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K-GROUPS OF REDUCED CROSSED PRODUCTS BY FREE GROUPS

by

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The starting point for the present paper was an open problem concerning projections in the reduced C^* -algebra of a free groups, which we solve by showing that there are no non-trivial projections in the reduced C^* -algebra of a free group.

Our results are in fact more general. For a C^*-algebra A endowed with an automorphic action of the free group on n generators F_n, we obtain a six terms cyclic exact sequence which can be used for the computation of the K-groups of the reduced crossed product of A by F_n. In particular for $C^*_r(F_n)$, the reduced C^*-algebra of F_n, we have $K_0(C^*_r(F_n)) \simeq \mathbb{Z}$ and $K_1(C^*_r(F_n)) \simeq \mathbb{Z}^n$. Now, J.Cohen has proved in [5] (see also [4]) that there are no non-trivial projections in the full C^*-algebra of the free group and quite recently J.Cuntz in [8] has proved that K_0 and K_1 of the full C^*-algebra of F_n are $\mathbb{Z}$ and respectively $\mathbb{Z}^n$. Thus our results show that the full group C^*-algebra and the reduced C^*-algebra of F_n have the same K-theory.

The present paper has three sections. § 1 and § 3 can be viewed as a generalization of the argument we gave in [17] for the proof of the exact sequence for the K-groups of crossed products by $\mathbb{Z}$. Crucial for our results, however, is § 2 which contains a study of the spectral subspaces of certain matrices with operator entries.

§ 1.

For a set I, we shall denote by $F_{(I)}$ the free group with card I generators, the generators being denoted by g_ι $(\iota \in I)$. As usual for $g \in F_{(I)}$ the length of the corresponding word will be denoted by $|g|$. In case I has a finite number n of elements, $F_{(I)}$ will be denoted by F_n.

We shall denote by $\mathcal{M}_n$ the C^*-algebra of $n \times n$ complex matrices and by 1_n its unit. Elements of $B \otimes \mathcal{M}_n$ (B a C^*-algebra) will be written either as elements of this tensor product or as $n \times n$ matrices with entries from B.

For $\mathcal{U} = \{u_\iota\}_{\iota \in I}$ a family of unitary elements of a C^*-algebra B, the corresponding homomorphism of $F_{(I)}$ will be written as $F_{(I)} \ni g \longmapsto u_g \in B$. Moreover if $A \subset B$ is a $*$-subalgebra of B, $A[\mathcal{U}]$ will denote the $*$-subalgebra generated by A and $\mathcal{U}$. Clearly, if $u_\iota A u_\iota^* = A$ for all $\iota \in I$, then every element of $A[\mathcal{U}]$ is of the form $\sum_{g \in F_{(I)}} a_g u_g$ where $a_g \in A$ are non-zero only for a finite number of $g \in F_{(I)}$.

For $\mathcal{U}$ as above, $\mathcal{U}_p$ will denote the family of unitaries $\{u_\iota \otimes 1_p\}_{\iota \in I}$ in $B \otimes \mathcal{M}_p$.

1.1. Lemma. *Let B be a unital C^*-algebra, $A \subset B$ a $*$-subalgebra with $1 \in A$, and let $\mathcal{U} = \{u_\iota\}_{\iota \in I}$ be a family of unitaries in B. Assume $u_\iota A u_\iota^* = A$ for all $\iota \in I$ and assume also that $A[\mathcal{U}]$ is dense in B. Then the group $K_1(B)$ is generated by the classes of invertible elements of the form*

$$1_B \otimes 1_n + x\,U$$

<u>where</u> $x \in A \otimes \mathcal{M}_n$ <u>is invertible and</u> $U \in B \otimes \mathcal{M}_n$ <u>is a unitary</u> <u>of the form</u> $U=(u_{ij})_{1\leq i,j\leq n}$ <u>with</u> $u_{ij}=0$ <u>if</u> $i\neq j$ <u>and</u> $u_{ii} \in \mathcal{U}$ $(1\leq i\leq n)$.

<u>Proof</u>. Let $\Gamma \subset K_1(B)$ denote the subgroup generated by the elements $[1_B \otimes 1_n + x\,U]$. Remark that if $x \in A\otimes\mathcal{M}_n$ is unitary, then $[1_B \otimes 1_n + 2xU] = [x] + \sum_{i=1}^{n} [u_{ii}]$, so that the classes $([u_\iota])_{\iota\in I}$ are clearly in Γ and also the range of $K_1(A) \longrightarrow K_1(B)$ is in Γ. Since $(A \otimes \mathcal{M}_p)[\mathcal{U}_p]$ is a dense $*$-subalgebra of $B \otimes \mathcal{M}_p$, it will be sufficient to prove that $[y]\in\Gamma$ when y is an invertible element of $(A\otimes\mathcal{M}_p)[\mathcal{U}_p]$.

Thus, let $y \in (A \otimes \mathcal{M}_p)[\mathcal{U}_p]$ be invertible. Write

$$y= \sum_{g\in J} a_g(u_g \otimes 1_p)$$

where $J \subset F_{(I)}$ is a finite subset and $a_g \in A \otimes \mathcal{M}_p$. It is easily seen that we can find $s > 0$, elements $\{y_j\}_{j=0}^{s}$ of $(A\otimes\mathcal{M}_p)[\mathcal{U}_p]$, unitaries $\{u_{\iota(j)}\}_{j=0}^{s}$ in $\mathcal{U}$ and integers $\{\varepsilon(j)\}_{j=1}^{s}$ such that: $y_0=y, y_s=0$, $\varepsilon(j) \in \{-1,1\}$ and $y_{j-1}-y_j(u_{\iota(j)}^{\varepsilon(j)} \otimes 1_p)\in A\otimes\mathcal{M}_p$ for all $j=1,\dots,s$.

Let a_j, $0 \leq j < s$ denote $y_j-y_{j+1}(u_{\iota(j+1)}^{\varepsilon(j+1)} \otimes 1_p) \in A \otimes \mathcal{M}_p$, $a_s=0$ (in general, many of the elements a_j will be zero).

We have:

$$\begin{pmatrix} a_0 & a_1 & \cdots & \cdots & a_s \\ -u^{\varepsilon(1)}_{\iota(1)} \otimes 1_p & 1\otimes 1_p & & & 0 \\ & -u^{\varepsilon(2)}_{\iota(2)} \otimes 1_p & 1\otimes 1_p & & \vdots \\ & & \ddots & \ddots & \\ 0 & & & -u^{\varepsilon(s)}_{\iota(s)} \otimes 1_p & 1\otimes 1_p \end{pmatrix} =$$

$$= \begin{pmatrix} 1\otimes 1_p & y_1 & \cdots & y_s \\ & 1\otimes 1_p & & 0 \\ & & \ddots & \\ 0 & & & 1\otimes 1_p \end{pmatrix} \begin{pmatrix} y & & & \\ & 1\otimes 1_p & & 0 \\ & & \ddots & \\ & 0 & & 1\otimes 1_p \end{pmatrix}$$

$$\begin{pmatrix} 1\otimes 1_p & & & & \\ -u^{\varepsilon(1)}_{\iota(1)} \otimes 1_p & 1\otimes 1_p & & & 0 \\ & -u^{\varepsilon(2)}_{\iota(2)} \otimes 1_p & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & -u^{\varepsilon(s)}_{\iota(s)} \otimes 1_p & 1\otimes 1_p \end{pmatrix}$$

The first and the third of the matrices in the right-hand side of the above equality are of the form identity plus nilpotent, so that their classes in K_1 are trivial. Thus in order to prove that $[y] \in \Gamma$ it will be sufficient to prove that the class of the matrix appearing in the left hand side of this equality is in Γ. One more reduction is possible. We may replace this matrix by the matrix which is obtained by multiplying it to the *left* by

$$\begin{pmatrix} 1 \otimes 1_p & & & O \\ & v_1 & & \\ & & \ddots & \\ O & & & v_s \end{pmatrix}$$

where

$$v_j = \begin{cases} 1 \otimes 1_p & \text{if} \quad \varepsilon(j) \leq 0 \\ u_{i(j)}^{-\varepsilon(j)} \otimes 1_p & \text{if} \quad \varepsilon(j) \geq 0 \end{cases}$$

Indeed, tha class of the matrix by which we multiply is in Γ.

The matrix which we obtain after corrying out the multiplication can be written in the form

$$U^* S + T$$

where $S \in 1_B \otimes \mathcal{M}_{p(s+1)}$, $T \in A \otimes \mathcal{M}_{p(s+1)}$ and U is a diagonal matrix, the diagonal entries of which are in $\mathcal{U}$. Since the matrix S has scalar entries, there is an invertible matrix $S_\varepsilon \in 1_B \otimes \mathcal{M}_{p(s+1)}$ close enough to S, so that $[U^*S + T] = [U^* S_\varepsilon + T]$. But then for $m = p(s+1)$ $[U^* S_\varepsilon + T] = [U^* + T S_\varepsilon^{-1}] = [U^*] + [1_B \otimes 1_m + T S_\varepsilon^{-1} U]$. Now clearly $[U^*] \in \Gamma$ and $1_B \otimes 1_m + T S_\varepsilon^{-1} U$ can be written as $1_B \otimes 1_m + yU$ with $y \in A \otimes \mathcal{M}_m$, but y is not necessarily invertible. The fact that y is not invertible can be overcome by noting that for $\varepsilon > 0$ and

$\delta > 0$ small enough, we have:

$$[1_B \otimes 1_m + yU] = \left[1_B \otimes 1_{2m} + \begin{pmatrix} \varepsilon & y^* \\ 0 & y \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\right] =$$

$$= \left[1_B \otimes 1_{2m} + \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\right] =$$

$$= \left[1_B \otimes 1_{2m} + \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \delta\sqrt{-1} & y \\ y^* & \delta\sqrt{-1} \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\right]$$

and $\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \delta\sqrt{-1} & y \\ y^* & \delta\sqrt{-1} \end{pmatrix}$ is invertible.

Q.E.D.

1.2. Lemma. Let B be a unital C^*-algebra, $A \subset B$ a $*$-subalgebra and $\mathcal{U} = \{u_\iota\}_{\iota \in I}$ a family of unitaries in B. Let further $x \in A \otimes \mathcal{M}_n$ and let $U \in B \otimes \mathcal{M}_n$ be a diagonal matrix, with diagonal entries in $\mathcal{U}$, so that $1_B \otimes 1_n + xU$ is invertible. Assume, moreover there is a homomorphism $\beta : \mathbb{T} \longrightarrow \mathrm{Aut}\, B$ such that $\beta(z)a = a$ for $a \in A$ and $\beta(z)u_\iota = zu_\iota$ for $\iota \in I$ and $z \in \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Then, there is $0 < \lambda < 1$ such that the spectrum of xU is contained in

$$\{\zeta \in \mathbb{C} \mid |\zeta| < \lambda\} \cup \{\zeta \in \mathbb{C} \mid |\zeta| > \lambda^{-1}\}$$

Proof. Let β_n denote the action of $\mathbb{T}$ on $B \otimes \mathcal{M}_n$. Then

$$\beta_n(z)(\zeta 1_B \otimes 1_n - xU) =$$

$$= \zeta 1_B \otimes 1_n - z\, x\, U$$

where $\zeta \in \mathbb{C}$ and $z \in \mathbb{T}$. Hence ζ is in the spectrum of $x\,U$ if and only if $z^{-1}\zeta$ is in the spectrum of $x\,U$ and the lemma follows from the fact that -1 is not in the spectrum of xU.

Q.E.D.

§ 2.

Throughout, in this section, A will denote a unital C^*-algebra endowed with an action $\alpha : F_n \longrightarrow \mathrm{Aut}\, A$ of F_n on A. By B we shall denote the reduced crossed product $A \times_{\alpha r} F_n$.

We shall assume that A is a C^*-algebra of operators on a Hilbert space $\mathcal{K}$ and the reduced crossed product $A \times_{\alpha r} F_n$ will be identified with the C^*-algebra of operators on $\ell^2(F_n, \mathcal{K}) \simeq$ $\simeq \ell^2(F_n) \otimes \mathcal{K}$ generated by the operators $\pi(a)(a \in A)$ and $u_g(g \in F_n)$ defined by

$$(\pi(a)k)(g) = (\alpha(g^{-1})a)k(g)$$

$$(u_g k)(h) = k(g^{-1}h)$$

where $g, h \in F_n$ and $k \in \ell^2(F_n, \mathcal{K})$.

We shall denote u_{g_i} by u_i for $1 \leq i \leq n$. Clearly B is generated by $\pi(A)$ and $\mathcal{U} = \{u_i\}_{1 \leq i \leq n}$.

In order to study $K_1(B)$ we must consider in view of Lemma 1.1 invertible matrices

$$1_B \otimes 1_m + xU \in B \otimes \mathcal{M}_m$$

where $x \in \pi(A) \otimes \mathcal{M}_m$ is invertible and $U \in B \otimes \mathcal{M}_m$ is diagonal with diagonal entries in $\mathcal{U}$. Note also that there is a homomorphism $\beta : \mathbb{T} \longrightarrow \mathrm{Aut}\, B$ such that $\beta(z)\pi(a) = \pi(a)$ and $\beta(z)u_i = zu_i$.

Hence by Lemma 1.2 there is $0 < \lambda < 1$ such that the spectrum of xU is contained in $\{\zeta \in \mathbb{C} \mid |\zeta| < \lambda\} \cup \{\zeta \in \mathbb{C} \mid |\zeta| > \lambda^{-1}\}$

This section is devoted to the study of the spectral subspaces for xU corresponding to $\{\zeta \in \mathbb{C} \mid |\zeta| < \lambda\}$ and respectively $\{\zeta \in \mathbb{C} \mid |\zeta| > \lambda^{-1}\}$.

We will consider $B \otimes \mathcal{M}_m$ represented on

$\mathcal{H} = \ell^2(F_n, \mathcal{K}) \otimes \mathbb{C}^m \simeq \ell^2(F_n, \mathcal{K}^m)$.

By $\mathcal{H}_0 \subset \mathcal{H}$ we shall denote the subspace of $\mathcal{H} = \ell^2(F_n, \mathcal{K}^m)$ of the elements $\xi : F_n \longrightarrow \mathcal{K}^m$ such that $g \neq e \Longrightarrow \xi(g) = 0$, where $e \in F_n$ is the neutral element of F_n. Remark also that $\mathcal{H}_0$ is a reducing and separating subspace for $\pi(A) \otimes \mathcal{M}_m$. Even more, denoting by P_0 the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_0$, we have for $y \in B \otimes \mathcal{M}_m$:

$$yP_0 = P_0 y \iff y \in \pi(A) \otimes \mathcal{M}_m$$

Let $F_n^+ \subset F_n$ be the semigroup in F_n generated by $\{e\} \cup \{g_1, \dots, g_n\}$. By $\mathcal{H}^+ \subset \mathcal{H}$ we shall denote the subspace $\ell^2(F_n^+, \mathcal{K}^m)$ of $\ell^2(F_n, \mathcal{K}^m)$ i.e., those $\xi : F_n \longrightarrow \mathcal{K}^m$ such that $g \notin F_n^+ \Rightarrow \xi(g) = 0$. It is easily seen that $\mathcal{H}^+$ is invariant for U, $u_i \otimes 1_m$ and reducing for $\pi(A) \otimes \mathcal{M}_m$.

By $\mathcal{H}_+$ and $\mathcal{H}_-$ (respectively by $\mathcal{H}_{+*}$ and $\mathcal{H}_{-*}$) we shall denote the spectral subspaces of xU (respectively of $(xU)^*$) corresponding to $\{\zeta \in \mathbb{C} \mid |\zeta| > \lambda^{-1}\}$ and $\{\zeta \in \mathbb{C} \mid |\zeta| < \lambda\}$. We have:

$$\mathcal{H}_- = \left\{ \xi \in \mathcal{H} \mid \overline{\lim_{k \to \infty}} \|(xU)^k \xi\|^{1/k} < \lambda \right\}$$

$$\mathcal{H}_+ = \left\{ \xi \in \mathcal{H} \mid \overline{\lim_{k \to \infty}} \|(U^* x^{-1})^k \xi\|^{1/k} < \lambda \right\}$$

$$\mathcal{H}_-^\perp = \mathcal{H}_{+*} = \left\{ \xi \in \mathcal{H} \mid \overline{\lim_{k \to \infty}} \|(x^{*-1} U)^k \xi\|^{1/k} < \lambda \right\}$$

$$\mathcal{H}_+^\perp = \mathcal{H}_{-*} = \left\{ \xi \in \mathcal{H} \mid \overline{\lim_{k \to \infty}} \|(U^* x^*)^k \xi\|^{1/k} < \lambda \right\}$$

Remark also that in the above equalities λ may be replaced by any number λ' such that $\lambda \leq \lambda' \leq 1$. The spectral projections of xU corresponding to $\{\zeta \in \mathbb{C} \mid |\zeta| > \lambda^{-1}\}$ and respectively $\{\zeta \in \mathbb{C} \mid |\zeta| < \lambda\}$ will be denoted by P_+ and respectively P_-.

2.1. Lemma. Let $k \in \mathbb{N}$. Then there are $X_k, Y_k \in \pi(A) \otimes \mathcal{M}_m$, $X_k \geq 0, Y_k \geq 0$ such that

$$X_k \mid \mathcal{H}^+ = ((xU) \mid \mathcal{H}^+)^{*k}((xU) \mid \mathcal{H}^+)^k$$

$$Y_k \mid \mathcal{H}^+ = ((x^{*-1}U) \mid \mathcal{H}^+)^{*k}((x^{*-1}U) \mid \mathcal{H}^+)^k$$

and moreover $X_k = Y_k^{-1}$. In particular for $\xi \in \mathcal{H}_0$ we have

$$\| (xU)^k \xi \|^2 = \langle X_k \xi, \xi \rangle$$

$$\| (x^{*-1}U)^k \xi \|^2 = \langle Y_k \xi, \xi \rangle$$

Proof. We have $(xU)^k = (b(i,j))_{1 \leq i,j \leq m}$ with $b(i,j) \in B$. Computing, it is easily seen that

$$b(i,j) = \sum_{\substack{g \in F_n^+ \\ |g|=k}} u_g \, \pi(a(i,j,g))$$

where $a(i,j,g) \in A$. For $g \in F_n^+$ let $s_g = u_g \mid \ell^2(F_n^+, \mathcal{K})$. It is easily seen that the s_g's are isometries and

$$|g| = |g'|, \; g, g' \in F_n^+ \Rightarrow s_{g'}^* s_g = \begin{cases} 1 & \text{if } g' = g \\ 0 & \text{if } g' \neq g \end{cases}$$

Using this fact it is immediate that

$$((xU) \mid \mathcal{H}^+)^{*k}((xU) \mid \mathcal{H}^+)^k = X_k \mid \mathcal{H}^+$$

for some $X_k \in \pi(A) \otimes \mathcal{M}_m$, $X_k \geq 0$. Also clearly for $\xi \in \mathcal{H}_0$ we have

$$\| (xU)^k \xi \|^2 = \| ((xU) \mid \mathcal{H}^+)^k \xi \|^2 = \langle X_k \xi, \xi \rangle$$

The corresponding relations for Y_k are obtained in the same way.

To prove that $X_k = Y_k^{-1}$ we proceed as follows. Let

$$T_1 = (xU)\,|\,\mathcal{H}^+$$
$$T_2 = (x^{*\,-1}U)\,|\,\mathcal{H}^+.$$

We have

$$X_k\,|\,\mathcal{H}^+ = T_1^{*\,k} T_1^{\,k}$$
$$Y_k\,|\,\mathcal{H}^+ = T_2^{*\,k} T_2^{\,k}$$

Now

$$T_2^* T_1 = (U\,|\,\mathcal{H}^+)^* (x^{*-1}\,|\,\mathcal{H}^+)^* (x\,|\,\mathcal{H}^+)(U\,|\,\mathcal{H}^+) =$$

$$= (U\,|\,\mathcal{H}^+)^* (x^{-1}\,|\,\mathcal{H}^+)(x\,|\,\mathcal{H}^+)(U\,|\,\mathcal{H}^+) =$$

$$= (U\,|\,\mathcal{H}^+)^* (U\,|\,\mathcal{H}^+) = I_{\mathcal{H}^+}$$ and hence $T_2^{*k} T_1^{\,k} = I_{\mathcal{H}^+}$

Let

$T_1^k = W(T_1^{*\,k} T_1^k)^{1/2} = W(X_k^{1/2}\,|\,\mathcal{H}^+)$ be the polar decomposition of T_1^k. From $T_2^{*\,k} T_1^k = I_{\mathcal{H}^+}$ we infer $T_2^{*k} = (X_k^{-1/2}\,|\,\mathcal{H}^+)W^*$ and the invertibility of X_k. But now, this gives:

$$Y_k\,|\,\mathcal{H}^+ = T_2^{*\,k} T_2^{\,k} = (X_k^{-1/2}\,|\,\mathcal{H}^+) W^* W (X_k^{-1/2}\,|\,\mathcal{H}^+) = X_k^{-1}\,|\,\mathcal{H}^+$$

Now since $\mathcal{H}^+$ is separating for $\pi(A) \otimes \mathcal{M}_m$ this gives $Y_k = X_k^{-1}$.

Q.E.D.

2.2. Lemma. We have

$$\mathcal{H}_0 = \mathcal{H}_0 \cap \mathcal{H}_- + \mathcal{H}_0 \cap \mathcal{H}_-^{\perp}$$

Proof. Let $\mathcal{H}_{01}$ and $\mathcal{H}_{02}$ denote the subspaces $\mathcal{H}_0 \cap \mathcal{H}_-$ and respectively $\mathcal{H}_0 \ominus (\mathcal{H}_0 \cap \mathcal{H}_-)$. It will be clearly sufficient to prove that $\mathcal{H}_{02} \subset \mathcal{H}_-^{\perp}$

It will be convenient to use the notations $\tilde{X}_k, \tilde{Y}_k$ for $X_k | \mathcal{H}_o$ and $Y_k | \mathcal{H}_o$.

Since the spectrum of $(xU)|\mathcal{H}_+$ is contained in $\{\zeta \in \mathbb{C} \mid |\zeta| > \lambda^{-1}\}$ there is $C_1 > 0$ such that for all $\eta \in \mathcal{H}_+$ we have

$$\|(x\,U)^k \eta\| \geqslant C_1 \lambda^{-k} \|\eta\| \quad , \qquad (k \in \mathbb{N})$$

Thus for $\xi \in \mathcal{H}_o$ we have

$$\|(xU)^k \xi\| \geqslant \|P_+\|^{-1} \|P_+ (xU)^k \xi\| = \|P_+\|^{-1} \|(xU)^k P_+ \xi\| \geqslant$$
$$\geqslant C_1 \|P_+\|^{-1} \lambda^{-k} \|P_+ \xi\| .$$

Hence, using Lemma 2.1 for $\xi \in \mathcal{H}_o$ we have

$\langle X_k \xi, \xi \rangle \geqslant 2C_2 \lambda^{-2k} \|P_+ \xi\|^2$ where $C_2 = \frac{1}{2} C_1^2 \|P_+\|^{-2}$. This can be also written in the form

$$\tilde{X}_k \geqslant 2C_2 \lambda^{-2k} R$$

where $R = (P_+ | \mathcal{H}_o)^* (P_+ | \mathcal{H}_o) \in L(\mathcal{H}_o)$.

Concerning the operator R, we note that Ker R= $\mathcal{H}_{o1}$

Now for $2\varepsilon_k = \|\tilde{Y}_k\|^{-1}$, taking into acount that $\tilde{X}_k = \tilde{Y}_k^{-1}$ we have $\tilde{X}_k \geqslant 2\varepsilon_k I$

and hence

$$\tilde{X}_k \geqslant \varepsilon_k I + C_2 \lambda^{-2k} R$$

so that

$$\tilde{Y}_k = \tilde{X}_k^{-1} \leqslant (\varepsilon_k I + C_2 \lambda^{-2k} R)^{-1} .$$

Let $\mathcal{H}_\delta$ $(\delta > 0)$ be the spectral subspace of R corresponding to $[\delta, \infty)$ .For $\xi \in \mathcal{H}_\delta$ we have

$$\langle \tilde{Y}_k \xi, \xi \rangle \leqslant \langle (\varepsilon_k I + C_2 \lambda^{-2k} R)^{-1} \xi \, , \, \xi \rangle \leqslant$$
$$\leqslant \left(\sup_{t \geqslant \delta} (\varepsilon_k + C_2 \lambda^{-2k} t)^{-1}\right) \|\xi\|^2 \leqslant$$

$$\leq C_2^{-1} \lambda^{2k} \delta^{-1} \|\xi\|^2.$$

and hence

$$\overline{\lim_{k\to\infty}} \|(x^{*-1}U)^k \xi\|^{2/k} = \overline{\lim_{k\to\infty}} \langle \tilde{Y}_k \xi, \xi \rangle^{1/k} \leq$$

$$\leq \overline{\lim_{k\to\infty}} (C_2^{-1} \lambda^{2k} \delta^{-1} \|\xi\|^2)^{1/k} = \lambda^2.$$

Thus $\mathcal{H}_\delta \subset \mathcal{H}_-^\perp$ and since $\overline{\bigcup_{\delta>0} \mathcal{H}_\delta} = \mathcal{H}_o \ominus \operatorname{Ker} R = \mathcal{H}_{o2}$

we have proved that $\mathcal{H}_{o2} \subset \mathcal{H}_-^\perp$.

Q.E.D.

2.3. Lemma. Let Q_- denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_-$. Then we have

$$Q_- \in \pi(A) \otimes \mathcal{M}_m \text{ and } UQ_- U^* \in \pi(A) \otimes \mathcal{M}_m$$

Thus, writing

$$U = \begin{pmatrix} u_{i_1} & & & 0 \\ & u_{i_2} & & \\ & & \ddots & \\ 0 & & & u_{i_m} \end{pmatrix}$$

and $Q_- = (q_{ij})_{1\leq i,j\leq m}$ we have

$$q_{ij} \in \pi(A) \text{ for all } 1 \leq i,j \leq m$$

and

$$i_s \neq i_t \implies q_{i_s,i_t} = 0$$

Proof. Since Q_- is the self-adjoint projection onto the range of the projection $P_- \in B \otimes \mathcal{M}_m$, it follows that $Q_- \in B \otimes \mathcal{M}_m$. The assertion of Lemma 2.2 means precisely that Q_- commutes with the orthogonal projection P_o of $\mathcal{H}$ onto $\mathcal{H}_o$, and this in turn means that $Q_- \in \pi(A) \otimes \mathcal{M}_m$

On the other hand, since xU is invertible we have $xU\mathcal{H}_- = \mathcal{H}_-$ and hence $U\mathcal{H}_- = x^{-1}\mathcal{H}_-$. This shows that UQ_-U^* is the self-adjoint projection onto the range of $x^{-1}Q_-x \in$

$\in \pi(A) \otimes \mathcal{M}_m$ and hence $UQ_-U^* \in \pi(A) \otimes \mathcal{M}_m$

That $q_{ij} \in \pi(A)$ is obvious. To see that $i_s \neq i_t \Rightarrow q_{i_s,i_t} = 0$, note that $UQU^* \in \pi(A) \otimes \mathcal{M}_m$ implies

$$u_{i_s} q_{i_s,i_t} u_{i_t}^* \in \pi(A)$$

which for $i_s \neq i_t$ implies $q_{i_s,i_t} = 0$.

Q.E.D.

2.4. Lemma. The group $K_1(A \times_{\alpha,r} F_n)$ is generated by classes of unitary elements of the form

$$(1 \otimes 1_{mn} - Q) + QvUQ$$

where $Q = Q_1 \oplus \dots \oplus Q_n$ with $Q_j \in \pi(A) \otimes \mathcal{M}_n$ self-adjoint projections, $U = (u_1 \otimes 1_m) \oplus \dots \oplus (u_n \otimes 1_m)$ and $v \in \pi(A) \otimes \mathcal{M}_{mn}$ a partial isometry such that $vv^* = Q$, $v^*v = UQU^*$.

Proof. In view of Lemma 1.1 it will be sufficient to prove that the class of

$$y = 1 \otimes 1_m + xU$$

is in the group generated by the classes of the unitaries described in the statement of the lemma. Now in view of Lemma 2.3 and taking into acount the fact that $[u_j]_{1 \leq j \leq n}$ are in the group generated by the special unitaries, we see that we may assume that

$$U = (u_1 \otimes 1_p) \oplus (u_2 \otimes 1_p) \oplus \dots \oplus (u_n \otimes 1_p) \text{ and that}$$

$$Q_- = Q_{-1} \oplus \dots \oplus Q_{-n}$$

where $Q_{-j} \in \pi(A) \otimes \mathcal{M}_p$.

Let now $Q=1\otimes 1_{np}-Q_-$ and $Q_j=1\otimes 1_p-Q_{-j}$. Since the operators $y|Q_-\mathcal{H}$ and $Qy|Q\mathcal{H}$ are *invertible* it is easily seen that

$$y_t=Q_-yQ_-+tQ_-yQ+QyQ$$

is invertible for $0\leq t\leq 1$. Hence $[y]=[Q_-yQ_-+QyQ]=[Q_-yQ_-+Q]+$ $+[Q_-+QyQ]$. Since the spectrum of $Q_-xU|Q_-\mathcal{H}$ is contained in $\{\zeta\in\mathbb{C}\mid|\zeta|<1\}$ we see that

$$Q_-(1\otimes 1_{np}+\varepsilon xU)Q_-+Q$$

is invertible for $0\leq\varepsilon\leq 1$ and hence $[Q_-yQ_-+Q]=0$ so that $[y]=[(1\otimes 1_{np}-Q)+QyQ]$. Since the spectrum of $Q\,xU|Q\mathcal{H}$ is contained in $\{\zeta\in\mathbb{C}\mid|\zeta|>1\}$ we see that

$$(1\otimes 1_{np}-Q)+Q(\varepsilon 1\otimes 1_{np}+xU)Q$$

is invertible for $0\leq\varepsilon\leq 1$ and hence

$$[y]=[(1\otimes 1_{np}-Q)+QxUQ].$$

Since $(1\otimes 1_{np}-Q)+QxUQ$ is invertible we infer that there is a partial isometry $v\in\pi(A)\otimes\mathcal{M}_{np}$ such that

$$QxUQU^*=v((QxUQU^*)^*(QxUQU^*))^{1/2}$$

and

$$v^*v=UQU^*,\ vv^*=Q.$$

Let D denote $((QxUQU^*)^*(QxUQU^*))^{1/2}$.

Then there is $0<\delta<1$ such that

$$\delta\,UQU^*\leq D\leq\delta^{-1}UQU^*$$

We have

$$QxUQ=QvDUQ$$

Remark now that

$(1\otimes 1_{np}-Q)+Qv(\varepsilon UQU^*+(1-\varepsilon)D)UQ$ is invertible for $0\leq\varepsilon\leq 1$.

It follows that

$$[y]=[(1\otimes 1_{np}-Q)+QvUQ]$$

Q.E.D.

§3.

In this section we shall generalize the Toeplitz-extension for crossed products by $\mathbb{Z}$ $(\simeq F_1)$ of [17] to a Toeplitz-extension for reduced crossed products by F_n and use it to derive a cyclic six terms exact sequence for reduced crossed products by F_n.

As in the preceding section we shall consider a unital C^*-algebra A with an action $\alpha : F_n \longrightarrow \mathrm{Aut}\,A$. We shall assume that A is a C^*-algebra of operators on a Hilbert space $\mathcal{K}$ and the reduced crossed product $A\times_{\alpha r}F_n$ will be defined as a C^*-algebra of operators on $\ell^2(F_n,\mathcal{K})\simeq\ell^2(F_n)\otimes\mathcal{K}$. It will be convenient to assume that the action $\alpha : F_n \longrightarrow \mathrm{Aut}A$ is implemented by a unitary repredsentation $g\longmapsto v_g$ of F_n on $\mathcal{K}$, i.e.

$$v_g\, a\, v_g^* = \alpha(g)a$$

for $a\in A, g\in F_n$.

Let $F_{n,j}$ denote the subset of F_n consisting of the elements $g_{i_1}^{m_1}\dots g_{i_k}^{m_k}$ $(i_1\neq i_2, i_2\neq i_3,\dots,i_{k-1}\neq i_k, k\geqslant 1, m_1\neq 0, m_2\neq 0,\dots$ $\dots, m_k\neq 0)$ such that $i_k=j$ and $m_k>0$. It is easily seen that

$$i\neq j \implies g_i\cdot F_{n,j}=g_i^{-1}\cdot F_{n,j}=F_{n,j}$$

$$g_j F_{n,j} = F_{n,j} \setminus \{g_j\}$$

$$g_j^{-1} F_{n,j} = F_{n,j} \cup \{e\}$$

Let $\mathcal{H} = (\ell^2(F_n, \mathcal{K}))^n$ and let $\mathcal{H}_\tau \subset \mathcal{H}$ be the subspace $\mathcal{H}_\tau = \bigoplus_{j=1}^{n} \ell^2(F_{n,j}, \mathcal{K})$. The reduced crossed product $A \times_{\alpha r} F_n$ will act on $\mathcal{H}$ by the n-fold multiple of its representation on $\ell^2(F_n, \mathcal{K})$ and for $x \in A \times_{\alpha r} F_n$ the corresponding operator on $\mathcal{H}$ will be denoted by $x \otimes 1_n$.

On $\mathcal{H}_\tau$ we shall consider the representation of A which is the restriction to $\mathcal{H}_\tau$ of $\pi \otimes 1_n$, i.e.

$$\rho(a)\left(\bigoplus_{j=1}^{n} k_j\right) = \bigoplus_{j=1}^{n} k'_j$$

where

$$k'_j(g) = (\alpha(g^{-1})a)k_j(g)$$

for $1 \le j \le n, g \in F_{n,j}$.

We define also the isometries $S_i \in L(\mathcal{H}_\tau)$ by

$$S_i\left(\bigoplus_{j=1}^{n} k_j\right) = \bigoplus_{j=1}^{n} k'_j$$

where

$$k'_j(g) = k_j(g_i^{-1} g) \text{ if } i \neq j$$

and

$$k'_i(g) = \begin{cases} k_i(g_i^{-1} g) & \text{for } g \in g_i F_{n,i} \\ 0 & \text{for } g = g_i \end{cases}$$

We have

$$I - S_i S_i^* = E_i$$

where E_i is the orthogonal projection of $\mathcal{H}_\tau$ onto the subspace of those $(\bigoplus_{j=1}^{n} k_j) \in \mathcal{H}_\tau$ such that $j \neq i \Rightarrow k_j=0$ and $g \notin F_{n,i} \setminus \{g_i\}$

$\Rightarrow k_i(g)=0$. Note also that

$$S_i^* \rho(a) S_i = \rho(\alpha(g_i^{-1})a)$$

With these preparations we can now define what we shall call the Toeplitz algebra for the reduced crossed product $A \times_{\alpha r} F_n$. This is the C^*-algebra $\mathcal{T}$ in $L(\mathcal{H}_\tau)$ generated by $\rho(A)$ and $(S_i)_{1 \leq i \leq n}$. It is also easily seen that the closed two-sided ideal $\mathcal{J}$ in $\mathcal{T}$ generated by $(E_i)_{1 \leq i \leq n}$ is

$$\bigoplus_{j=1}^{n} (A \otimes K(\ell^2(F_{n,j})))$$

where $\mathcal{H}_\tau$ has been identified with $\bigoplus_{j=1}^{n} (\mathcal{K} \otimes \ell^2(F_{n,j}))$ and where $K(\mathcal{X})$ for a Hilbert space $\mathcal{X}$ denotes the C^*-algebra of compact operators on $\mathcal{X}$. Since $K(\mathcal{X})$ depends up to isomorphism only on the dimension of $\mathcal{X}$, we shall write sometimes simply K instead of $K(\mathcal{X})$ when $\mathcal{X}$ is separable infinite-dimensional and when this won't lead to confusions.

3.1. Lemma. *There exists a homomorphism $p: \mathcal{T} \longrightarrow A \times_{\alpha r} F_n$ such that $p(S_i)=u_i$ and $p(\rho(a))=\pi(a)$ for $1 \leq i \leq n$ and $a \in A$. For this homomorphism we have $\operatorname{Ker} p = \mathcal{J}$ where $\mathcal{J}$ is the closed two-sided ideal of $\mathcal{T}$ generated by $(E_i)_{1 \leq i \leq n}$ and is isomorphic to $(A \otimes K)^n$.*

Proof. Define $W \in L(\mathcal{H})$ by

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$$W\left(\bigoplus_{j=1}^{n} k_j\right)=\bigoplus_{j=1}^{n} k'_j$$

where

$$k'_j(g)=v_{g_j}k_j(gg_j)$$

It is easily seen that W is in the commutant of $(A \times_{\alpha\gamma} F_n)\otimes 1_n$.

Moreover we have for $m \longrightarrow \infty$

$$W^m\tilde{S}_i W^{*m} \xrightarrow{s} u_i \otimes 1_n$$

$$W^m\tilde{S}_i^* W^{*m} \xrightarrow{s} u_i^* \otimes 1_n$$

$$W^m\tilde{\rho}(a)W^{*m} \xrightarrow{s} \pi(a)$$

$$W^m\tilde{E}_i W^{*m} \xrightarrow{s} 0$$

where for $T \in L(\mathcal{H}_\tau)$ we denote by $\tilde{T} \in L(\mathcal{H})$ the operator $T \oplus 0_{\mathcal{H}\ominus\mathcal{H}_\tau}$. Since $\rho(A) \cup \{S_i\}_{1\le i\le n} \cup \{S_i^*\}_{1\le i\le n}$ is a set of generators for $\mathcal{T}$ as a Banach algebra, we infer that

$$\text{s-lim}_{m\to\infty} W^m\tilde{T}W^{*m}$$

exists for all $T \in \mathcal{T}$ and the map

$$T \longmapsto \text{s-lim}_{m\to\infty} W^m\tilde{T}W^{*m}$$

is a unital $*$-homomorphism of $\mathcal{T}$ into $(A \times_{\alpha\gamma} F_n)\otimes 1_n \simeq A \times_{\alpha\gamma} F_n$. This is the homomorphism p we were looking for, as can be easily checked, and is clearly onto and unique.

Since $p(E_i)=0$ it is obvious that $\operatorname{Ker} p \supset \mathcal{J}$.

To see that $\operatorname{Ker} p \subset \mathcal{J}$ we proceed as follows. For $x \in A \times_{\alpha\gamma} F_n$ we define

$$\Phi(x)=P_{\tau}(x\otimes 1_n)|_{\mathcal{H}_{\tau}}\in L(\mathcal{H}_{\tau})$$

where P_{τ} is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_{\tau}$. It is easily seen that Φ is a completely positive map and $\Phi(A\times_{\alpha\gamma}F_n)\subset\mathcal{T}$ (this must be checked only on the dense $*$-subalgebra of $A\times_{\alpha\gamma}F_n$ generated by $\pi(A)\cup\{u_i\}_{1\le i\le n}$). Moreover it is easy to check that $\Phi(xy)-\Phi(x)\Phi(y)\in\mathcal{J}$ and $p(\Phi(x))=x$ (again these assertions must be checked only on the $*$-subalgebra generated by $\pi(A)\cup\{u_i\}_{1\le i\le n}$). From $\Phi(xy)-\Phi(x)\Phi(y)\in\mathcal{J}$ we infer that $q\circ\Phi$ is a unital $*$-homomorphism of $A\times_{\alpha\gamma}F_n$ into $\mathcal{T}/\mathcal{J}$, where q is the canonical $*$-homomorphism $q:\mathcal{T}\longrightarrow\mathcal{T}/\mathcal{J}$. Moreover $q\circ\Phi$ is onto since $\Phi(A\times_{\alpha\gamma}F_n)\supset\rho(A)\cup\{S_i\}_{1\le i\le n}$.

Consider now $y\in\operatorname{Ker}p$. Then there is $z\in A\times_{\alpha\gamma}F_n$ such that $(q\circ\Phi)(z)=q(y)$ or equivalently $\Phi(z)-y\in\mathcal{J}$. But then

$$z=p(\Phi(z)-y)\in p(\mathcal{J})=\{0\}$$

so that $q(y)=0$, which means $y\in\mathcal{J}$

Q.E.D.

The preceding lemma can be summarized by the exact sequence

$$0\longrightarrow(A\otimes K)^n\xrightarrow{\ \psi\ }\mathcal{T}\xrightarrow{\ p\ }A\times_{\alpha\gamma}F_n\longrightarrow 0$$

For the next lemmas we shall need the unitaries

$$\Omega_i=\begin{pmatrix}S_i & E_i\\ 0 & S_i^*\end{pmatrix}\in\mathcal{T}\otimes\mathcal{M}_2$$

3.2. Lemma. The following diagram is commutative

$$\begin{array}{ccc} K_1((A\otimes K)^n) & \xrightarrow{\psi_*} & K_1(\mathcal{T}) \\ \wr \big| & & \big\uparrow \rho_* \\ (K_1(A))^n & \xrightarrow{\beta_1} & K_1(A) \end{array}$$

where

$$\beta_1(\gamma_1 \oplus \ldots \oplus \gamma_n) = \sum_{i=1}^{n} (\gamma_i - (\alpha(g_i^{-1}))_* \gamma_i)$$

for $\gamma_i \in K_1(A)$ $(1 \le i \le n)$.

Proof. The Lemma is a statement concerning elements of $K_1(A)$, i.e. concerning classes of unitaries from the algebras $A \otimes \mathcal{M}_k$. Since replacing $(A, \alpha, \mathcal{T})$ by $(A\otimes \mathcal{M}_k, \alpha \otimes id_k, \mathcal{T}\otimes \mathcal{M}_k)$ we have the same situation, it is easily seen that it is sufficient to prove the lemma only for the classes of unitaries from A.

Let $v \in A$ be unitary and consider the element

$$\omega = \underbrace{0 \oplus \ldots \oplus 0}_{(i-1)\text{-times}} \oplus [v] \oplus \underbrace{0 \oplus \ldots \oplus 0}_{(n-i)\text{-times}} \in (K_1(A))^n.$$

Taking the corresponding element in $K_1((A \otimes K)^n)$ and then applying ψ_* it is easily seen that the element of $K_1(\mathcal{T})$ we get in this way is

$$[\rho(v)E_i + (I-E_i)] =$$
$$= [\rho(v)E_i + S_i S_i^*]$$

On the other hand

$$(\rho_* \circ \beta_1)(\omega) = \rho_*([v] - [\alpha(g_i^{-1})v]) =$$
$$= [\rho(v)] - [\rho(\alpha(g_i^{-1})v)]$$

We have

$$[\rho(v)] - [\rho(\alpha(g_i^{-1})v)] = [\rho(v)] - [S_i^* \rho(v) S_i] =$$

$$= [\rho(v)] - \left[\Omega_i \begin{pmatrix} S_i^* \rho(v) S_i & 0 \\ 0 & I \end{pmatrix} \Omega_i^* \right] =$$

$$= [\rho(v)] - \left[\begin{pmatrix} \rho(v) S_i S_i^* + E_i & 0 \\ 0 & I \end{pmatrix}\right] =$$

$$= [\rho(v)] - [\rho(v) \; S_i S_i^* + E_i] =$$

$$= [\rho(v)(\rho(v) S_i S_i^* + E_i)^{-1}] =$$

$$= [\rho(v)(\rho(v^*) S_i S_i^* + E_i)] = [\rho(v) E_i + S_i S_i^*]$$

which concludes the proof.

Q.E.D.

3.3. Lemma. The following diagram is commutative:

$$\begin{tikzcd} K_o((A \otimes K)^n) \arrow[r, "\psi_*"] \arrow[d, phantom, "\wr"] & K_o(\mathcal{J}) \\ (K_o(A))^n \arrow[r, "\beta_o"] & K_o(A) \arrow[u, "\rho_*"'] \end{tikzcd}$$

where $\beta_o(\gamma_1 \oplus \ldots \oplus \gamma_n) = \sum_{i=1}^{n} (\gamma_i - (\alpha(g_i^{-1}))_* \gamma_i)$

for $\gamma_i \in K_o(A)$, $(1 \leq i \leq n)$.

Proof. As in the proof of Lemma 3.2 it will be sufficient to prove the lemma only for the classes of self-adjoint projections from A.

Let $q \in A$ be a self-adjoint projection and consider the element

$$\omega = \underbrace{0 \oplus \ldots \oplus 0}_{(i-1)\text{-times}} \oplus [q] \oplus \underbrace{0 \oplus \ldots \oplus 0}_{(n-i)\text{-times}} \in (K_o(A))^n$$

Taking the corresponding element in $K_o((A \otimes K)^n)$ and then applying γ_* it is easily seen that the element of $K_o(\mathcal{T})$ we get in this way is

$$[\rho(q) E_i] .$$

On the other hand

$$(\rho_* \circ \beta_o)(\omega) = \rho_*([q] - [\alpha(g^{-1})q]) =$$

$$= [\rho(q)] - [\rho(\alpha(g_i^{-1})q)]$$

We have

$$[\rho(\alpha(g_i^{-1})q)] =$$

$$= \left[\Omega_i \begin{pmatrix} \rho(\alpha(g_i^{-1})q) & 0 \\ 0 & 0 \end{pmatrix} \Omega^*_i\right] =$$

$$= \left[\Omega_i \begin{pmatrix} S_i^* \rho(q) S_i & 0 \\ 0 & 0 \end{pmatrix} \Omega^*_i\right] =$$

$$= \left[\begin{pmatrix} \rho(q) S_i S_i^* & 0 \\ 0 & 0 \end{pmatrix}\right] =$$

$$= [\rho(q) S_i S_i^*] = [\rho(q)] - [\rho(q) E_i]$$

Q.E.D.

3.4. <u>Lemma</u>. <u>The homomorphism</u>

$$\rho_* : K_1(A) \longrightarrow K_1(\mathcal{T})$$

<u>is surjective</u>.

<u>Proof</u>. Consider the diagram

$$\begin{tikzcd}
K_1((A \otimes K)^n) \arrow[r, "\gamma_*"] \arrow[d, dash, "\wr"'] & K_1(\mathcal{T}) \arrow[r, "p_*"] & K_1(A \times_{\alpha\gamma} F_n) \arrow[r, "\delta"] & K_0((A \otimes K)^n) \\
(K_1(A))^n \arrow[r, "\beta_1"] & K_1(A) \arrow[u, "\rho_*"'] \arrow[ur, "\pi_*"'] & &
\end{tikzcd}$$

The top row of the diagram is an exact sequence, being a segment of the exact sequence of K-theory applied to the Toeplitz extension. Also, because of Lemma 3.2, the square in this diagram is commutative. We also have $\pi_* = p_* \circ \rho_*$.

It is easily seen from these facts, that all we have to prove, is that $\operatorname{Im} \pi_* \supset \operatorname{Ker} \delta$. We shall use to this end Lemma 2.4.

Since the sum of the classes of two generators of the form considered in Lemma 2.4 is again the class of such a generator, we infer that every element in $K_1(A \times_{\alpha\gamma} F_n)$ is the difference of the classes of two such generators.

On the other hand we assert that for a generator

$$\omega = [(1 \otimes 1_{mn} - Q) + QvUQ]_1$$

(where $Q = Q_1 \oplus \dots \oplus Q_n$ with $Q_j \in \pi(A) \otimes \mathcal{M}_m$ self-adjoint projections, $U = (u_1 \otimes 1_m) \oplus \dots \oplus (u_n \otimes 1_m)$ and $v \in \pi(A) \otimes \mathcal{M}_{mn}$ is a partial isometry with $vv^* = Q, v^*v = UQU^*$), after identifying $K_0((A \otimes K)^n)$ with $(K_0(\pi(A)))^n$, we have

$$-\delta\omega = [Q_1]_o \oplus \dots \oplus [Q_n]_o$$

Indeed, the completely positive map $\Phi \otimes id_{mn}$ when applied to $(1 \otimes 1_{mn} - Q) + QvUQ$ provides a lifting of this unitary to an isometric element of $\mathcal{T} \otimes \mathcal{M}_{mn}$. But for such elements, δ is computed as the index of the isometric lifting and our assertion follows easily.

Returning to the description of Ker δ, an element of $K_1(A \times_{\alpha r} F_n)$ can always be written in the form

$$\omega = [(1 \otimes 1_{mn} - Q) + QvUQ]_1 -$$

$$- [(1 \otimes 1_{mn} - Q') + Q'v'UQ']_1$$

where $U = (u_1 \otimes 1_m) \oplus \dots \oplus (u_n \otimes 1_m)$, $Q = Q_1 \oplus \dots \oplus Q_n$, $Q' = Q'_1 \oplus \dots \oplus Q'_n$ with $Q_j, Q'_j \in \pi(A) \otimes \mathcal{M}_m$ self-adjoint projections and $v, v' \in \pi(A) \otimes \mathcal{M}_{mn}$ are partial isometries such that $vv^* = Q$, $v'v'^* = Q'$, $v^*v = UQU^*$ $v'^*v' = UQ'U^*$. Then $\omega \in$ Ker δ if and only if in $K_o(\pi(A))$ we have $[Q_j]_o = [Q'_j]_o$ for $1 \leq j \leq n$. In fact if $\omega \in$ Ker δ one may assume that there are *unitary elements* $w_j \in \pi(A) \otimes \mathcal{M}_m$ $(1 \leq j \leq n)$ such that $w_j Q_j w_j^* = Q'_j$. Indeed, replacing m by m+p and Q_j by $Q_j \oplus (1 \otimes 1_p)$, Q'_j by $Q'_j \oplus (1 \otimes 1_p)$ and v by $v \oplus (1 \otimes 1_{np})$, v' by $v' \oplus (1 \otimes 1_{np})$ the class ω remains unchanged and we have $\underbrace{\text{and for } p \text{ great enough the } w_j\text{'s exist}}$

$$\omega = [w((1 \otimes 1_{mn} - Q) + QvUQ)w^*((1 \otimes 1_{mn} - Q') + Q'v'UQ')^*]_1 =$$

$$= [(1 \otimes 1_{mn} - Q') + Q'wvUw^*Q'U^*v'^*Q']_1 =$$

$$= [(1 \otimes 1_{mn} - Q') + Q'wv(Uw^*U^*)(UQ'U^*)v'^*Q']_1 \quad \text{where } w = w_1 \oplus \dots \oplus w_n.$$

But since $w = w_1 \oplus \dots \oplus w_n$, $Q' = Q'_1 \oplus \dots \oplus Q'_n$ and $U = (u_1 \otimes 1m) \oplus \dots \oplus (u_n \otimes 1m)$ we have $Uw^*U^* \in \pi(A) \otimes \mathcal{M}_{mn}$ and

$UQ'U^* \in \pi(A) \otimes \mathcal{M}_{mn}$ so that $\omega \in \pi_* K_1(A)$.

Q.E.D.

Before passing to the next lemma it will be convenient to make some definitions.

By $\check{F}_{n,j}$ we shall denote the subset of F_n consisting of the elements $g_{i_1}^{m_1} \ldots g_{i_k}^{m_k}$ $(i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{k-1} \neq i_k, k \geqslant 1, m_1 \neq 0, m_2 \neq 0, \ldots, m_k \neq 0)$ such that $i_k = j$ and $m_k < 0$. It is easily seen that F_n is the disjoint union of $F_{n,i}$ $(1 \leq i \leq n)$, $\check{F}_{n,i}$ $(1 \leq i \leq n)$ and $\{e\}$.

We shall denote by $P_{j,g}$ and $\check{P}_{j,g}$ the orthogonal projections of $\ell^2(F_n, \mathcal{K})$ onto the subspaces $\ell^2(F_{n,j}g, \mathcal{K})$ and respectively $\ell^2(\check{F}_{n,j}g, \mathcal{K})$ $(1 \leq j \leq n, g \in F_n)$. Remark that

$$[P_{j,g}, A \times_{\alpha\gamma} F_n] \subset A \otimes K(\ell^2(F_n))$$

$$[\check{P}_{j,g}, A \times_{\alpha\gamma} F_n] \subset A \otimes K(\ell^2(F_n))$$

$$[P_{j,g}, \pi(A)] = \{0\}$$

$$[\check{P}_{j,g}, \pi(A)] = \{0\}$$

Further, for $g \in F_n$, we shall denote by Q_g the orthogonal projection of $\ell^2(F_n, \mathcal{K})$ onto $\ell^2(\{g\}, \mathcal{K})$. Clearly we have

$$Q_g \in A \otimes K(\ell^2(F_n))$$

$$[Q_g, \pi(A)] = \{0\}$$

We shall also need the unitaries R_g $(g \in F_n)$ acting on $\ell^2(F_n, \mathcal{K})$ which are defined by

$$(R_g k)(h) = v_g k(hg)$$

Then $g \longmapsto R_g$ is a unitary representation of F_n and

$$[R_g, A \times_{\alpha\gamma} F_n] = \{0\}$$

Moreover, we have

$$R_g P_{j,h} = P_{j,hg^{-1}} R_g$$

$$R_g \check{P}_{j,h} = \check{P}_{j,hg^{-1}} R_g$$

$$R_g Q_h = Q_{hg^{-1}} R_g$$

Remark now the following decomposition of $F_{n,j}$ into disjoint subsets

$$F_{n,j} = \{g_j\} \cup \Big(\bigcup_{i \neq j} (F_{n,i} g_j \cup \check{F}_{n,i} g_j) \Big) \cup F_{n,j} g_j$$

and remark also that

$$F_{n,i} g_j = (F_n \setminus \check{F}_{n,i}) g_i g_j$$

This gives the following way of writing the above decomposition of $F_{n,j}$ into disjoint subsets

$$F_{n,j} = \{g_j\} \cup F_{n,j} g_j \cup \Big(\bigcup_{i \neq j} ((F_n \setminus \check{F}_{n,i}) g_i g_j \cup \check{F}_{n,i} g_j) \Big)$$

Correspondingly we define an isometric operator

$$V : \ell^2(F_{n,n}, \mathcal{K}) \oplus (\ell^2(F_n, \mathcal{K}))^{n-1} \longrightarrow \ell^2(F_{n,n}, \mathcal{K})$$

by the formula

$$V(\xi \oplus \eta_1 \oplus \ldots \oplus \eta_{n-1}) =$$

$$= R_{g_n^{-1}} \xi + \sum_{i=1}^{n-1} (R_{g_n^{-1} g_i^{-1}} (I - \check{P}_{i,e}) \eta_i + R_{g_n^{-1}} \check{P}_{i,e} \eta_i).$$

Remark that the range of V is

$\ell^2(F_{n,n} \setminus \{g_n\}, \mathcal{K})$ and that for $a \in A$ $(\pi(a) \mid \ell^2(F_{n,n}, \mathcal{K}))V =$

$$= V((\pi(a) \mid \ell^2(F_{n,n}, \mathcal{K})) \oplus \underbrace{\pi(a) \oplus \ldots \oplus \pi(a)}_{(n-1)\text{-times}})$$

Moreover for $x \in A \times_{\alpha\gamma} F_n$ we have

$$(P_{n,e}\, x|\ell^2(F_{n,n}\mathcal{K}))-V((P_{n,e}\, x|\ell^2(F_{n,n}\mathcal{K}))\oplus \underbrace{x\oplus\ldots\oplus x}_{(n-1)\text{-times}})V^* \in$$
$$\in \quad A\otimes K(\ell^2(F_{n,n})).$$

To check this fact, remark first that for $x\in\pi(A)\cup\{u_1,\ldots,u_n\}$ the above relation holds. Next note that the map $A\times_{\alpha_r} F_n \ni x \longmapsto (P_{n,e}\, x|\ell^2(F_{n,n}\mathcal{K}))$ is a completely positive map, which modulo $A\otimes K(\ell^2(F_{n,n}))$ is a $*$ - homomorphism. Thus our assertion will follow after checking that

$$V(A\otimes K(\ell^2(F_{n,n})))\oplus 0 \oplus \ \ldots\oplus 0)V^* \subset A\otimes K(\ell^2(F_{n,n})).$$

Since V intertwines

$$(\pi(a)\,|\,\ell^2(F_{n,n}\mathcal{K}))\oplus\pi(a)\oplus\ldots\oplus\pi(a) \quad \text{and} \quad (\pi(a)|\ell^2(F_{n,n}\mathcal{K})), \text{ we}$$

see that it will be sufficient to check that

$$V((1\otimes K(\ell^2(F_{n,n})))\oplus 0\oplus\ldots\oplus 0)V^* \subset 1\otimes K(\ell^2(F_{n,n}))$$

or equivalently, to check that

$$R_{g_n^{-1}}(1\otimes K(\ell^2(F_{n,n})))R_{g_n} \subset 1\otimes K(\ell^2(F_{n,n}))$$

which is quite easy.

Now, we define

$$\tilde{V}: \mathcal{H}_\tau\oplus(\ell^2(F_n\mathcal{K}))^{n-1} \longrightarrow \mathcal{H}_\tau \quad \text{by}$$

$$\tilde{V}(\xi\oplus\eta) = \xi\oplus V\eta \quad \text{where} \quad \xi\in\bigoplus_{j=1}^{n-1}\ell^2(F_{n,j},\mathcal{K})$$

and $\quad \eta \in \ell^2(F_{n,n}\mathcal{K})\oplus(\ell^2(F_n,\mathcal{K}))^{n-1}.$

Then, $\tilde{V}$ is isometric and $\tilde{V}\tilde{V}^* = I_{\mathcal{H}_\tau} - E_n$. Moreover, for $x\in\mathcal{T}$ we have

$$x-\tilde{V}(x\oplus\underbrace{p(x)\oplus\ldots\oplus p(x)}_{(n-1)\text{-times}})\tilde{V}^* \in \mathcal{J}$$

and for $a\in A$

$$\rho(a)-\tilde{V}(\rho(a)\oplus\pi(a)\oplus\ldots\oplus\pi(a))\tilde{V}^{*}=$$
$$=\rho(a)(I-\tilde{V}\tilde{V}^{*})=\rho(a)E_n.$$

3.5. Lemma. The homomorphism

$$\rho_{*}:K_1(A)\longrightarrow K_1(\mathcal{T})$$

is an isomorphism.

Proof. That ρ_* is surjective has been already established in Lemma 3.4, so it will be sufficient to prove that ρ_* is injective

Using several times the argument that the Toeplitz-extension for $(A\otimes \mathcal{M}_n, \alpha\otimes id_n)$ coincides with the Toeplitz-extension of (A,α) tensored by $\mathcal{M}_n$ it is easily seen that it will be sufficient to prove the following fact: if $v_0, v_1 \in A$ are unitaries and if $[0,1]\ni t\longmapsto w_t\in\mathcal{T}$ is a continuous functions with values unitary elements of $\mathcal{T}$, such that $w_0=\rho(v_0)$ and $w_1=\rho(v_1)$, then in $K_1(A)$ we have $[v_0]=[v_1]$.

To this end consider

$$\tilde{w}_t=E_n+\tilde{V}(w_t\oplus\underbrace{p(w_t)\oplus\ldots\oplus p(w_t)}_{(n-1)-\text{times}})\tilde{V}^{*}$$

Remark that $\tilde{w}_t$ is unitary, depends continuously on t and $\tilde{w}_t-w_t\in\mathcal{J}$. Moreover $\tilde{w}_0=E_n+\rho(v_0)(1-E_n)$ and $\tilde{w}_1=E_n+\rho(v_1)(1-E_n)$. Thus for

$$y_t=\tilde{w}_t^{*}w_t$$

we have

$$y_0=\rho(v_0)E_n+(1-E_n)$$
$$y_1=\rho(v_1)E_n+(1-E_n)$$
$$y_t\in 1+\mathcal{J}$$

and $t \longmapsto y_t$ is a continuous function with values unitaries in $1+\mathcal{J}$. Thus we have $[y_0]=[y_1]$ in $K_1(\mathcal{J})$. Taking into acount the isomorphisms $K_1(\mathcal{J}) \simeq K_1((A\otimes K)^n) \simeq (K_1(A))^n$ this gives precisely

$$\underbrace{0\oplus \cdots \oplus 0}_{(n-1)\text{-times}} \oplus [v_0] = \underbrace{0\oplus \quad \oplus 0}_{(n-1)\text{-times}} \oplus [v_1]$$

in $(K_1(A))^n$ and hence $[v_0]=[v_1]$.

Q.E.D.

3.6. <u>Theorem</u>. *The diagram*

$$\begin{tikzcd}
(K_0(A))^n \arrow[r, "\beta_0"] & K_0(A) \arrow[r, "\pi_*"] & K_0(A \times_{\alpha r} F_n) \arrow[d] \\
K_1(A \times_{\alpha r} F_n) \arrow[u] & K_1(A) \arrow[l, "\pi_*"'] & (K_1(A))^n \arrow[l, "\beta_1"']
\end{tikzcd}$$

where the vertical arrows correspond to the connecting homomorphisms in the exact sequence for the Toeplitz extension (modulo the isomorphisms $K_j((A\otimes K)^n) \simeq (K_j(A))^n$, $j=0,1$) and where $\beta_j(\gamma_1 \oplus \ldots \oplus \gamma_n) = (\gamma_1 - (\alpha(g_1^{-1}))_* \gamma_1) + \ldots + (\gamma_n - (\alpha(g_n^{-1}))_* \gamma_n)$ for $\gamma_i \in K_j(A)$ $(1 \le i \le n, j=0,1)$, is an exact sequence.

<u>Proof</u>. Looking at the diagram in the proof of Lemma 3.4 and using Lemma 3.2 and Lemma 3.5 we have that the sequence

$$K_0(A \times_{\alpha r} F_n) \longrightarrow (K_1(A))^n \xrightarrow{\beta_1} K_1(A) \xrightarrow{\pi_*} K_1(A \times_{\alpha r} F_n) \longrightarrow K_0(A)$$

is exact. This shows also, that the theorem will be proved if we can establish the analogue of Lemma 3.5, with K_1 replaced by K_0 (the analogue of Lemma 3.2 is just Lemma 3.3).

To this end, remark first that

$(id_{C(\mathbb{T})}\otimes \rho)_*: K_1(C(\mathbb{T})\otimes A) \longrightarrow K_1(C(\mathbb{T})\otimes \mathcal{T})$

is an isomorphism. Indeed, applying Lemma 3.5 to $(C(\mathbb{T})\otimes A, id_{C(\mathbb{T})}\otimes\alpha)$ instead of (A,α), we obtain precisely this fact.

Now, related to the periodicity theorem, there is a natural isomorphism

$$K_1(C(\mathbb{T})\otimes M) \simeq K_0(M)\oplus K_1(M)$$

for any unital C^*-algebra. This together with the fact that

$(id_{C(\mathbb{T})}\otimes \rho)_*: K_1(C(\mathbb{T})\otimes A) \longrightarrow K_1(C(\mathbb{T})\otimes\mathcal{T})$

is an isomorphism, immediately gives that $\rho_*: K_0(A) \longrightarrow K_0(\mathcal{T})$ is an isomorphism.

Q.E.D.

3.7. __Corollary__. <u>We have</u> $K_0(C_r^*(F_n)) \simeq \mathbb{Z}$ <u>the generator being</u> $[1]$. <u>Also, we have</u> $K_1(C_r^*(F_n)) \simeq \mathbb{Z}^n$ <u>the generators being</u> $[u_1], \ldots, [u_n]$.

<u>Proof</u>. That $K_0(C_r^*(F_n)) \simeq \mathbb{Z}$ with generator $[1]$ follows immediately from Theorem 3.6 applied, to the case when $A = \mathbb{C}$. Also, from Theorem 3.6 we get that the index map $K_1(C_r^*(F_n)) \longrightarrow (K_0(\mathbb{C}))^n \simeq \mathbb{Z}$ is an isomorphism. The computation of this index map in the proof of Lemma 3.4 immediately shows that the classes $[u_1], \ldots [u_n]$ are mapped into the generators of $\mathbb{Z}^n$.

Q.E.D.

3.8. __Corollary__. <u>There is no non-trivial projection in</u> $C_r^*(F_n)$.

<u>Proof</u>. There is a faithful trace-state on $C_r^*(F_n)$. The

range of the homomorphism $K_0(C_r^*(F_n)) \longrightarrow \mathbb{R}$ induced by this trace-state is $\mathbb{Z}$ because of Corollary 3.7. Hence the trace of a projection in $C_r^*(F_n)$ can be only 0 or 1 and hence the only projections in $C_r(F_n)$ are 0 and 1.

Q.E.D.

3.9. Remark. Taking inductive limits the abalogs of Theorem 3.6 and Corollaries 3.7 and 3.8 for free groups with an infinite number of generators are immediately obtained.

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