsilon(T)) = 0 \qquad \text{in } \Omega$$

$$(4.9) \qquad B_i^* p_{\varepsilon,i} = \partial\psi_{i\varepsilon}(u_{i\varepsilon}^*) + u_{i\varepsilon} - u_i^* \quad \text{in } \Sigma_i; i = 1,2$$

where $p_{\varepsilon i}$ is the restriction of p_ε to $L^2(\Sigma_i)$.The solution p_ε is meant in the sense of Definition 1 and the symbol $(p_\varepsilon)_t \in L^2(0,T;(H^1(\Omega))')$ is used for the derivative of p_ε in the sense of $(H^1(\Omega))'$ - valued distributions on $]0,T[$.

LEMMA 1 <u>For</u> $\varepsilon \to 0$ <u>one has</u>

$$(4.10) \qquad u_{i\varepsilon} \longrightarrow u_i^* \qquad \text{strongly in } L^2(\Sigma_i); i = 1,2$$

$$(4.11) \qquad y_\varepsilon \longrightarrow y^* \qquad \text{weakly in } W(Q) \text{ and strongly in } L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$$

$$(4.12) \qquad \beta_i^\varepsilon(y_\varepsilon) \longrightarrow f_i - B_i u_i^* - \frac{\partial y^*}{\partial \nu} \text{ weakly in } L^2(\Sigma_i); i = 1,2$$

<u>Proof</u> We have

$$F_\varepsilon(u_{1\varepsilon}, u_{2\varepsilon}) \le \frac{1}{2}\int_Q h\,|z - y_d|^2\,dxdt + \sum_{i=1}^{2} \psi_{i\varepsilon}(u_i^*) + \varphi_\varepsilon(z_\varepsilon(T))$$

where $z_\varepsilon = K_\varepsilon(B_1u_1^* + f_1, B_2u_2^* + f_2)$.

According to Proposition 1,we have

$$(4.13) \qquad z_\varepsilon \longrightarrow y^* \text{ strongly in } C([0,T];L^2(\Omega)).$$

Since $\psi_{i\varepsilon} \le \psi_i$ and $\varphi_\varepsilon \le \varphi$ it follows that

$$(4.14) \qquad \limsup_{\varepsilon \to 0} F_\varepsilon(u_{1\varepsilon}, u_{2\varepsilon}) \le F(u_1^*, u_2^*).$$

In particular it follows that $\{u_{i\varepsilon}\}$ are bounded in $L^2(\Sigma_i); i = 1,2$. Thus without any loss of generality we may assume that

$$(4.15) \qquad u_{i\varepsilon} \longrightarrow \tilde{u}_i^* \qquad \text{weakly in } L^2(\Sigma_i); i = 1,2 .$$

Then according to Proposition 2, we have

$$(4.16)\qquad y_\varepsilon \longrightarrow \tilde{y}^* = K(B_1\tilde{u}_1^* + f_1, B_2\tilde{u}_2^* + f_2) \text{ weakly in } W(Q)$$

and strongly in $L^2(Q)$

On the other hand, since the functions ψ_i and φ are weakly lower semicontinuous, and (see [4] ,p.107)

$$\psi_{i\varepsilon}(u_i) = \varepsilon\|\psi_{i\varepsilon}(u_i)\|_i^2/2\varepsilon + \psi_i((1+\varepsilon\partial\psi_i)^{-1}u_i)$$

$$\varphi_\varepsilon(y) = \varepsilon\|\partial\varphi_\varepsilon(y)\|^2_{L^2(\Omega)}/2\varepsilon + \varphi((1+\varepsilon\partial\varphi)^{-1}y)$$

it follows by (4.15) and (4.16) ,

$$\liminf_{\varepsilon\to 0} \psi_{i\varepsilon}(u_{i\varepsilon}) \geq \psi_i(\tilde{u}_i^*) \quad , \; i = 1,2$$

$$\liminf_{\varepsilon\to 0} \varphi_\varepsilon(y_\varepsilon(T)) \geq \varphi(\tilde{y}^*(T)) \quad .$$

Along with (4.7) and (4.14) the latter imply (4.10). Next by Proposition 2 it follows (4.11) and (4.12) thereby completing the proof.

LEMMA 2 <u>The exists</u> $C > 0$ <u>independent of</u> ε <u>such that</u>

$$(4.17)\qquad \|p_\varepsilon(t)\|_{L^2(\Omega)} + \|p_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq C; t \in [0,T]$$

$$(4.18)\qquad \int_{\Sigma_i} |(\beta_i^\varepsilon)'(y_\varepsilon)p_\varepsilon| \, d\sigma\, dt \leq C \quad ; \; i = 1,2$$

$$(4.19)\qquad \|(p_\varepsilon)_t\|_{L^2(0,T;H^{-1}(\Omega))} \leq C .$$

<u>Proof.</u> Without no loss of generality we may assume that p_ε is a regular solution to (4.8), i.e., $p_\varepsilon \in H^{1,2}(Q)$. Then multiplying equation (4.8) by p_ε and integrating on $Q_t = \Omega \times]t,T[$, it follows by the Green formula ,

$$(4.20)\qquad \frac{1}{2}\|p_\varepsilon(t)\|^2_{L^2(\Omega)} + \int_t^T a(p_\varepsilon, p_\varepsilon)ds \leq \frac{1}{2}\|p_\varepsilon(T)\|^2_{L^2(\Omega)} +$$

$$+ \int_{Q_t} h\,|y_\varepsilon - y_d|\,|p_\varepsilon|\,dxdt, \quad t \in [0,T].$$

Let ζ be a C^1 approximation of the function sgn. We multiply equation (4.8) by $\zeta(p_\varepsilon)$ and integrate over Q. Using once again the Green formula and letting ζ tend to sgn we get

$$(4.21) \quad \sum_{i=1}^{2} \int_{\Sigma_i} |(\beta_i^\varepsilon)'(y_\varepsilon)p_\varepsilon|\,d\sigma\,dt \le \int_Q h\,|y_\varepsilon - y_d|\,dxdt$$

$$+ \int_\Omega |p_\varepsilon(x,T)|\,dx .$$

On the other hand, since $y_\varepsilon(T) \longrightarrow y^*(T)$ in $L^2(\Omega)$ and

$$\|\partial\varphi_\varepsilon(y)\|_{L^2(\Omega)} \le \inf\left\{\|w\|_{L^2(\Omega)} ; w \in \partial\varphi(y)\right\}$$

it follows by (4.8) that $\{p_\varepsilon(T)\}$ is bounded in $L^2(\Omega)$ (Here we have also used the fact that $\partial\varphi$ is locally bounded on $L^2(\Omega)$.) Then by (4.20) and (4.21) we get (4.17). Next by (4.8) we get estimate (4.19) there by completing the proof of Lemma 1.

It follows by (4.17) and (4.18) that $\{p_\varepsilon\}$ is precompact in $L^2(0,T;H^\delta(\Omega))$ where $\frac{1}{2} < \delta < 1$. Hence there exists $p \in L^2(0,T;H^1(\Omega))$ with $p_t \in L^2(0,T;H^{-1}(\Omega))$ such that for some sequence $\varepsilon \longrightarrow 0$, one has

$$(4.22) \quad p_\varepsilon \longrightarrow p \quad \text{weakly in } L^2(0,T;H^1(\Omega))$$

$$\text{strongly in } L^2(0,T;H^\delta(\Omega))$$

$$\text{and weak star in } L^\infty(0,T;L^2(\Omega))$$

$$(4.23) \quad (p_\varepsilon)_t \to p_t \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)) .$$

Here p_t denotes the derivative of p(t) in the sense of $H^{-1}(\Omega)$ valued distributions on $]0,T[$. Then it follows that p(t)) is absolutely continuous from $[0,T]$ to $H^{-1}(\Omega)$ and by (4.22) we see that p(t) is weakly continuous from $[0,T]$ to $L^2(\Omega)$, i.e., $p \in C_w([0,T]\,;\,L^2(\Omega))$.

In particular, we may infer that

(4.24) $\quad p_\varepsilon(t) \longrightarrow p(t) \qquad$ weakly in $L^2(\Omega)$ for every $t \in [0,T]$.

Since $\{\partial\varphi_\varepsilon(y_\varepsilon(T))\}$ is bounded in $L^2(\Omega)$ and $y_\varepsilon(T) \longrightarrow y^*(T)$ strongly in $L^2(\Omega)$, it follows by (4.8) that

$$p(T) + \partial\varphi(y^*(T)) \ni 0 \qquad \text{in } L^2(\Omega).$$

Next by (4.22) and the trace theorem it follows that

(4.25) $\quad p_\varepsilon \longrightarrow p$ strongly in $L^2(\Sigma)$

which along with (4.9) and (4.10) implies

(4.26) $\quad B_i^* p_i \in \partial\psi_i(u_i^*)\ , \ i = 1,2$

where p_i is the restriction of p to Σ_i, $i = 1,2$.

Finally, it follows by (4.18) that there exist two bounded Radon measures $\mu_p^i \in M(\Sigma_i)$ on $\overline{\Sigma}_i$, $i = 1,2$ such that

(4.27) $\quad (\beta_i^\varepsilon)'(y_\varepsilon)p_\varepsilon \longrightarrow \mu_p^i$ weak star in $M(\Sigma_i)$; $i = 1,2$.

Thus letting ε tend to zero in (4.8) we see that p is a solution to

$$p_t - Ap = h(y^* - y_d) \qquad \text{in } Q$$

(4.28) $$\frac{\partial p}{\partial \nu} + \mu_p^i = 0 \qquad \text{in } \Sigma_i;\ i = 1,2$$

$$p(T) + \partial\varphi(y^*(T)) \ni 0 \qquad \text{in } \Omega.$$

Equation (4.28) must be interpreted of course in the following sense (see (3.7))

(4.29) $$\int_Q p\chi_t\,dxdt + \int_0^T a(p,\chi)dt + \sum_{i=1}^{2} \int_{\Sigma_i} \mu_p^i \chi\, d\sigma \ +$$

$$+ \int_Q h(y^* - y_d)\eta\, dxdt + \int_\Omega \zeta\, \eta(x,T)dx = 0,$$

for all $\eta \in C^1(\bar{Q})$ such that $\eta(x,0) = 0$, $x \in \Omega$. Here ζ is an element of $L^2(\Omega)$ satisfying equation (3.9).

Summarising we have proved the following intermediate result.

PROPOSITION 3 Let y^*, u_1^*, u_2^* be optimal in problem (3.1). Then under assumptions $1^\circ \sim 7^\circ$ there exists a function

$$p \in C_w([0,T];\ L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$$

with $p_t \in L^2(0,T;H^{-1}(\Omega))$, which satisfies system (4.28) and equations (4.26). Moreover, p is the limit in the sense of (4.22). (4.23), (4.24), (4.25) and (4.27) of the sequence $\{p_\varepsilon\}$ of solutions to (4.8).

5 PROOF OF THEOREM 1

We begin with a technical result concerning the generalized gradients. Let β be a locally Lipschitzian function on real axis and let β^ε be the function defined by formula (2.7), i.e.,

$$(5.1) \qquad \beta^\varepsilon(r) = \int_{-\infty}^{\infty} \beta_\varepsilon(r - \varepsilon\theta)\,\rho(\theta)d\theta\ ; \quad r \in R,\ \varepsilon > 0$$

where $\beta_\varepsilon = \varepsilon^{-1}(1 - (1+\varepsilon\beta)^{-1})$.

By $\partial\beta$ we shall denote the generalized gradient of β (see (1.8)).

LEMMA 3. Let E be a locally compact space and let ν a positive measure on E such that $\nu(E) < \infty$. Let $\{y_\varepsilon\} \subset L^1(E)$ be a sequence such that for $\varepsilon \to 0$,

$$(5.2) \qquad y_\varepsilon \longrightarrow y \text{ strongly in } L^1(E) \text{ and}$$

$$(5.3) \qquad (\beta^\varepsilon)'(y_\varepsilon) \longrightarrow g \text{ weakly in } L^1(E).$$

Then

(5.4) $$g(x) \in \partial\beta(y(x)) \quad \nu\text{-a.e. } x \in E .$$

Proof. By $L^1(E)$ we have denoted the space of all real-valued ν-measurable functions $y(x)$ defined ν-a.e. on E such that $|y(x)|$ is ν-integrable over E.

Selecting a subsequence of $\{y_\varepsilon\}$ we may assume that

(5.5) $$y_\varepsilon(x) \longrightarrow y(x) \quad \nu\text{-a.e.} \quad x \in E.$$

Next by (5.3) and the Mazur theorem it follows that

(5.6) $$g = \lim_{m\to\infty} g_m \quad \text{strongly in } L^1(E)$$

where $\{g_m\} \subset L^1(E)$ are of the form

(5.7) $$g_m = \sum_{j\in I_m} \alpha_m^j \, (\beta^{\varepsilon_j})'(y_{\varepsilon_j}).$$

Here I_m is a finite subset of natural numbers in the interval $[m, \infty[$ and $\alpha_m^j \geqslant 0$, $\sum_{j\in I_m} \alpha_m^j = 1$.

According to (5.6) we may also assume without any loss of generality that

(5.8) $$g_m(x) \longrightarrow g(x) \quad \nu\text{-a.e.} \quad x \in E.$$

We fix $x \in E$ such that (5.5) and (5.8) hold, and consider a sequence $\{z_n\}$ of real numbers such that $\beta'(z_n)$ exist and $z_n \to y(x)$ for $n \to \infty$. We set $y_j = y_{\varepsilon_j}(x)$ and notice that by (5.1) we have

(5.9) $$(\beta^{\varepsilon_j})'(y_j) = \varepsilon_j^{-1} \int_{-\infty}^{\infty} \beta_{\varepsilon_j}(y_j - \varepsilon_j\theta)\,\rho(\theta)\,d\theta .$$

On the other hand, we have

$$\beta(z_j) = \beta((1 + \varepsilon_j\beta)^{-1}(y_j - \varepsilon_j\theta)) + \beta'(z_j)(z_j -$$

$$- (1 + \varepsilon_j \beta)^{-1}(y_j - \varepsilon_j \theta)) + \omega_j(\theta)(z_j -$$

$$- (1 + \varepsilon_j \beta)^{-1}(y_j - \varepsilon_j \theta))$$

where $\omega_j(\theta) \longrightarrow 0$ for $\delta_j = z_j - (1 + \varepsilon_j \beta)^{-1}(y_j - \varepsilon_j \theta) \longrightarrow 0.$

Along with (5.9) the latter yields

$$(5.10)\quad (\beta^{\varepsilon_j})'(y_j) = \beta'(z_j) - \beta'(z_j) \int_{-\infty}^{\infty} \beta_{\varepsilon_j}(y_j - \varepsilon_j \theta)\, \rho'(\theta)d\theta -$$

$$- \varepsilon^{-1} \int_{-\infty}^{\infty} \omega_j(\theta)(z_j - (1 + \varepsilon_j \beta)^{-1}(y_j - \varepsilon_j \theta))\rho'(\theta)d\theta$$

Since β is locally Lipschitzian, it follows by (5.5) that $\beta_{\varepsilon_j}(y_j - \varepsilon_j \theta) \longrightarrow \beta(y(x))$ uniformly in θ on $[-1,1]$.

On the other hand, z_j can be chosen sufficiently close to y_j in a such a way,

$$|y_j - z_j| / \varepsilon_j \longrightarrow 0 \quad \text{for } j \longrightarrow \infty .$$

Thus $\delta_j \to 0$ for $j \to \infty$ and (5.10) yields

$$|(\beta^{\varepsilon_j})'(y_j) - \beta'(z_j)| \longrightarrow 0 \quad \text{for } j \longrightarrow \infty .$$

Along with (5.7) and definition of $\partial\beta$, the latter yields $g(x) \in \partial\beta(y(x))$ as claimed.

Now we continue the proof of Theorem 1 by observing that by condition (3.2) it follows after some calculations involving (5.1), the estimate

$$(5.11)\qquad (\beta_i^{\varepsilon})'(y)y \le C(|\beta_i^{\varepsilon}(y)| + y^2 + 1); i = 1,2 \; ; y \in R$$

where $C > 0$ is independent of ε.

For each $\varepsilon > 0$ and natural number n, we set

$$E_n^{\varepsilon} = \{ (\sigma,t) \in \Sigma \; ; \; |y_{\varepsilon}(\sigma,t)| \le n \}$$

where y_ε are defined as in Section 4.

Let Σ_0 be an arbitrary measurable subset of Σ_i where i = 1 or 2. By (5.11) we have

$$(5.12)\qquad \int_{\Sigma_0} |p_\varepsilon|\,|(\beta_i^\varepsilon)'(y_\varepsilon)|\,d\sigma\,dt \le \int_{E_n^\varepsilon\cap\Sigma_0} |p_\varepsilon|\,|(\beta_i^\varepsilon)'(y_\varepsilon)|\,d\sigma\,dt + \int_{CE_n^\varepsilon\cap\Sigma_0} |p_\varepsilon|\,|(\beta_i^\varepsilon)'(y_\varepsilon)|\,d\sigma\,dt \le$$

$$C_n \int_{\Sigma_0} |p_\varepsilon|\,d\sigma\,dt + Cn^{-1}\int_{CE_n^\varepsilon\cap\Sigma_0} |p_\varepsilon|\,|\beta_i^\varepsilon(y_\varepsilon)|\,d\sigma\,dt +$$

$$+ C\int_{\Sigma_0} |p_\varepsilon|\,|y_\varepsilon|\,d\sigma\,dt + Cn^{-1}$$

where $CE_n = \Sigma\setminus E_n^\varepsilon$. Inasmuch as by Lemmas 1,2, $\{\beta_i^\varepsilon(y_\varepsilon)\}$ is bounded in $L^2(\Sigma_i)$ and $\{y_\varepsilon\}$, $\{p_\varepsilon\}$ are strongly convergent in $L^2(\Sigma)$, it follows by (5.12) that for each $\eta > 0$ there exists $\delta(\eta)$ such that

$$\int_{\Sigma_0} |p_\varepsilon\,(\beta_i^\varepsilon)'(y_\varepsilon)|\,d\sigma\,dt \le \eta \qquad \text{for } m(\Sigma_0) \le \delta(\eta)$$

(m denotes the Lebesgue measure on Σ). In other words, the family $\left\{\int_{\Sigma_0} p_\varepsilon\,(\beta_i^\varepsilon)'(y_\varepsilon)\,d\sigma\,dt;\ \Sigma_0\subset\Sigma_i\right\}$ is equicontinuous.

Hence by the Dunford-Pettis criterion, the family $\{p_\varepsilon\,(\beta_i^\varepsilon)'(y_\varepsilon)\}$ is weakly compact in $L^1(\Sigma_i)$. Then by (4.27) it follows that $\mu_p^i \in L^1(\Sigma_i)$ and

$$(5.13)\qquad (\beta_i^\varepsilon)'(y_\varepsilon)\,p_\varepsilon \to \mu_p^i \text{ weakly in } L^1(\Sigma_i)\ ;\ i = 1,2 .$$

On the other hand it follows by (4.11) that $\{y_\varepsilon\}$ is convergent in $L^2(0,T;H^{1/2}(\Gamma))$.

Thus selecting a subsequence if necessary we have

(5.14) $$y_\varepsilon(x,t) \longrightarrow y^*(x,t) \quad \text{a.e. } (x,t) \in \Sigma_i; i = 1,2$$

and by Egorov's theorem, for each $\eta > 0$ there exists a measurable subset $E^i_\eta \subset \Sigma_i$ such that $m(\Sigma \setminus E^i_\eta) \le \eta$, y_ε is bounded on E^i_η and

(5.15) $$y_\varepsilon(x,t) \longrightarrow y^*(x,t) \text{ uniformly on } E^i_\eta; \; i = 1,2 .$$

Next, since $\{(\beta^\varepsilon_i)'(y_\varepsilon)\}$ are uniformly bounded on E^i_η we may assume (extracting further subsequence) that

(5.16) $$(\beta^\varepsilon_i)'(y_\varepsilon) \longrightarrow g_i \quad \text{weakly in } L^1(E^i_\eta)$$

(actually weak-star in $L^\infty(E^i_\eta)$). Then by Lemma 3 it follows that

$$g(x,t) \in \partial\beta_i(y^*(x,t)) \quad \text{a.e. } (x,t) \in E^i_\eta; \; i = 1,2.$$

Now by (4.22) and the Egorov theorem we may assume that $p_\varepsilon \longrightarrow p$ uniformly on E^i_η. Along with (5.15) and (5.16) the latter implies that $\mu^i_p = g_i p$ on E^i_η. Hence

$$\mu^i_p(x,t) \in p(x,t)\,\partial\beta_i(y^*(x,t)) \quad \text{a.e. } (x,t) \in E^i_\eta .$$

Since $m(\Sigma_i \setminus E^i_\eta)$ can be made arbitrarily small we may conclude that

$$\mu^i_p(x,t) \in p(x,t)\,\partial\beta_i(y^*(x,t)) \quad \text{a.e. } (x,t) \in \Sigma_i; \; i = 1,2.$$

Thus the conclusions of Theorem 1 follow by Proposition 3.

PROOF OF THEOREM 2

If β is the graph defined by (1.6) then $\beta_\varepsilon(r) = -\varepsilon^{-1}r^-$ for $r \in R$, and

$$\beta^\varepsilon(r) = \varepsilon^{-1}\int_{\varepsilon^{-1}r}^{\infty}(r - \varepsilon\theta)\rho(\theta)\,d\theta \quad , r \in R$$

respectively (we set $\dot{\beta}^{\varepsilon} = (\beta^{\varepsilon})'$)

$$\dot{\beta}^{\varepsilon}(r) = \varepsilon^{-1} \int_{\varepsilon^{-1}r}^{\infty} \rho(\theta)d\theta .$$

Hence

$$(6.1) \quad |y_{\varepsilon} \dot{\beta}^{\varepsilon}(y_{\varepsilon})p_{\varepsilon} - p_{\varepsilon} \beta^{\varepsilon}(y_{\varepsilon})| = |p_{\varepsilon} \int_{\varepsilon^{-1}y_{\varepsilon}}^{\infty} \rho(\theta)d\theta| \leq \varepsilon|\dot{\beta}^{\varepsilon}(y_{\varepsilon})p_{\varepsilon}| .$$

On the other hand, arguing as in [1],[2] we find that

$$(6.2) \quad |p_{\varepsilon} \beta^{\varepsilon}(y_{\varepsilon})| \leq 2\varepsilon |\dot{\beta}^{\varepsilon}(y_{\varepsilon})p_{\varepsilon}| (\zeta_{\varepsilon} + \varepsilon^{-1}|y_{\varepsilon}|\gamma_{\varepsilon}), \quad \text{a.e. on } \Sigma$$

where

$$\zeta_{\varepsilon}(\sigma,t) = \begin{cases} 0 & \text{if } |y_{\varepsilon}(\sigma,t)| > \varepsilon \\ 1 & \text{if } |y_{\varepsilon}(\sigma,t)| \leq \varepsilon \end{cases}$$

and

$$\gamma_{\varepsilon}(\sigma,t) = \begin{cases} 0 & \text{if } y_{\varepsilon}(\sigma,t) > -\varepsilon \\ 1 & \text{if } y_{\varepsilon}(\sigma,t) \leq -\varepsilon. \end{cases}$$

Inasmuch as by (4.12), $\{\beta^{\varepsilon}(y_{\varepsilon})\}$ is bounded in $L^2(\Sigma)$ and by Lemma 2, $\{\dot{\beta}^{\varepsilon}(y_{\varepsilon})p_{\varepsilon}\}$ is bounded in $L^1(\Sigma)$ we see by (6.2) that on some subsequence $\varepsilon \to 0$ we have

$$(6.3) \quad p_{\varepsilon} \beta^{\varepsilon}(y_{\varepsilon}) \longrightarrow 0 \quad \text{a.e. on } \Sigma.$$

On the other hand, we know that $\beta^{\varepsilon}(y_{\varepsilon}) \to u^* - f - \frac{\partial y^*}{\partial \nu}$ weakly in $L^2(\Sigma)$ and $p_{\varepsilon} \to p$ strongly in $L^2(\Sigma)$. This implies that the sequence $\{p_{\varepsilon} \beta^{\varepsilon}(y_{\varepsilon})\}$ is weakly convergent in $L^1(\Sigma)$ to $p(B_1u_1^* - \frac{\partial y^*}{\partial \nu} - f_1)$ and by (6.3) it follows that

$$(6.4) \quad p(B_1u_1^* - \frac{\partial y^*}{\partial \nu} - f_1) = 0 \quad \text{a.e. on } \Sigma$$

and therefore

$$p_\varepsilon \beta^\varepsilon(y_\varepsilon) \longrightarrow 0 \text{ strongly in } L^1(\Sigma).$$

Then by (6.1) we see that

(6.5) $$y_\varepsilon \dot{\beta}^\varepsilon(y_\varepsilon)p_\varepsilon \to 0 \text{ strongly in } L^1(\Sigma).$$

Next by the Egorov theorem, for each $\eta > 0$, $\exists E_\eta$ a measurable subset of Σ such that $m(\Sigma \setminus E_\eta) \leq \eta$, $y_\varepsilon \longrightarrow y^*$ uniformly on E_η and y^* is continuous on E_η. Along with (6.5) the latter yields

(6.6) $$\lim_{\varepsilon \to 0} y^* \dot{\beta}^\varepsilon(y_\varepsilon)p_\varepsilon = 0 \text{ strongly in } L^1(E_\eta).$$

Denote by $E_{\eta,\delta}$ the following subset of Σ

$$E_{\eta,\delta} = \{(\sigma,t) \in E_\eta ; |y^*(\sigma,t)| \geq \delta\}.$$

Next, by Proposition 3 it follows that there exists a measure $\mu \in M(\Sigma)$ such that (see (4.27))

(6.7) $$p_\varepsilon \dot{\beta}^\varepsilon(y_\varepsilon) \longrightarrow \mu \qquad \text{weak-star in } M(\Sigma).$$

Let $\mu = (\mu)_a + (\mu)_s$ be the Lebesgue decomposition of μ into its absolutely continuous part $(\mu)_a$ and the singular part $(\mu)_s$. By (6.6) and (6.7) we see that $\mu = 0$ on $E_{\eta,\delta}$. By definition of singular part we deduce that the supposrt of $(\mu)_s$ is concentrated in $E_\eta \cap \{(\sigma,t) \in \Sigma ; y^*(\sigma,t) > 0\}$. Since $m(\Sigma \setminus E_\eta) \longrightarrow 0$ for $\eta \longrightarrow 0$ we may conclude that

$$(\mu)_a = 0 \text{ on } \{(\sigma,t); y^*(\sigma,t) > 0\}.$$

Along with equations (4.26), (4.28) and (6.4) the latter completes the proof of Theorem 2.

<u>REMARK</u> Let us consider problem (3.1) with state system

$$y_t + Ay = 0 \qquad \text{in } Q$$

(6.8) $$y\left(\frac{\partial y}{\partial \nu} + \beta_0(y) - B_1u_1 - f_1\right) = 0 \,; y \geq 0, \frac{\partial y}{\partial \nu} +$$

$$+\beta_o(y) - B_1u_1 - f_1 \geq 0 \quad \text{in } \Sigma .$$

$$y(o) = y_o$$

where $u_1 \in U_1$ and A ,$B_1: U_1 \longrightarrow L^2(\Sigma)$, f_1,y_o satisfy conditions $1^o \sim 7^o$ .Here β_o is a differentiable,monotonically increasing and Lipschitzian function on R.

System (6.8) can be written in the form (1.1) where $\Gamma_1 = \Gamma$ and

$$(6.9) \quad \beta_1(r) = \begin{cases} \beta_o(r) & \text{if } r > 0 \\ ]-\infty,0] & \text{if } r = 0 \\ \phi & \text{if } r < 0 . \end{cases}$$

The prototype of this problem is the enzyme diffusion problem (1.4)

In order to obtain necessary conditions for optimality in this case,is more convenient to replace the approximating system (4.2) by

$$(6.10) \quad \begin{aligned} & y_t + Ay = 0 && \text{in } Q \\ & \frac{\partial y}{\partial \nu} + \beta_o(y) + \beta^{\varepsilon}(y) = B_1u_1 + f_1 && \text{in } \Sigma \\ & y\,o) = y_o . \end{aligned}$$

Obviously,Lemmas 1,2 as well as Proposition 3 remain valid and p_ε is in this case the solution to

$$(6.11) \quad \begin{aligned} & (p_\varepsilon)_t - Ap_\varepsilon = h(y_\varepsilon - y_d) && \text{in } Q \\ & \frac{\partial p_\varepsilon}{\partial \nu} + \beta_o'(y_\varepsilon)p_\varepsilon + (\beta^\varepsilon)'(y_\varepsilon)p_\varepsilon = 0 && \text{in } \Sigma \\ & p_\varepsilon(T) + \partial\varphi_\varepsilon(y_\varepsilon(T)) = 0. \end{aligned}$$

Then passing to limit it follows from preceeding proof that p is the solution to

$$(6.12) \quad \begin{aligned} & p_t - Ap = h(y^* - y_d) && \text{in } Q \\ & \left(\frac{\partial p}{\partial \nu}\right)_a + \beta_o'(y^*)p = 0 && \text{a.e.in} \{y^* > 0\} \\ & p = 0 && \text{in } \{y^* = 0\} \cap \{B_1u_1^* - \frac{\partial y^*}{\partial \nu} - \beta_o'(y^*)p - f_1 > 0\} \end{aligned}$$

still keeping equations (3.14) and(3.15).

7. SOME APPLICATIONS IN CONTROLLABILITY

Consider the control system

$$(7.1)\quad \begin{cases} y_t + Ay = 0 & \text{in } Q = \Omega \times]0,T[\\ \frac{\partial y}{\partial \nu} + \beta_1(y) \ni B_1 u & \text{in } \Sigma_1 = \Gamma_1 \times]0,T[\\ \frac{\partial y}{\partial \nu} + \beta_2(y) \ni 0 & \text{in } \Sigma_2 = \Gamma_2 \times]0,T[\\ y(x,0) = y_o(x) & x \in \Omega \end{cases}$$

where A is a linear second order,simmetric and elliptic differential operator on Ω and $\beta_i, i = 1,2$ are two maximal monotone graphs in $R \times R$ satisfying condition (2.2) .

Here B_1 is a linear continuous operator from the Hilbert space U_1 to $L^2(\Sigma_1)$, $y_o \in L^2(\Omega)$ and $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 are two disjoint and smooth parts of Γ .Further we shall assume that the boundary Γ as well as the coefficients of A are analytic.

As seen in Section 2,under these assumptions for every $u \in U_1$, (7.1) has a unique solution $y \in W(Q)$.

THEOREM 3 <u>Let system</u> (7.1) <u>satisfy the above conditions.In addition,assume that</u> Γ_1 <u>is open in</u> Γ <u>and the range of</u> B_1 <u>is dense in</u> $L^2(\Sigma_1)$.<u>Then system</u> (7.1) <u>is weakly controlable</u> ,i.e.,<u>for each</u> $y_1 \in L^2(\Omega)$ <u>there exists a sequence</u> $\{u_n\} \subset U_1$ <u>such that</u>

$$(7.3)\quad y_n(T) \longrightarrow y_1 \quad \text{weakly in } L^2(\Omega) \text{ and strongly in } H^{-1}(\Omega).$$

<u>Here</u> y_n <u>is the solution to</u> (7.1) <u>where</u> $u = u_n$.

<u>Proof</u>. Let y_1 be arbitrary but fixed in $L^2(\Omega)$ .Consider the following optimal control problem.Minimize

$$(7.4)\quad \frac{1}{2}\left(\|y(T) - y_1\|^2_{L^2(\Omega)} + \lambda \|u\|^2_1 \right)$$

on all of $u \in U_1$ and $y \in W(Q)$ subject to (7.1) This is a special case of problem (3.1) where $h \equiv 0$,

$$\psi_1(u) = \lambda \| u \|_1^2 /2 \quad \text{and} \quad \varphi(y) = \frac{1}{2} \| y - y_1 \|^2_{L^2(\Omega)}.$$

Clearly (7.4) has at least one solution (u_λ, y_λ).

By Proposition 3 it follows that there exists $p_\lambda \in L^2(0,T;H^1(\Omega)) \cap C_w([0,T];L^2(\Omega))$ with $(p_\lambda)_t \in L^2(0,T;H^{-1}(\Omega))$ such that

$$\text{(7.4)} \quad \begin{aligned} &(p_\lambda)_t - Ap_\lambda = 0 && \text{in } Q \\ &\frac{\partial p_\lambda}{\partial \nu} + \mu_\lambda^i = 0 && \text{in } \Sigma_i;\ i = 1,2 \end{aligned}$$

$$\text{(7.4)} \quad p_\lambda(T) + y_\lambda(T) - y_1 = 0 \qquad \text{in } L^2(\Omega)$$

$$\text{(7.6)} \quad B_1^* p_{\lambda,1} = \lambda u_\lambda \qquad \text{in } L^2(\Sigma_1)$$

where $\mu_\lambda^i \in M(\Sigma_i)$ $i=1,2$.

In other words, we have

$$\text{(7.7)} \quad \int_Q p_\lambda \gamma_t \, dxdt + \int_0^T a(p, \gamma)dt + \mu_\lambda^1(\gamma) + \mu_\lambda^2(\gamma)) + $$

$$+ \int_\Omega p(x,T)\gamma(x,T)dt = 0$$

for all $\gamma \in C^1(\bar{Q})$ such that $\gamma(x,0) = 0$, $x \in \Omega$. Here $\mu_\lambda^i(\gamma)$ denotes the value of μ_λ^i at the trace of γ on Σ_i.

Next by (7.3) it follows that

$$\| y_\lambda(T) - y_1 \|^2_{L^2(\Omega)} + \lambda \| u_\lambda \|_1^2 \le C$$

and therefore

$$\text{(7.8)} \quad \lambda u_\lambda \longrightarrow 0 \quad \text{strongly in } L^2(\Sigma_1)$$

On the other hand, p_λ is the limit in the sense of (4.22), (4.25), (4.27) of a sequence $\{p_{\lambda,\varepsilon}\}_{\varepsilon > 0} \subset W(Q)$ satisfying

(4.8) and (4.9) where $h = 0$ and ψ_1, ψ are defined as above. Then it follows by estimates (4.20),(4.21) that

$$(7.9) \quad \|p_\lambda(t)\|_{L^2(\Omega)} + \|p_\lambda\|_{L^2(0,T;H^1(\Omega))} + \|(p_\lambda)_t\|_{L^2(0,T;H^{-1}(\Omega))} \leq C, \quad t \in [0,T].$$

and by (4.27)

$$(7.10) \quad \|\mu_\lambda^i\|_{M(\Sigma_i)} \leq C \qquad i = 1,2; \quad \lambda > 0.$$

Hence $\{p_\lambda\}$ is strongly compact in $L^2(0,T;H^\delta(\Omega)) \cap C([0,T]; H^{-1}(\Omega))$; $0 < \delta < 1$. Thus there exists a function p such that

$$(7.11) \quad p_\lambda \longrightarrow p \quad \text{strongly in } L^2(0,T;H^\delta(\Omega)) \cap C([0,T];H^{-1}(\Omega))$$

$$\text{and weakly in } L^2(0,T;H^1(\Omega))$$

$$(7.12) \quad \mu_\lambda^i \longrightarrow \mu^i \quad \text{weak star in } M(\Sigma_i);\ i = 1,2.$$

By (7.9) and (7.11) it follows that

$$(7.13) \quad p_\lambda(t) \longrightarrow p(t) \quad \text{weakly in } L^2(\Omega) \text{ for every } t \in [0,T]$$

Letting $\lambda \to 0$ in (7.3) we see that p is the solution to

$$(7.14) \quad p_t - Ap = 0 \qquad \text{in } Q$$

$$\frac{\partial p}{\partial \nu} + \mu^i = 0 \qquad \text{in } \Sigma_i;\ i = 1,2$$

$$(7.15) \quad B_1^* p_1 = 0$$

where $p_1 \in L^2(0,T;H^{1/2}(\Gamma_1))$ is the restriction of p to Σ_1.

Since the range of B_1 is dense in $L^2(\Sigma_1)$ we may infer by (7.15) that $p_1 = 0$, i.e., (7.16) $\quad p = 0 \quad$ a.e. on Σ_1.

Next it follows by (7.11),(7.12) and (7.16) that

$$\mu^1(\varkappa) = 0$$

for all $\varkappa \in C^1(\bar{Q})$ such that $\varkappa(x,0) = \varkappa(x,T) = 0$ for $x \in \Omega$ and $\varkappa(\sigma,t) = 0$ for $(\sigma,t) \in \Sigma_2$. Hence $\mu^1 = 0$ on $\Gamma_1 \times]0,T[$.

We have therefore proved that

$$p_t - Ap = 0 \quad \text{in } Q$$

$$\frac{\partial p}{\partial \nu} = 0 \quad \text{in } \Gamma_1 \times]0,T[$$

$$p = 0 \quad \text{in } \Gamma_1 \times]0,T[.$$

Since $m(\Gamma_1) > 0$ it follows by a result due to Mizohata already quoted that $p = 0$. Thus the conclusions of Theorem 3 follow by (7.5),(7.11) and (7.13).

REMARK. The above problem has been studied by different methods by Henry [9] who has shown in particular that if $\beta_2 = 0$ and β_1 is continuous then (7.1) is strongly controlable in $L^2(\Omega)$. It is tempting to hope that the same might be true under present assumptions.

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