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ON PERIODIC DISTRIBUTION GROUPS

Ioana Cioránescu

We give a spectral characterization of the infinitesimal generator of a periodic distribution group generalizing some results of Hram Bart [1] on periodic groups of class (C_o).

1. INTRODUCTION

Let X be a Banach space and A a closed and densely defined operator on X; then A is said to be *well-posed for the abstract Cauchy problem* in the sense of distributions if there exists $E\varepsilon$ $\varepsilon L(\mathcal{D};L(X))$ satisfying the following conditions:

(i) $\operatorname{supp}E\subset[0,+)$;

(ii) $E'-AE=\delta\otimes I_x$; $E'-EA=\delta\otimes I_{D(A)}$

where $\mathcal{D}$ is the test functions space of L. Schwartz, E' is the derivative of E, I_x and $I_{D(A)}$ are the identities on X and on the domain D(A) of A, respectively.

Following J.L. Lions we shall call E in the above definition a *distribution semi-group* and A its *infinitesimal generator* [8].

An $L(X)$-valued distribution E is called a *distribution group* if

(a) $E(\varphi*\psi)=E(\varphi)\ E(\psi)$, for every $\varphi,\psi\varepsilon\mathcal{D}$;

(b) $E=E_++E_-$ where E_+ and $\check{E}_-$ are distribution semigroups (where $\check{E}$ is defined by $\check{E}(\varphi)=E(\check{\varphi})$, $\varphi\varepsilon\mathcal{D}$ and $\check{\varphi}(t)=\varphi(-t)$).

A distribution group E is called *tempered* if E_+, $E\varepsilon L(S;L(X))$, S being the space of rapidly decreasing test functions.

By a result of Lions [8] the generator A of a tempered distribution group has purely imaginary spectrum; a complete characterization of the generator of a tempered distribution group was given in [4], namely we have:

THEOREM 1.1. *A densely defined and closed operator* A *with purely imaginary spectrum is the generator of a tempered distribution group if and only if there are* $n_o, m_o \varepsilon \mathbb{N}$ *such that*

(1.1) $$||R(\lambda;A)|| \leq \text{const.}(1+|\lambda|)^{n_o}|\text{Re}\lambda|^{-m_o} \quad \text{for } \text{Re}\lambda \neq 0.$$

Moreover we have

(1.2) $$R(\lambda;A)=\begin{cases} E_+(e^{-\lambda t}) & \text{for } \text{Re}\lambda>0 \\ -E_-(e^{-\lambda t}) & \text{for } \text{Re}\lambda<0 \end{cases}$$

and

(1.3) $$E(\varphi)=\frac{1}{2\pi}\lim_{\varepsilon\to 0_+}\int_{-\infty}^{+\infty}[R(\varepsilon+it;A)-R(-\varepsilon+it;A)]\hat{\varphi}(t)dt, \quad \varphi\varepsilon\mathcal{D} \quad (\textit{where}$$

$$\hat{\varphi}(t)=\int_{-\infty}^{+\infty}e^{ist}\varphi(s)ds).$$

Let E be a tempered distribution group and consider

$$\sigma=\{E(\varphi);\ \varphi\varepsilon\mathcal{D}\} \text{ and } R=\{R(\lambda;A);\ \text{Re}\lambda\neq 0\}.$$

Denoting by B^c the commutant of a set $B\subset L(X)$ we can easily get from (1.2) and (1.3) that

(1.4) $$\sigma^c=R^c \text{ and } \sigma^{cc}=R^{cc}.$$

Let us put $B=\sigma^{cc}=R^{cc}$; then B is a strongly closed subalgebra of $L(X)$ containing the identity, $B\supset\sigma\cup R$ and the spectrum of each $B\varepsilon B$ with respect to B coincides with $\sigma(B)$.

Let $M=\{m\}$ be the set of maximal ideals of B and $B\to B(m)$ the Gelfand representation of B in the space $C(M)$ of continuous functions on M; then for $B\varepsilon B$, $\sigma(B)=B(M)=\{B(m);\ m\varepsilon M\}$.

As $B\subset R$, by a well-known result [7], there are $M_1, M_2\subset M$, such that

$$M=M_1\cup M_2, \quad M_1\cap M_2=\emptyset$$

and a function $\alpha\varepsilon C(M)$ such that

(1.5) $$R(\lambda;A)(m)=\begin{cases} (\lambda-\alpha(m))^{-1}, & m\varepsilon M_1 \\ 0, & m\varepsilon M_2 \end{cases}, \quad \text{Re}\lambda\neq 0$$

and $\sigma(A)=\alpha(M_1)$.

Let us put $A_n=nAR(n;A)$, $n\varepsilon\mathbb{N}$; there $A_n\varepsilon L(X)$, $\lim\limits_{n\to\infty} R(\lambda;A_n)=$

$R(\lambda;A)$ and by a result of H.Fattorini [5]:

$$(1.6)\qquad E(\varphi)=\lim_{n\to+\infty}\int_{-\infty}^{+\infty} e^{tA_n}\varphi(t)dt, \qquad \varphi\in\mathcal{D}.$$

Then putting $\alpha_n(m)=A_n(m)$, $m\in M$, it is clear by (1.5) that $\alpha_n(m)\to\alpha(m)$ $(n\to\infty)$ uniformly on M_1 and using (1.6) we finally get

$$(1.7)\qquad E(\varphi)(m)=\int_{-\infty}^{+\infty} e^{t\alpha(m)}\varphi(t)dt, \qquad \varphi\in\mathcal{D}\ ,\ m\in M_1\ .$$

Let us remark that the relation (1.7) is valid for arbitrary distribution semi-groups (see [2]).

2. THE SPECTRAL CHARACTERIZATION OF PERIODIC DISTRIBUTION GROUPS

Let X be a Banach space and E an $L(X)$-valued distribution; we say that E is:

(α) *periodic* if $E(\varphi)=E(\tau_T\varphi)$, for some $T>0$ and every $\varphi\in\mathcal{D}$, where $\tau_T\varphi(t)=\varphi(t-T)$;

(β) *strongly periodic* if for every $x\in X$, the X-valued distribution E_x defined by $E_x(\varphi)=E(\varphi)x$, $\varphi\in\mathcal{D}$, is periodic;

(γ) *weakly periodic* if for every $x^*\in X^*$ and $x\in X$, the scalar distribution x^*Ex defined by $x^*Ex(\varphi)=x^*E(\varphi)x$, $\varphi\in\mathcal{D}$, is periodic.

Let us denote by P_T the space of infinitely differentiable functions of period $T>0$ and let us call a T-*unitary function* a function $\xi\in\mathcal{D}$ such that $\sum_{-\infty}^{+\infty}\xi(t-nT)=1$, $t\in\mathbb{R}$. Then each periodic distribution E can be extended to the space P_T by the formula $E(\Theta)=E(\xi\Theta)$, $\Theta\in P_T$, independently of the unitary function ξ (see [9]).

We have the following essential result which can be proved exactly as in the scalar case [9]:

$E\in L(\mathcal{D};L(X))$ is periodic of period $T>0$ *if and only if*

$$(2.1)\qquad E=\sum_{-\infty}^{+\infty} A_n e^{in\omega t},\ \omega=\frac{2\pi}{T}$$

where the convergence holds in $L(S;L(X))$ *and*

$$(2.2)\qquad A_n=\frac{1}{T}E(e^{-in\omega t}),\qquad n\in\mathbb{Z}$$

are in $L(X)$ *such that the sequence* $\{\|A_n\|\}_{n\in\mathbb{Z}}$ *is of slow growth* *(that is* $||A_n|| \leq \text{const.}|n|^k$, *for a given* $k \in \mathbb{N}$).

The series (2.1) is called *the Fourier series of* E and the operators A_n given by (2.2) are called *the Fourier coefficients of* E.

It is clear that each periodic vector-valued distribution is tempered.

In the case of $L(X)$-valued functions the above three notions of periodicity are equivalent, as was proved in [1], Theorem 2.1. Using the Fourier expansion (2.1) and a similar argument as in [1], we obtain the following result:

PROPOSITION 2.1. *Let* $E \in L(\mathcal{D}; L(X))$; *then the following three statements are equivalent:*

(α) *E is periodic;*

(β) *E is strongly periodic;*

(γ) *E is weakly periodic.*

Further we restrict ourselves for simplicity to the case when E has period 2π. Our main result is:

THEOREM 2.2. *A closed and densely defined operator* A *is the generator of a periodic distribution group of period* 2π *if and only if*

a) $\sigma(A) \subset i\mathbb{Z}$ *and consists of poles of order* $\leq m_o$ *of the resolvent which satisfy*

$$||R(\lambda;A)|| \leq \text{const.}(1+|\lambda|)^{n_o}, \qquad \mathrm{Re}\lambda > \varepsilon_o$$

for some $n_o, m_o \in \mathbb{N}$, $\varepsilon_o > 0$.

b) *the set of eigenvectors of* A *spans a dense subspace in* X.

PROOF. *Necessity*. Let A be the generator of the periodic distribution group E, of period 2π. Then using (1.7) and the periodicity of E, we have:

$$(2.3) \qquad \int_{-\infty}^{+\infty} e^{t\alpha(m)} \varphi(t)dt = \int_{-\infty}^{+\infty} e^{(t+2\pi)\alpha(m)} \varphi(t)dt, \qquad \varphi \in \mathcal{D}, \ m \in M_1.$$

For each $m \in M$ there is $\varphi_m \in \mathcal{D}$ with $\int_{-\infty}^{+\infty} e^{t\alpha(m)} \varphi(t)dt \neq 0$ such that (2.3) gives $e^{2\pi\alpha(m)} = 1$, $\forall m \in M_1$, that is $\alpha(m) = ki$, $k \in \mathbb{Z}$, $m \in M_1$. As $\sigma(A) = \alpha(M_1)$, the first part from (a) results. The second part is a consequence of Theorem 1.1.

In order to prove the necessity of (b), let us recall the following result of D.Fujiwara [6]:

If A is the generator of a tempered distribution group E *then denoting by* $D_\infty=\bigcap_{n=0}^{\infty} D(A^n)$*, endowed with the Fréchet topology given by the norms* $\{\|A^n x\|\}_{n\in\mathbb{N}}$*, the restriction* $A|D_\infty$ *generates an equi-continuous group* $\{T_t\}_{t\in\mathbb{R}}$ *in* $L(D_\infty)$*; moreover, we have*

$$E(\varphi)x=\int_{-\infty}^{+\infty}\varphi(t)T_t x\,dt, \qquad \varphi\in\mathcal{D},\quad x\in D_\infty. \tag{2.4}$$

Then for $\lambda\in\mathbb{C}$ and $x\in D_\infty$ we put (as in [1]):

$$B_{\lambda,t}x=e^{\lambda t}\int_0^t e^{-\lambda s}T_s x\,ds.$$

A simple computation gives

$$(\lambda-A)B_{\lambda,t}x=e^{\lambda t}x-T_t x$$

that is

$$(\lambda-A)B_{\lambda,2\pi}x=(e^{2\pi\lambda}-1)x, \qquad x\in D_\infty.$$

Hence for $x\in D_\infty$ and λ outside $i\mathbb{Z}$, we have

$$R(\lambda;A)x=B_{\lambda,2\pi}/e^{2\pi\lambda}-1. \tag{2.5}$$

The above relation shows that on D_∞ the resolvent has simple poles at each $\lambda=mi$, $m\in\mathbb{Z}$. For $m\in\mathbb{Z}$, let P_m be the residue of $R(\lambda;A)$ at mi; it is well known that P_m is a non-zero projection called *the spectral projection associated with* mi *and* A.

From (2.5) we immediately get

$$P_m x=\frac{1}{2\pi}\int_0^{2\pi} e^{-mit}T_t x\,dt, \qquad x\in D_\infty.$$

Let ξ be a 2π-unitary function; then:

$$P_m x=\frac{1}{2\pi}\int_0^{2\pi}\sum_{-\infty}^{+\infty}\xi(t-2n\pi)e^{-mit}T_t x\,dt=\frac{1}{2\pi}\sum_{-\infty}^{+\infty}\int_n^{n+1}\xi(s)e^{-mis}T_s x\,ds=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\xi(s)e^{-mis}T_s x\,ds, \qquad x\in D_\infty$$

so that by (2.4) we get:

$$P_m x = \mathcal{E}(e^{-mit})x, \quad x \in D_\infty.$$

But D_∞ is dense in X, hence

$$(2.6) \qquad P_m = \mathcal{E}(e^{-imt}).$$

Moreover, for each $x \in D_\infty$

$$T_t x = \sum_m e^{imt} P_m x, \quad t \in \mathbb{R}.$$

Taking $t=2\pi$, we get:

$$(2.7) \qquad x = \sum_m P_m x, \qquad x \in D_\infty.$$

Let us further denote by R(B), respectively N(B) the image, respectively the null space of the operator B. Then it is clear that $R(P_m | D_\infty) = R(P_m)$ and $N(mi-A) \subset D_\infty$, $\forall\, m \in \mathbb{Z}$. Thus a simple argument shows that $R(P_m) = N(mi-A)$ and so part (b) of the necessity follows from (2.7).

Sufficiency. By (a) it is clear that A generates a tempered distribution group $\mathcal{E}$. Take $x \in N(mi-A)$, $m \in \mathbb{Z}$; then $x \in D_\infty$ and clearly $T_t x = e^{mit}x$. By (2.4), we have:

$$\mathcal{E}(\varphi)x = \int_{-\infty}^{+\infty} e^{imt}\varphi(t)x\,dt, \quad \varphi \in \mathcal{D},\ x \in N(mi-A).$$

This means that $\mathcal{E}x$ is 2π-periodic for $x \in \bigcup_{m \in \mathbb{Z}} N(mi-A)$ and condition (b) implies the desired conclusion.

From the above proof, we see that also the following holds:

COROLLARY. *Let $\mathcal{E}$ be a periodic distribution group (of period 2π) and P_m the m-the Fourier coefficient of $\mathcal{E}$; then:*

(i) *P_m is a projection and coincide with the residue of $R(\lambda;A)$ at the point mi;*

(ii) $\sum_{m=-\infty}^{+\infty} P_m x = x$, $\forall\, x \in D_\infty$.

REMARK. The above theorem and corollary generalize Theorem 3.1 from [1]: in the case of periodic groups of class (C_o), $\sigma(A)$ consists of simple poles of $R(\lambda;A)$ at $\lambda=mi$, $m \in \mathbb{Z}$ and $\sum_{m=-\infty}^{+\infty} P_m x = x$

for all $x \in D(A)$, P_m being the residue of $R(\lambda;A)$ at mi.

3. AN EXEMPLE OF A PERIODIC DISTRIBUTION GROUP

Let $\sigma(t)=\frac{3\pi-t}{2}$ for $0\le t<2\pi$ and extended with period 2π on all $\mathbb{R}$; then

1) $\sigma(t)>0, \ \forall\, t\in\mathbb{R}$
2) σ is continuous on each interval $(2n\pi, 2(n+1)\pi)$
3) $\sigma\in L^\infty(\mathbb{R})$
4) the function $1/\sigma$ is periodic and has the same above three properties.

Let $C_{2\pi}$ be the Banach space of all bounded functions on $\mathbb{R}$, which are periodic with period 2π and continuous on each interval of the form $(2n\pi, 2(n+1)\pi)$, $n\in\mathbb{Z}$, endowed with the usual supremum norm. We have

PROPOSITION 3.1. *The map defined by*

$$\text{(3.1)} \qquad E(\varphi)f=\frac{\varphi*\sigma f}{\sigma}, \quad \varphi\in\mathcal{D},\ f\in C_{2\pi}$$

is a periodic distribution group in $L(C_{2\pi})$ *with generator*

$$\text{(3.2)} \qquad \begin{cases} Af=-\frac{d}{dt}(\sigma f)/\sigma \\ D(A)=\{f\in C_{2\pi}\ ,\ d/dt(\sigma f)\in C_{2\pi}\}. \end{cases}$$

PROOF. One can easily verify that $E\in L(\mathcal{D};L(C_{2\pi}))$ and that E is periodic; moreover $E(\varphi*\psi)=E(\varphi)E(\psi)$, $\varphi,\psi\in\mathcal{D}$. For $\varphi\in\mathcal{D}$, let us denote by

$$\varphi_+(t)=\begin{cases}\varphi(t) & t>0\\ 0 & t<0\end{cases} \quad\text{and}\quad \varphi_-(t)=\begin{cases}0 & t>0\\ \varphi(t) & t<0.\end{cases}$$

Then putting $E_+(\varphi)f=(\varphi_+*\sigma f)/\sigma$ and

$E_-(\varphi)f=(\varphi_-*\sigma f)/\sigma$ for $\varphi\in\mathcal{D}$, $f\in C_{2\pi}$, $E=E_++E_-$,

a simple computation shows that E_+ and $\check{E}_-$ are distribution semi-groups with generator A, respectively -A, defined by (3.2) (let us remark that A is closed and D(A) is dense in $C_{2\pi}$).

Let us put:

$$\text{(3.3)} \qquad (R(\lambda;A)f)(s)=\frac{1}{\sigma(s)}\int_0^{+\infty} e^{-\lambda t}\sigma(s-t)f(s-t)\,dt, \quad \mathrm{Re}\lambda>0,$$

and let us estimate the right hand side of (3.3).

We recall that for $0<t<2\pi$ holds

(3.4) $$\sigma(\pi)=\pi+\sin t+\frac{\sin 2t}{2}+\frac{\sin 3t}{3}+\dots,$$

and that the equality (3.4) is valid a.e. on $\mathbb{R}$ and the convergence holds in $\mathcal{D}'$.

A simple computation gives

$$\int_0^{+\infty} e^{-\lambda t}\sin n(s-t)\,dt=\frac{\lambda\sin ns-n\cos ns}{\lambda^2+n^2}$$

and taking into account the positivity of σ and the relation (3.4), we obtain, for $\mathrm{Re}\lambda>0$

$$\|R(\lambda;A)f\|\le\mathrm{const.}\frac{|\lambda|+1}{\mathrm{Re}\lambda^2}\|f\|.$$

(We used the formula $\sum_{n=1}^{\infty}\frac{\cos ns}{\alpha^2+n^2}=\frac{\pi}{2\alpha}\frac{\mathrm{ch}\,\alpha(\pi-s)}{\mathrm{sh}\,\alpha\pi}-\frac{1}{2\alpha^2}$, $0<s<2\pi$).

A similar estimation holds for $\mathrm{Re}\lambda<0$ and this implies that the distribution group E is not usual, that is it does not coincide with a group of continuous operators in $L(C_{2\pi})$. This follows also from the fact that the group $\{T_t\}_{t\in\mathbb{R}}$ generated by A on D_∞ is given by:

(3.5) $$(T_t f)(s)=\frac{\sigma(s-t)f(s-t)}{\sigma(s)},\quad f\in C_{2\pi}$$

and it is clear that $D(T_t)\neq C_{2\pi}$.

Let us finally remark that choosing the function σ in a convenient way, many other periodic distribution groups can be constructed as above; by (3.5), it is clear that they are generalizing the group of translations. In a similar way general tempered distribution semi-groups in $L(L^2)$ were constructed in [3].

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