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1. INTRODUCTION. Let X and Y be complex Hilbert spaces. We denote by $\mathcal{C}(X,Y)$ the set of all closed linear operators, defined on linear submanifolds on X, with values in Y. Let $\mathcal{B}(X,Y)$ be the subset of those operators from $\mathcal{C}(X,Y)$ which are everywhere defined, hence continuous. We write simply $\mathcal{C}(X)$ for $\mathcal{C}(X,X)$ and $\mathcal{B}(X)$ for $\mathcal{B}(X,X)$. If $S\in\mathcal{C}(X,Y)$, let $D(S)$, $N(S)$ and $R(S)$ be respectively the _domain of definition_, the _null-space_ and the _range_ of S. We need also the notion of _reduced minimum modulus_ $\gamma(S)$ of S [5], which is given by the formula

$$\gamma(S)=\sup\{\gamma\geq 0,\ \|Sx\|\geq\gamma\|(1-P_{N(S)})x\|,\ x\in D(S)\},$$

where $P_{N(S)}$ is the orthogonal projection of X onto $N(S)$, provided that $S\neq 0$. When $S\subset 0$ then one defines $\gamma(S)=\infty$. It is easily seen that $\gamma(S)>0$ if and only if $R(S)$ is closed and in this case $\gamma(S)^{-1}$ is the norm of the operator $Sx\to(1-P_{N(S)})x$ from $R(S)$ into X.

Consider now a _(cochain) complex of Hilbert spaces_ [12] $(X,\alpha)=(X^p,\alpha^p)_{p\in Z}$, where Z is the ring of integers, X^p is a Hilbert space, $\alpha^p\in\mathcal{C}(X^p,X^{p+1})$ and $R(\alpha^p)\subset N(\alpha^{p+1})$ for all $p\in Z$. Let us denote by $\{H^p(X,\alpha)\}_{p\in Z}$ the cohomology of the complex (X,α), i.e.

$$H^p(X,\alpha)=N(\alpha^p)/R(\alpha^{p-1}),\quad p\in Z,$$

and by $\dim H^p(X,\alpha)$ the algebraic dimension of the linear space

$H^p(X,\alpha)$.

We recall that a complex of Hilbert spaces $(X,\alpha)=(X^p,\alpha^p)_{p\in Z}$ is said to be Fredholm [12] if $\inf\{\gamma(\alpha^p);\ p\in Z\}>0$, $\dim H^p(X,\alpha)<\infty$ for each $p\in Z$ and $H^p(X,\alpha)\neq 0$ only for a finite number of indices. In this case we may define the index of (X,α) by the formula

$$\text{ind}\ (X,\alpha)=\sum_{p\in Z}(-1)^p \dim H^p(X,\alpha),$$

and the number ind (X,α), which is in fact the Euler characteristic of the complex (X,α), is invariant under small or compact perturbations (see [12] for details).

There is a more general concept of complex of Hilbert spaces, called semi - Fredholm, for which the index, possibly infinite, still makes sense (see [12] for a precise definition). We shall work in the last section with a particular semi-Fredholm complex of Hilbert spaces which is not Fredholm.

We note that for a Fredholm complex (X,α) the quotient space $H^p(X,\alpha)$ is isomorphic to the subspace $N(\alpha^p)\ominus R(\alpha^{p-1})$ for all $p\in Z$, so that $H^p(X,\alpha)$ will be given this meaning in the sequel. It is easily seen that there is no essential loss of generality in assuming that $D(\alpha^p)$ is dense in X^p for every $p\in Z$ so that we shall work only with complexes $(X,\alpha)=(X^p,\alpha^p)_{p\in Z}$ having this property.

The aim of this work is to consider tensor products of Fredholm complexes of Hilbert spaces and to prove a variant of the Künneth formula [6] for them. In order to state the main result we need some more notations and definitions. For any pair of Hilbert spaces X and Y we denote by $X \bar{\otimes} Y$ the completion, with respect to the canonical Hilbert norm, of the algebraic tensor product $X \otimes Y$. Take now two complexes of Hilbert spaces $(X,\alpha)=(X^p,\alpha^p)_{p\in Z}$ and $(Y,\beta)=(Y^q,\beta^q)_{q\in Z}$. By analogy with the algebraic case [6] we shall define the <u>tensor product</u> $(X \bar{\otimes} Y, \alpha \bar{\otimes} \beta)$ of the complexes (X,α) and (Y,β) as the complex of Hilbert spaces $(Z,\lambda)=(Z^r,\lambda^r)_{r\in Z}$, where

$$Z^r = \bigoplus_{p+q=r} (X^p \bar{\otimes} Y^q)$$

and λ^r is given, roughly speaking, by the formula

$$\lambda^r | D(\alpha^p) \otimes D(\beta^q) = (\alpha^p \otimes 1_q) + (-1)^p (1_p \otimes \beta^q)$$

(see the next section for a precise definition) for all p, q and r in Z, p+q=r, where 1_p and 1_q are the identities on X^p and Y^p, respectively.

The main result of the paper is the following:

THEOREM. <u>Consider two Fredholm complexes of Hilbert spaces</u> (X,α) <u>and</u> (Y,β). <u>Then their tensor product</u> $(X \bar{\otimes} Y, \alpha \bar{\otimes} \beta)$ <u>is Fredholm and has the properties</u>:

(1) $$H^r(X \bar{\otimes} Y, \alpha \bar{\otimes} \beta) = \bigoplus_{p+q=r} (H^p(X,\alpha) \otimes H^q(Y,\beta)),$$

<u>for all</u> $r \in Z$ (the tensor Künneth formula);

(2) $$\text{ind}\,(X \bar{\otimes} Y, \alpha \bar{\otimes} \beta) = \text{ind}\,(X,\alpha) \,.\, \text{ind}\,(Y,\beta) \,.$$

The next section contains the auxiliary results needed for the proof of Theorem as well as the proof itself.

The last section contains two applications. The first one is a consequence of our results applied to the tensor products of commuting systems of linear closed operators. The second application is the solution of the $\bar{\partial}$-problem for vector - valued square integrable exterior forms on strongly pseudoconvex domains, using some well-known results in the scalar case [4].

2. PROOF OF THE MAIN RESULT. We obtain our Theorem, stated in the Introduction, as a consequence of some auxiliary results. The first result transforms some information connected with a complex of Hilbert spaces into an equivalent property valid for a certain operator (see [11] for a similar but not identical procedure).

2.1. PROPOSITION. Let $(X,\alpha)=(X^p,\alpha^p)_{p\in Z}$ be a complex of Hilbert spaces. Then there exist a Hilbert space H_α and a densely defined operator $T_\alpha\in C(H_\alpha)$ such that $R(T_\alpha)\subset N(T_\alpha)$, with the following properties:

(1) $\gamma_\alpha:=\inf\{\gamma(\alpha^p);\ p\in Z\}=\gamma(T_\alpha)$; in particular, $\gamma_\alpha>0$ iff $R(T_\alpha)$ is closed;

(2) $\gamma_\alpha>0$ and $H^p(X,\alpha)=0$ for all p iff $R(T_\alpha)=N(T_\alpha)$;

(3) (X,α) is Fredholm iff $T_\alpha+T_\alpha^*$ is Fredholm.

Proof. We define the Hilbert space

$$H_\alpha=\bigoplus_{p\in Z} X^p$$

and the operator T_α on H_α by the relation

$$T_\alpha\Big(\bigoplus_{p\in Z} x_p\Big)=\bigoplus_{p\in Z}\alpha^p x_p$$

where

(2.1) $\bigoplus_{p\in Z} x_p\in H_\alpha$, $x_p\in D(\alpha^p)$, $p\in Z$, $\sum_{p\in Z}\|\alpha^p x_p\|^2<\infty$.

It is easily seen that T_α is a densely defined closed operator whose domain of definition is given by (2.1). It is also clear that

(2.2) $N(T_\alpha)=\bigoplus_{p\in Z} N(\alpha^p)$

and that $R(T_\alpha)\subset N(T_\alpha)$.

Assume now $\gamma_\alpha>0$. In this case we have the equality

(2.3) $R(T_\alpha)=\bigoplus_{p\in Z} R(\alpha^p)$

Plainly, $R(T_\alpha)$ is contained in $\bigoplus_{p\in Z} R(\alpha^p)$. Conversely, take $\bigoplus_{p\in Z} y_{p+1}\in\bigoplus_{p\in Z} R(\alpha^p)$, therefore $y_{p+1}=\alpha^p x_p$, for all p. As we may take $x_p\in N(\alpha^p)^\perp$, we have $\|x_p\|\le\gamma(\alpha^p)^{-1}\|y_{p+1}\|$, hence

$$\sum_{p\in Z}\|x_p\|^2\leqslant\sup_{p\in Z}\gamma(\alpha^p)^{-1}\sum_{p\in Z}\|\alpha^p x_p\|^2$$

so $\bigoplus_{p\in Z} x_p\in D(T_\alpha)$ and $\gamma(T_\alpha)\ge\gamma_\alpha$.

Supposing $R(T_\alpha)$ closed and taking $y_{p+1}\in R(\alpha^p)$ we can find $x_p\in D(\alpha^p)$ with $y_{p+1}=\alpha^p x_p$ and $\|x_p\|\le\gamma(T_\alpha)^{-1}\|y_{p+1}\|$; as $p\in Z$ is arbitrary, we infer $\gamma_\alpha\ge\gamma(T_\alpha)$. Note that if either γ_α or $\gamma(T_\alpha)$ is null, the above argument shows that the other has to be null too, consequently $\gamma_\alpha=\gamma(T_\alpha)$.

If $\gamma_\alpha>0$ and $H^p(X,\alpha)=0$ for all p, then, by (2.2) and (2.3) we obtain that $R(T_\alpha)=N(T_\alpha)$. Conversely, if $R(T_\alpha)=N(T_\alpha)$ then

$\gamma(T_\alpha)>0$, which implies (2.3) by the previous argument, whence $H^p(X,\alpha)=0$ for all $p\in Z$, concluding the proof of the second assertion.

The operator $A_\alpha=T_\alpha+T_\alpha^*$ is self-adjoint [10] and satisfies

$$\overline{R(A_\alpha)}=\overline{R(T_\alpha)}\oplus\overline{R(T_\alpha^*)}.$$

Note that $R(T_\alpha)$ is closed iff $R(A_\alpha)$ is closed; on the other hand one can see that

$$N(T_\alpha+T_\alpha^*)=N(T_\alpha)\cap N(T_\alpha^*)=\bigoplus_{p\in Z}N(\alpha^p)\cap N((\alpha^{p-1})^*)=\bigoplus_{p\in Z}(N(\alpha^p)\ominus R(\alpha^{p-1})).$$

From these facts it is plain now that (X,α) is Fredholm iff A_α is Fredholm, which completes the proof.

2.2. COROLLARY. <u>A complex of Hilbert spaces</u> (X,α) <u>is Fredholm and exact iff</u> $(T_\alpha+T_\alpha^*)^{-1}\in\mathcal{B}(H_\alpha)$, <u>where</u> H_α <u>and</u> T_α <u>are given by Proposition 2.1</u>.

<u>Proof</u>. We have $R(T_\alpha)\subset N(T_\alpha)$. The equality holds iff $(T_\alpha+T_\alpha^*)^{-1}\in\mathcal{B}(H_\alpha)$, as shown in [10, Lemma 3.1].

2.3. <u>Remark</u>. The construction from Proposition 2.1 does not yield a bounded operator T_α even in case $\alpha^p\in\mathcal{B}(X^p,X^{p+1})$, for every $p\in Z$, unless $\sup\|\alpha^p\|<\infty$. It is therefore natural, from our standpoint, to work only with closed operators.

Let X_1, X_2, Y_1, Y_2 be Hilbert spaces, $S_1\in\mathcal{C}(X_1,Y_1)$ and $S_2\in\mathcal{C}(X_2,Y_2)$ be densely defined operators. Then the operator $S_1\otimes S_2$, defined on $D(S_1)\otimes D(S_1)$, is closable (as a consequence of the fact that $S_1^*\otimes S_2^*$ is defined on the dense subspace $D(S_1^*)\otimes D(S_2^*)$, S_j^* being the adjoint of S_j, $j=1,2$); we denote by $S_1 \bar{\otimes} S_2$ its canonical closure (see also [8]).

2.4. LEMMA. Consider two Hilbertspaces X, Y and take $S \in C(X)$ densely defined, with $R(S)$ closed. Then $(S \bar{\otimes} 1_Y)^* = S^* \bar{\otimes} 1_Y$, $R(S \bar{\otimes} 1_Y) = R(S) \bar{\otimes} Y$, $N(S \bar{\otimes} 1_Y) = N(S) \bar{\otimes} Y$ and $\gamma(S \bar{\otimes} 1_Y) = \gamma(S)$, where 1_Y is the identity on Y.

Proof. The equality $(S \bar{\otimes} 1_Y)^* = S^* \bar{\otimes} 1_Y$ follows from [8, Chapt.9] (and it is not connected with the assumption that R(S) be closed).

The inclusion $N(S) \bar{\otimes} Y \subset N(S \bar{\otimes} 1_Y)$ is obvious. Conversely, if $\xi \in N(S \bar{\otimes} 1_Y)$ we can find a sequence $\xi_k \in D(S) \otimes Y$ such that $\xi_k \to \xi$ and $(S \otimes 1_Y)\xi_k \to 0$ as $k \to \infty$. Moreover, we may represent

$$\xi_k = \sum_{j \in I_k} x_j \otimes y_j \ , \quad k \in \mathbb{Z} \ ,$$

with $\{y_j\}_{j \in I_k}$ an orthonormal system, I_k being a finite family of indices. Write then $x_j = x_j' + x_j''$ with $x_j' \in N(S)$ and $x_j'' \in N(S)^\perp$, hence $\|x_j''\| \le \gamma(S)^{-1} \|Sx_j\|$, for all $j \in I_k$. Set

$$\xi_k' = \sum_{j \in I_k} x_j' \otimes y_j, \quad \xi_k'' = \sum_{j \in I_k} x_j'' \otimes y_j$$

and note that

$$\|\xi_k''\|^2 = \sum_{j \in I_k} \|x_j''\|^2 \le \gamma(S)^{-1} \sum_{j \in I_k} \|Sx_j\|^2 = \gamma(S)^{-1} \|(S \otimes 1_Y)\xi_k\|^2$$

therefore $\xi_k'' \to 0$ as $k \to \infty$, showing that $\xi \in N(S) \bar{\otimes} Y$.

Let us prove now that $R(S \bar{\otimes} 1_Y)$ is closed. For, consider $\eta = \sum_{j \in I} Sx_j \otimes y_j$, with I finite and $\{y_j\}_{j \in I}$ an orthonormal system.

Then, as above, if $\xi = \sum_{j\in I} x_j \otimes y_j$ with $x_j \in N(S)^\perp$ we have $\|\xi\| \le \gamma(S)^{-1}\|\eta\|$. In particular, $R(S \bar{\otimes} 1_Y)$ is closed and $\gamma(S \bar{\otimes} 1_Y) \ge \gamma(S)$. In fact, this is actually an equality. Indeed, if $\eta = Sx \otimes y$ with $\|y\|=1$ then $\xi = x \otimes y \in N(S \bar{\otimes} 1_Y)^\perp$ when $x \in N(S)^\perp$ and

$$\|x\| = \|\xi\| \le \gamma(S \bar{\otimes} 1_Y)^{-1}\|\eta\| = \gamma(S \bar{\otimes} 1_Y)^{-1}\|Sx\|,$$

whence $\gamma(S) \ge \gamma(S \bar{\otimes} 1_Y)$.

Finally, from these arguments we infer that

$$R(S \bar{\otimes} 1_Y) = N(S^* \bar{\otimes} 1_Y)^\perp = R(S) \bar{\otimes} Y,$$

which concludes the proof of the lemma.

2.5. Remark. Lemma 2.4 provides a different proof for Theorem 2.7 of [10], having a less specific character.

2.6. LEMMA. *Let $(X,\alpha) = (X^p, \alpha^p)_{p\in Z}$ and $(Y,\beta) = (Y^q, \beta^q)_{q\in Z}$ be complexes of Hilbert spaces and denote by $\{H_\alpha, T_\alpha\}$, $\{H_\beta, T_\beta\}$ the Hilbert spaces and the operators given by Proposition 2.1 for (X,α) and (Y,β) respectively. Define then $\tau_\alpha \in \mathcal{B}(H_\alpha)$ by the relation*

$$\tau_\alpha\left(\bigoplus_{p\in Z} x_p\right) = \bigoplus_{p\in Z} (-1)^p x_p \tag{2.4}$$

If $A_{\alpha,\beta} = (T_\alpha + T_\alpha^*) \bar{\otimes} 1_\beta + \tau_\alpha \bar{\otimes} (T_\beta + T_\beta^*)$ then $A_{\alpha,\beta}$ is self-adjoint in $H_\alpha \bar{\otimes} H_\beta$ and satisfies

$$\|A_{\alpha,\beta}\xi\|^2 = \|((T_\alpha + T_\alpha^*) \bar{\otimes} 1_\beta)\xi\|^2 + \|(\tau_\alpha \bar{\otimes} (T_\beta + T_\beta^*))\xi\|^2 \tag{2.5}$$

for every $\xi \in D(A_{\alpha,\beta})$, where 1_β is the identity on H_β.

Proof. It is known that $A_\alpha = T_\alpha + T_\alpha^*$ is self-adjoint [10, Lemma 2.4] hence $A_\alpha \bar{\otimes} 1_\beta$ is self-adjoint, by Lemma 2.4. Similarly

$\tau_\alpha \bar{\otimes} A_\beta$ is self-adjoint, where $A_\beta = T_\beta + T_\beta^*$. Let E_α and E_β be the spectral measures [3] of A_α and A_β, respectively. The proof of the lemma will be obtained in several steps.

1°. If σ is any bounded Borel set in $\mathbb{R}$ and $\xi \in H_\alpha \bar{\otimes} H_\beta$ then $\xi_\sigma = (E_\alpha(\sigma) \bar{\otimes} E_\beta(\sigma))\xi \in D(A_{\alpha,\beta})$. Indeed, the operators $A_\alpha E_\alpha(\sigma)$ and $A_\beta E_\beta(\sigma)$ are bounded [3] and we infer that $(A_\alpha \bar{\otimes} 1_\beta + \tau_\alpha \bar{\otimes} A_\beta)\xi_\sigma =$ $(A_\alpha E_\alpha(\sigma) \bar{\otimes} E_\beta(\sigma) + \tau_\alpha E_\alpha(\sigma) \bar{\otimes} A_\beta E_\beta(\sigma))\xi$.

2°. If we fix $\xi \in H_\alpha \bar{\otimes} H_\beta$ then for any $\varepsilon > 0$ there exists a bounded Borel set $\sigma \subset \mathbb{R}$ large enough such that $||\xi - \xi_\sigma|| < \varepsilon$. Indeed, $\sigma \to E_\alpha(\sigma) \bar{\otimes} 1_\beta$ and $\sigma \to 1_\alpha \bar{\otimes} E_\beta(\sigma)$ are two commuting spectral measures in $H_\alpha \bar{\otimes} H_\beta$, therefore their product is a spectral measure on $\mathbb{R} \times \mathbb{R}$ [3]. In particular, there exists a Borel set σ with the required property.

3°. If $\xi \in D(A_{\alpha,\beta})$ and $\xi_k = \xi_{\sigma_k}$, where σ_k is the interval $[-k,k]$, k natural, then $\xi_k \to \xi$ and $A_{\alpha,\beta}\xi_k \to A_{\alpha,\beta}\xi$ as $k \to \infty$. Indeed, $\xi_k \to \xi$ by 2°. Then we have $(A_\alpha \bar{\otimes} 1_\beta)\ (E_\alpha(\sigma_k) \bar{\otimes} 1_\beta)\xi \to$ $\to (A_\alpha \bar{\otimes} 1_\beta)\xi$, since $\sigma \to E_\alpha(\sigma) \bar{\otimes} 1_\beta$ is the spectral measure of $A_\alpha \bar{\otimes} 1_\beta$. Analogously, $(\tau_\alpha \bar{\otimes} A_\beta)(1_\alpha \bar{\otimes} E_\beta(\sigma_k))\xi \to (\tau_\alpha \bar{\otimes} A_\beta)\xi$, $\sigma \to 1_\alpha \bar{\otimes} E_\beta(\sigma)$ being the spectral measure of $1_\alpha \bar{\otimes} A_\beta$. Noticing that $(A_\alpha \bar{\otimes} 1_\beta)(E_\alpha(\sigma_k) \bar{\otimes} (E_\beta(\sigma_k) - 1_\beta))\xi =$

$= (E_\alpha(\sigma_k) \bar{\otimes} 1_\beta)(1_\alpha \bar{\otimes} (E_\beta(\sigma_k) - 1_\beta))(A_\alpha \bar{\otimes} 1_\beta)\xi$ and

$(\tau_\alpha \bar{\otimes} A_\beta)((E_\alpha(\sigma_k) - 1_\alpha) \bar{\otimes} E_\beta(\sigma_k))\xi =$

$= (\tau_\alpha \bar{\otimes} E_\beta(\sigma_k))((E_\alpha(\sigma_k) - 1_\alpha) \bar{\otimes} 1_\beta)(1_\alpha \bar{\otimes} A_\beta)\xi$,

we infer easily that $A_{\alpha,\beta}\xi_k \to A_{\alpha,\beta}\xi$ as $k \to \infty$.

4°. Let us show now that (2.5) holds. Indeed, if $\xi \in D(A_{\alpha,\beta})$

and $\sigma\subset\mathbb{R}$ is bounded, then we have

$$||A_{\alpha,\beta}\xi_\sigma||^2=||(A_\alpha\bar{\otimes}1_\beta)\xi_\sigma||^2+<(A_\alpha\bar{\otimes}1_\beta)\xi_\sigma,$$

$$(\tau_\alpha\bar{\otimes}A_\beta)\xi_\sigma>+<(\tau_\alpha\bar{\otimes}A_\beta)\xi_\sigma\ ,\ (A_\alpha\bar{\otimes}1_\beta)\xi_\sigma>+$$

$$+||(\tau_\alpha\bar{\otimes}A_\beta)\xi_\sigma||^2=||(A_\alpha\bar{\otimes}1_\beta)\xi_\sigma||^2+||(\tau_\alpha\bar{\otimes}A_\beta)\xi_\sigma||^2$$

since $(\tau_\alpha\bar{\otimes}A_\beta)\xi_\sigma\in D(A_\alpha\bar{\otimes}1_\beta)$, $(A_\alpha\bar{\otimes}1_\beta)\xi_\sigma\in D(\tau_\alpha\bar{\otimes}A_\beta)$ from 1° and $(A_\alpha\tau_\alpha\bar{\otimes}A_\beta)\xi_\sigma+(\tau_\alpha A_\alpha\bar{\otimes}A_\beta)\xi_\sigma=0$ by the property $A_\alpha\tau_\alpha+\tau_\alpha A_\alpha=0$. The relation (2.5) is then obtained by applying 3°.

5°. The operator $A_{\alpha,\beta}$ is closed. Indeed, this is a simple consequence of (2.5).

6°. We have only to show that $A_{\alpha,\beta}$ is self-adjoint. Indeed, $(A_\alpha\bar{\otimes}1_\beta)^2=A_\alpha^2\bar{\otimes}1_\beta$ is self-adjoint and $(\tau_\alpha\bar{\otimes}A_\beta)^2=1_\alpha\bar{\otimes}A_\beta^2$ is self--adjoint too. Moreover, the spectral measure of $A_\alpha^2\bar{\otimes}1_\beta$ commutes with the spectral measure of $1_\alpha\bar{\otimes}A_\beta^2$, therefore $A_\alpha^2\bar{\otimes}1_\beta+1_\alpha\bar{\otimes}A_\beta^2$ is also self-adjoint [7]. Plainly we have $A^*_{\alpha,\beta}\supset A_{\alpha,\beta}$. Taking $\eta\in D(A^*_{\alpha,\beta})$ and $\zeta=A^*_{\alpha,\beta}\eta$ such that the pair $\{\eta,\zeta\}$ be orthogonal on the graph of $A_{\alpha,\beta}$, we obtain that $\zeta\in D(A^*_{\alpha,\beta})$ and $(1+A^{*2}_{\alpha,\beta})\eta=0$. Notice that

$$A^2_{\alpha,\beta}\xi_\sigma=(A_\alpha^2\bar{\otimes}1_\beta)\xi_\sigma+(1_\alpha\bar{\otimes}A_\beta^2)\xi_\sigma$$

for every bounded $\sigma\subset\mathbb{R}$, therefore $A^2_{\alpha,\beta}\supset A_\alpha^2\bar{\otimes}1_\beta+1_\alpha\bar{\otimes}A_\beta^2$. Notice that $(A^*_{\alpha,\beta})^2\subset(A^2_{\alpha,\beta})^*\subset A_\alpha^2\bar{\otimes}1_\beta+1_\alpha\bar{\otimes}A_\beta^2$ and $A_\alpha^2\bar{\otimes}1_\beta+1_\alpha\bar{\otimes}A_\beta^2$ is positive, hence $\eta=0$, implying $A_{\alpha,\beta}=A^*_{\alpha,\beta}$.

2.7. LEMMA. <u>With the conditions of Lemma</u> 2.6, <u>if</u> $T_{\alpha\otimes\beta}=$ $=T_\alpha\otimes 1_\beta+\tau_\alpha\otimes T_\beta$ <u>is defined on</u> $D(T_\alpha)\otimes D(T_\beta)$ <u>then</u> $T_{\alpha\otimes\beta}$ <u>is</u>

closable in $H_\alpha \bar{\otimes} H_\beta$, its canonical closure $T_{\alpha \bar{\otimes} \beta}$ satisfies $R(T_{\alpha \bar{\otimes} \beta}) \subset N(T_{\alpha \bar{\otimes} \beta})$ and one has the equality

$$T_{\alpha \bar{\otimes} \beta} + T^*_{\alpha \bar{\otimes} \beta} = A_{\alpha,\beta}$$

Proof. Since $T^*_\alpha \otimes 1_\beta + \tau_\alpha \otimes T^*_\beta$ is defined on the dense subspace $D(T^*_\alpha) \otimes D(T^*_\beta)$, we derive that $T_{\alpha \otimes \beta}$ is closable. As $T_\alpha \tau_\alpha + \tau_\alpha T_\alpha = 0$, we infer that $R(T_{\alpha \bar{\otimes} \beta}) \subset N(T_{\alpha \bar{\otimes} \beta})$, from a similar property of $T_{\alpha \otimes \beta}$.

Note that $A_{\alpha,\beta} = T_{\alpha \bar{\otimes} \beta} + T^*_{\alpha \bar{\otimes} \beta}$ on $D(A_\alpha) \otimes D(A_\beta)$, $A_{\alpha,\beta}$ is self-adjoint by Lemma 2.6 and $T_{\alpha \bar{\otimes} \beta} + T^*_{\alpha \bar{\otimes} \beta}$ is self-adjoint by [10, Lema 2.4]. The operator $A_{\alpha,\beta}$ is, in fact, essentially self--adjoint on $D(A_\alpha) \otimes D(A_\beta)$, whence $A_{\alpha,\beta}$ and $T_{\alpha \bar{\otimes} \beta} + T^*_{\alpha \bar{\otimes} \beta}$ must be equal.

In order to define the tensor product of two complexes of Hilbert spaces we shall use a procedure suggested by Proposition 2.1 (the direct way is rather troublesome). An elementary but important step in this respect is the identification of the space

$$\bigoplus_{r \in Z} \bigoplus_{p+q=r} (X^p \bar{\otimes} Y^q)$$

with the space

$$(\bigoplus_{p \in Z} X^p) \bar{\otimes} (\bigoplus_{q \in Z} Y^q)$$

for any two families of Hilbert spaces $\{X^p\}_{p \in Z}$ and $\{Y^q\}_{q \in Z}$, which can be made in a natural way.

2.8. Definition. With the notations of Lemmas 2.6 and 2.7, the complex $(Z,\lambda)=(Z^r,\lambda^r)_{r\in\mathbb{Z}}$, where

$$Z^r=\bigoplus_{p+q=r}(X^p \mathbin{\bar{\otimes}} Y^q)$$

and $\lambda^r=T_{\alpha\bar{\otimes}\beta}|Z^r\cap D(T_{\alpha\bar{\otimes}\beta})$ (here Z^r is regarded as a subspace of $H_\alpha \bar{\otimes} H_\beta$), will be called the <u>tensor product</u> of the complexes (X,α) and (Y,β). The complex (Z,λ) will be also denoted by $(X \bar{\otimes} Y,\ \alpha \bar{\otimes} \beta)$.

It is clear that λ^r maps Z^r in Z^{r+1}. Furthermore, if H_λ and T_λ correspond to (Z,λ) in Proposition 2.1, then, with our identifications, $H_\lambda=H_\alpha \bar{\otimes} H_\beta$ and $T_\lambda=T_{\alpha\bar{\otimes}\beta}$. Indeed, the inclusion $T_\lambda\subset T_{\alpha\bar{\otimes}\beta}$ is obvious. Conversely, if $\xi=\bigoplus_{p\in\mathbb{Z}} x_p\in D(T_\alpha)$ and $\eta=\bigoplus_{q\in\mathbb{Z}} y_q\in D(T_\beta)$ then we can write, by the relations (2.1),

$$\|\bigoplus_{r\in\mathbb{Z}}\ \bigoplus_{p+q=r}(\alpha^p x_p\otimes y_q+(-1)^{p+1}x_{p+1}\otimes\beta^{q-1}y_{q-1}\|^2=$$

$$=\sum_{r\in\mathbb{Z}}\ \sum_{p+q=r}\|\alpha^p x_p\otimes y_q+(-1)^{p+1}x_{p+1}\otimes\beta^{q-1}y_{q-1}\|^2\le$$

$$\le(\|T_\alpha\xi\|\|\eta\|+\|\xi\|\|T_\beta\eta\|)^2<\infty\quad,$$

which shows that $\xi\otimes\eta\in D(T_\lambda)$. We have therefore $D(T_\alpha)\otimes D(T_\beta)\subset$ $\subset D(T_\lambda)$; as T_λ is closed, we obtain actually $T_\lambda=T_{\alpha\bar{\otimes}\beta}$.

2.9. LEMMA. <u>Let</u> (X,α) <u>and</u> (Y,β) <u>be two complexes of Hilbert spaces, with</u> (X,α) <u>Fredholm and exact. Then the tensor product</u> $(X \bar{\otimes} Y,\ \alpha \bar{\otimes} \beta)$ <u>is Fredholm and exact</u>.

<u>Proof</u>. We apply Corollary 2.2. Since $T_\alpha+T_\alpha^*$ has a continuous inverse, then $(T_\alpha+T_\alpha^*)\bar{\otimes} 1_\beta$ has a continuous inverse, therefore by (2.5) we deduce

$$||A_{\alpha,\beta}\xi||\geq||((T_\alpha+T_\alpha^*)\ \overline{\otimes}\ 1_\beta)\xi||\geq C||\xi||$$

for every $\xi\in D(A_{\alpha,\beta})$, where $C>0$ is a constant. By Lemma 2.6 the operator $A_{\alpha,\beta}$ is self-adjoint, hence $A_{\alpha,\beta}$ has a continuous inverse. We conclude, by Lemma 2.7 and Definition 2.8 that $(X\ \overline{\otimes}\ Y, \alpha\ \overline{\otimes}\ \beta)$ is Fredholm and exact.

Let us consider two complexes of Hilbert spaces $(X_o,\alpha_o)=(X_o^p,\alpha_o^p)_{p\in\mathbf{Z}}$ and $(X_1,\alpha_1)=(X_1^p,\alpha_1^p)_{p\in\mathbf{Z}}$. Then we may define their direct sum $(X_o\oplus X_1,\ \alpha_o\oplus\alpha_1)$ which is a complex $(X,\alpha)=(X^p,\alpha^p)_{p\in Z}$ given by $X^p=X_o^p\oplus X_1^p$ and $\alpha^p=\alpha_o^p\oplus\alpha_1^p$ for all $p\in\mathbf{Z}$. It is easily seen that if both (X_o,α_o) and (X_1,α_1) are Fredholm then (X,α) is Fredholm and ind $(X,\alpha)=$ind $(X_o,\alpha_o)+$ind (X_1,α_1). If $(Y,\beta)=(Y^q,\beta^q)_{q\in\mathbf{Z}}$ is another Fredholm complex and both $(X_o\ \overline{\otimes}\ Y,\ \alpha_o\ \overline{\otimes}\ \beta)$, $(X_1\ \overline{\otimes}\ Y,\ \alpha_1\ \overline{\otimes}\ \beta)$ are Fredholm then we have

(2.6) $$\text{ind }((X_o\oplus X_1)\ \overline{\otimes}\ Y,\ (\alpha_o\oplus\alpha_1)\ \overline{\otimes}\ \beta)=$$
$$=\text{ind }(X_o\ \overline{\otimes}\ Y,\ \alpha_o\ \overline{\otimes}\ \beta)+\text{ind }(X_1\ \overline{\otimes}\ Y,\ \alpha_1\ \overline{\otimes}\ \beta),$$

by the identification of the complex $((X_o\oplus X_1)\ \overline{\otimes}\ Y,\ (\alpha_o\oplus\alpha_1)\ \overline{\otimes}\beta)$ with the complex $((X_o\ \overline{\otimes}\ Y)\oplus(X_1\ \overline{\otimes}\ Y),\ (\alpha_o\ \overline{\otimes}\ \beta)\oplus(\alpha_1\ \overline{\otimes}\ \beta))$.

2.10. <u>Proof of Theorem</u>. If $(X,\alpha)=(X^p,\alpha^p)_{p\in\mathbf{Z}}$ is a Fredholm complex of Hilbert spaces, then we define $X_o^p=N(\alpha^p)\ominus R(\alpha^{p-1})$, $\alpha_o^p=0$, $X_1^p=X^p\ominus X_o^p$ and $\alpha_1^p=\alpha^p|X_1^p\cap D(\alpha^p)$, for all $p\in\mathbf{Z}$. Then $(X_o,\ \alpha_o)=(X_o^p,\alpha_o^p)_{p\in\mathbf{Z}}$ is a complex of finite dimensional Hilbert spaces of finite length with the property ind $(X_o,\alpha_o)=\sum_{p\in\mathbf{Z}}(-1)^p\dim X_o^p=$ $=$ind (X,α) while $(X_1,\alpha_1)=(X_1^p,\alpha_1^p)_{p\in\mathbf{Z}}$ is a Fredholm complex which is exact.

A similar decomposition can be obtained for another Fredholm complex $(Y,\beta)=(Y^q,\beta^q)_{q\in Z}$, namely $Y_o^q=N(\beta^q)\ominus R(\beta^{q-1})$, $\beta_o^q=0$, $Y_1^q=Y^q\ominus Y_o^q$ and $\beta_1^q=\beta^q|Y_1^q\cap D(\beta^q)$, for all $q\in Z$. Then we have the identification

$$(X\mathbin{\bar{\otimes}} Y,\ \alpha\mathbin{\bar{\otimes}}\beta)=$$

$$=((X_o\mathbin{\bar{\otimes}} Y_o)\oplus(X_o\mathbin{\bar{\otimes}} Y_1)\oplus(X_1\mathbin{\bar{\otimes}} Y_o)\oplus(X_1\mathbin{\bar{\otimes}} Y_1),$$

$$(\alpha_o\mathbin{\bar{\otimes}}\beta_o)\oplus(\alpha_o\mathbin{\bar{\otimes}}\beta_1)\oplus(\alpha_1\mathbin{\bar{\otimes}}\beta_o)\oplus(\alpha_1\mathbin{\bar{\otimes}}\beta_1)),$$

from which we derive the equality

$$\text{ind}\,(X\mathbin{\bar{\otimes}} Y,\ \alpha\mathbin{\bar{\otimes}}\beta)=\text{ind}\,(X_o\mathbin{\bar{\otimes}} Y_o,\alpha_o\mathbin{\bar{\otimes}}\beta_o)\ ,$$

obtained from (2.6) and Lemma 2.9. As $\alpha_o^p=0$, $\beta_o^q=0$ for all indices, we have

$$\text{ind}\,(X_o\mathbin{\bar{\otimes}} Y_o,\alpha_o\mathbin{\bar{\otimes}}\beta_o)=\sum_{r\in Z}(-1)^r\dim\bigoplus_{p+q=r}(X_o^p\otimes Y_o^q)=$$

$$=\sum_{r\in Z}(-1)^r\sum_{p+q=r}(\dim X_o^p)(\dim Y_o^q)=$$

$$=\Big(\sum_{p\in Z}(-1)^p\dim X_o^p\Big)\Big(\sum_{q\in Z}(-1)^q\dim Y_o^q\Big)=(\text{ind}\,(X,\alpha))\cdot(\text{ind}\,(Y,\beta)).$$

The equality

$$H^r(X\mathbin{\bar{\otimes}} Y,\ \alpha\mathbin{\bar{\otimes}}\beta)=\bigoplus_{p+q=r}(H^p(X,\alpha)\otimes H^q(Y,\beta)),\quad r\in Z\ ,$$

follows from the same argument.

We end this section with a result which is useful in some applications.

2.11. PROPOSITION. Let (X,α) and (Y,β) be two complexes of Hilbert spaces. Then $(X\mathbin{\bar{\otimes}} Y),\ \alpha\mathbin{\bar{\otimes}}\beta)$ is Fredholm and exact iff either (X,α) or (Y,β) is Fredholm and exact.

Proof. If (X,α) or (Y,β) is exact then, by Lemma 2.9,

$(X \overline{\otimes} Y, \alpha \overline{\otimes} \beta)$ is also exact.

Conversely, we shall use a procedure inspired from [1]. Assume that both $A_\alpha = T_\alpha + T_\alpha^*$ and $A_\beta = T_\beta + T_\beta^*$ are not invertible (we preserve the notations from Lemmas 2.6 and 2.7). Suppose that there exist sequences $\{\xi_k\}_k \subset H_\alpha$ and $\{\eta_k\}_k \subset H_\beta$ such that $||\xi_k|| = ||\eta_k|| = 1$, $A_\alpha \xi_k \to 0$ and $A_\beta \eta_k \to 0$, as $k \to \infty$. Then $||\xi_k \otimes \eta_k|| = 1$ and $A_{\alpha,\beta}(\xi_k \otimes \eta_k) \to 0$ as $k \to \infty$. Since A_α and A_β are self-adjoint and a self-adjoint operator has a continuous inverse if and only if it is bounded below, we obtain that $A_{\alpha,\beta}$ is not invertible, which is a contradiction.

Note that $(X \overline{\otimes} Y, \alpha \overline{\otimes} \beta)$ is zero iff either (X,α) or (Y,β) is zero, which completes the proof.

3. SOME APPLICATIONS. In this section we give two applications of the previous results.

1) The first application is related to the spectral theory of commuting systems of linear transformation. We recall some definitions and notations from [11] (see also [9] for bounded operators).

Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ denote a system of n indeterminates and $\Lambda[\sigma]$ the exterior algebra over $\mathbb{C}$ generated by $\sigma_1, \ldots, \sigma_n$. For any integer p, $0 \le p \le n$, $\Lambda^p[\sigma]$ will be the space of all homogeneous exterior forms of degree p in $\sigma_1, \ldots, \sigma_n$. For an arbitrary Hilbert space X, $\Lambda[\sigma, X]$ $(\Lambda^p[\sigma, X])$ will denote the tensor product $X \otimes \Lambda[\sigma]$ $(X \otimes \Lambda^p[\sigma])$. If X and Y are two Hilbert spaces, there is a natural identification between $\Lambda[\sigma, X] \overline{\otimes} \Lambda[\zeta, Y]$ and $\Lambda[(\sigma,\zeta), X \overline{\otimes} Y]$, where $\zeta = (\zeta_1, \ldots, \zeta_m)$ is another system of indeterminates (see also [1]). We consider as well the operators $S_j : \Lambda[\sigma] \to \Lambda[\sigma]$, $S_j\xi=\sigma_j\wedge\xi$, $\xi\in\Lambda[\sigma]$, which satisfy the anticommutation relations

$$S_jS_k+S_kS_j=0, \qquad j,k=1,\ldots,n.$$

In the following defintions X will be a fixed Hilbert space.

3.1. Definition [11]. We say that $a=(a_1,\ldots,a_n)\subset C(X)$ is a D-commuting system if there exists a dense subspace D of X in $\bigcap_{j=1}^{n} D(a_j)$ with the properties:

i) the restriction $\hat{\delta}_a=(a_1\otimes S_1+\ldots+a_n\otimes S_n)|\Lambda[\sigma,D]$ is closable;

ii) if δ_a is the canonical closure of $\hat{\delta}_a$ then $R(\delta_a)\subset N(\delta_a)$.

3.2. Definition [11]. Suppose that $a=(a_1,\ldots,a_n)\subset C(X)$ is a D-commuting system. Then a is called singular (nonsingular) if $R(\delta_a)\neq N(\delta_a)$ $(R(\delta_a)=N(\delta_a))$.

Notice that to each D-commuting system we can associate a complex of Hilbert spaces $(\Lambda^p[\sigma,X], \delta_a^p)_{p=0}^n$, where $\delta_a^p = =\delta_a|\Lambda^p[\sigma,X]\cap D(\delta_a)$, hence $a=(a_1,\ldots,a_n)$ is said to be Fredholm [11] if the corresponding complex is Fredholm (see also [2] for bounded operators).

The joint spectrum $\sigma_D(a,X)$ of a D-commuting system $a=(a_1,\ldots,a_n)\subset C(X)$ is the set of those points $z\in\mathbb{C}^n$ such that $z - a$ is singular [11].

If $a=(a_1,\ldots,a_n)\subset C(X)$ is Fredholm then one can define its index [11] by the equality

$$\operatorname{ind}_D a=\operatorname{ind}(\Lambda^p[\sigma,X], \delta_a^p)_{p=0}^n$$

Let Y be another fixed Hilbert space.

3.3. LEMMA. *Let* $a=(a_1,\ldots,a_n)\subset\mathcal{C}(X)$ *be a* D_a*-commuting system and let* $b=(b_1,\ldots,b_m)\subset\mathcal{C}(Y)$ *be a* D_b*-commuting system. Then*

$$a\bar{\otimes} b := (a_1\bar{\otimes} 1_Y,\ldots,a_n\bar{\otimes} 1_Y, 1_X\bar{\otimes} b_1,\ldots,1_X\bar{\otimes} b_m)\subset\mathcal{C}(X\bar{\otimes} Y)$$

is a $D_a\otimes D_b$ *- commuting system and*

$$\sigma_{D_a\otimes D_b}(a\bar{\otimes} b;\, X\bar{\otimes}Y)\subset\sigma_{D_a}(a,X)\times\sigma_{D_b}(b,Y)\ .$$

Proof. If $\sigma=(\sigma_1,\ldots,\sigma_n)$ is a system of indeterminates associated with $a=(a_1,\ldots,a_n)$ then the pair $\{\Lambda[\sigma,X],\ \delta_a\}$ is associated with the complex $(\Lambda^p[\sigma,X],\delta_a^p)_{p=0}^n$ in the sense of Proposition 2.1. Similarly, if $\zeta=(\zeta_1,\ldots,\zeta_m)$ is another system of indeterminates associated with $b=(b_1,\ldots,b_m)$ then the pair $\{\Lambda[\zeta,Y],\ \delta_b\}$ is connected with the complex $(\Lambda^q[\zeta,Y],\delta_b^q)_{q=0}^m$ in the same way. Notice that there is a natural identification between $\Lambda[\sigma,X]\bar{\otimes}\Lambda[\zeta,Y]$ and $\Lambda[(\sigma,\zeta),\ X\bar{\otimes} Y]$ and we have, with this identification,

$$\hat{\delta}_{a\bar{\otimes} b}\theta=(\hat{\delta}_a\otimes 1_{\Lambda[\zeta,Y]})\theta+(\tau\otimes\hat{\delta}_b)\theta$$

for all $\theta\in\Lambda[\sigma,D_a]\otimes\Lambda[\zeta,D_b]=\Lambda[(\sigma,\zeta),\ D_a\otimes D_b]$, where τ is given by (2.4) for $\Lambda[\sigma,X]$ (see also [1]). The operator $\delta_a\otimes 1_{\Lambda[\zeta,Y]}+\tau\otimes\delta_b$, defined on $D(\delta_a)\otimes D(\delta_b)$, is closable by Lemma 2.7, therefore $\hat{\delta}_{a\bar{\otimes} b}$ is closable. In fact, since δ_a is the closure of $\hat{\delta}_a$ on $\Lambda[\sigma,D_a]$ and δ_b is the closure of $\hat{\delta}_b$ on $\Lambda[\zeta,D_b]$, one can easily check that the canonical closure $\delta_{a\bar{\otimes} b}$ of $\hat{\delta}_{a\bar{\otimes} b}$ is equal to the canonical closure of $\delta_a\otimes 1_{\Lambda[\zeta,Y]}+\tau\otimes\delta_b$, hence, by Lemma 2.7, $R(\delta_{a \bar{\otimes} b}) \subset N(\delta_{a \bar{\otimes} b})$, showing that $a \bar{\otimes} b$ is a $D_a \otimes D_b$--commuting system, the set $D_a \otimes D_b \subset \bigcap_{j=1}^{n} D(a_j \bar{\otimes} 1_Y) \cap \bigcap_{k=1}^{m} D(1_X \bar{\otimes} b_k)$ being obviously dense in $X \bar{\otimes} Y$.

In order to prove the second statement, take

$$(z,w) \notin \sigma_{D_a}(a,X) \times \sigma_{D_b}(b,Y) .$$

We may assume without any loss of generality that $z=0$, $w=0$. If $0 \notin \sigma_{D_a}(a,X)$, then by Proposition 2.1 and Lemma 2.9 we infer that $a \bar{\otimes} b$ is nonsingular. The same thing happens when $0 \notin \sigma_{D_b}(b,Y)$, therefore $(0,0) \notin \sigma_{D_a \otimes D_b}(a \bar{\otimes} b, X \bar{\otimes} Y)$.

The next result will not be used in our further arguments. It is, however, of the same type as the other assertions stated in this context (see also [1] for bounded operators).

3.4. LEMMA. <u>Let</u> $a=(a_1,\dots,a_n) \subset C(X)$ <u>be a</u> D_a<u>-commuting system and define</u> $\tilde{a}=(a_1 \bar{\otimes} 1_Y,\dots,a_n \bar{\otimes} 1_Y) \subset C(X \bar{\otimes} Y)$. Then $\tilde{a}$ is a $D_a \otimes Y$-<u>commuting system and, for</u> $Y \neq 0$, $\tilde{a}$ <u>is nonsingular iff</u> a <u>is nonsingular</u>.

<u>Proof</u>. Let $(\Lambda^p[\sigma,X], \delta_a^p)_{p=0}^n$ be the complex associated with $a=(a_1,\dots,a_n)$ and let $(Y^q,0)_{q\geq 0}$ be the complex with $Y^o=Y$ and $Y^q=0$ if $q\geq 1$. The tensor product of these complexes is $(\Lambda^p[\sigma,X] \bar{\otimes} Y, \delta_a^p \bar{\otimes} 1_Y)_{p=0}^n$. Notice that $\Lambda^p[\sigma,X] \bar{\otimes} Y$ may be identified with $\Lambda^p[\sigma,X \bar{\otimes} Y]$, while $\delta_a^p \bar{\otimes} 1_Y$ becomes $\delta_{\tilde{a}}^p$, therefore $\tilde{a}$ is a $D_a \otimes Y$ - commuting system. Moreover, by Propositions 2.11 and 2.1 we infer that $\tilde{a}$ is nonsingular iff a is nonsingular.

The hypothesis $Y \neq 0$ is used in order to obtain the nonsingularity of a from that of $\tilde{a}$, noting that $(Y^q,0)_{q\geq 0}$ is not exact when $Y^{o}=Y\neq 0$.

The next statement extends the main result from [1].

3.5. PROPOSITION. Let $a=(a_1,\ldots,a_n)\subset C(X)$ be a D_a - commuting system and let $b=(b_1,\ldots,b_m)\subset C(Y)$ be a D_b - commuting system. Then the following assertions hold true:

1) $\sigma_{D_a\otimes D_b}(a\bar{\otimes} b,\ X\bar{\otimes} Y)=\sigma_{D_a}(a,X)\times\sigma_{D_b}(b,Y)$

2) If a and b are Fredholm then $a\bar{\otimes} b$ is Fredholm and

$$\mathrm{ind}_{D_a\otimes D_b}(a\bar{\otimes} b)=\mathrm{ind}_{D_a}a\cdot\mathrm{ind}_{D_b}b$$

Proof. For the first statement we have only to prove that $\sigma_{D_a}(a,X)\times\sigma_{D_b}(b,Y)\subset\sigma_{D_a\otimes D_b}(a\bar{\otimes} b,\ X\bar{\otimes} Y)$ the opposite inclusion following from Lemma 3.3.

Indeed, if $0\notin\sigma_{D_a\otimes D_b}(a\bar{\otimes} b,\ X\bar{\otimes} Y)$, then by Proposition 2.11 we have either $0\notin\sigma_{D_a}(a,X)$ or $0\notin\sigma_{D_b}(b,Y)$ hence $0\notin\sigma_{D_a}(a,X)\times\sigma_{D_b}(b,Y)$.

The second assertion follows from our Theorem, via the identification described in the proof of Lemma 3.3.

3.6. PROPOSITION. Let $X_1,\ldots,X_n$ be Hilbert spaces and let $a_j\in C(X_j)$, $j=1,\ldots,n$, be densely defined operators. If $X=X_1\bar{\otimes}\ldots\bar{\otimes}X_n$ and $\tilde{a}_j\in C(X)$ is the canonical closure of the operator

$$1_1\otimes 1_2\otimes\ldots\otimes 1_{j-1}\otimes a_j\otimes 1_{j+1}\otimes\ldots\otimes 1_n$$

then $\tilde{a}=(\tilde{a}_1,\ldots,\tilde{a}_n)$ is a D-commuting system, where $D=D(a_1)\otimes\ldots\otimes D(a_n)$, and

$$\sigma_D(\tilde{a},X)=\sigma_{D(a_1)}(a_1,X_1)\times\ldots\times\sigma_{D(a_n)}(a_n,X_n)\ .$$

Proof. Both statements are obtained by means of an inductive argument. Indeed, for n=2 this is a special case of Proposition 3.5.

Assume now that the statements hold true for any n-1 operators, $n\geq 3$. Then if we denote by b_j the canonical closure of the operator $1_1\otimes\ldots\otimes a_j\otimes\ldots\otimes 1_{n-1}$ $(j=1,\ldots,n-1)$, we obtain that $\tilde{a}=(b_1\bar{\otimes}1_n,\ldots,b_{n-1}\bar{\otimes}1_n,\tilde{1}_{n-1}\bar{\otimes}a_n)$, where $\tilde{1}_{n-1}$ stands for $1_1\bar{\otimes}\ldots\bar{\otimes}1_{n-1}$. If $b=(b_1,\ldots,b_{n-1})$, $D(b)=D(a_1)\otimes\ldots\otimes D(a_{n-1})$ and $\tilde{X}_{n-1}=X_1\bar{\otimes}\ldots\bar{\otimes}X_{n-1}$ then we obtain by Proposition 3.5 and by the induction hypothesis that

$$\sigma_{D(b)\otimes D(a_n)}(b\bar{\otimes}a_n,X)=\sigma_{D(b)}(b,\tilde{X}_{n-1})\times\sigma_{D(a_n)}(a_n,X_n)=$$

$$=\sigma_{D(a_1)}(a_1,X_1)\times\ldots\times\sigma_{D(a_{n-1})}(a_{n-1},X_{n-1})\times\sigma_{D(a_n)}(a_n,X_n)\ .$$

2) The second application is related to the behaviour of the $\bar{\partial}$-operator in strongly pseudoconvex domains. Namely, we prove a result concerning the cohomology of the Cauchy-Riemann complex of H-valued square integrable exterior forms on such a domain, H being an arbitrary Hilbert space, possessing information about the scalar-valued exterior forms.

Let $\Omega\subset\mathbb{C}^n$ be a strongly pseudoconvex domain. We denote by $\Lambda^p[\Omega]$ the Hilbert space of all (0,p) exterior forms on Ω, which are square integrable. Let $\bar{\partial}^p$ be the restriction of the $\bar{\partial}$-operator on $\Lambda^p[\Omega]$. When Ω is an arbitrary strongly pseudoconvex manifold, it is known that

(3.1) $$\dim N(\bar{\partial}^p)/R(\bar{\partial}^{p-1})<\infty,\ p\geq 1.$$

As $\Omega \subset \mathbb{C}^n$, we have actually $R(\bar\partial^{p-1}) = N(\bar\partial^p)$, $p \geq 1$, via the Grauert theorem about holomorphic convexity of strongly pseudoconvex manifolds and Theorem B of Cartan (see [4] for some details). With our terminology, the Cauchy - Riemann complex $(\Lambda^p[\Omega], \bar\partial^p)_{p=0}^n$ is semi-Fredholm [11].

Take now an arbitrary Hilbert space H. One can consider again the space $\Lambda^p[\Omega,H]$ of all H-valued (0,p) exterior forms on Ω, which are square integrable. In this context, the $\bar\partial$-operator, denoted by $\bar\partial_H$, can be constructed in an independent way. Let $\bar\partial^p_H$ be the restriction of $\bar\partial_H$ on $\Lambda^p[\Omega,H]$. One can see that $\Lambda^p[\Omega,H] = \Lambda^p[\Omega] \bar\otimes H$ and $\bar\partial^p_H = \bar\partial^p \bar\otimes 1_H$, where 1_H is the identity on H (see [10] for details concerning the $\bar\partial$-operator in Hilbert spaces). We shall prove the following:

3.7. PROPOSITION. <u>If $\Omega \subset \mathbb{C}^n$ is a strongly pseudoconvex domain then $(\Lambda^p[\Omega,H], \bar\partial^p_H)_{p=0}^n$ is a semi - Fredholm complex of Hilbert spaces with the property</u>

$$R(\bar\partial^{p-1}_H) = N(\bar\partial^p_H), \quad p \geq 1 \quad .$$

<u>Proof</u>. Let us define $X^o = \Lambda^o[\Omega] \ominus N(\bar\partial^o)$, $X^p = \Lambda^p[\Omega]$ $p \geq 1$, $\alpha^o = \bar\partial^o | X^o$ and $\alpha^p = \bar\partial^p$, $p \geq 1$. Then $(X,\alpha) = (X^p, \alpha^p)_{p=0}^n$ is a complex of Hilbert spaces which is Fredholm and exact at each stage. Consider then the complex $(H,0) = (H^p, 0)_{p \geq 0}$, where $H^o = H$ and $H^p = 0$ for $p \geq 0$. By Lemma 2.9, the tensor product $(X \bar\otimes H, \alpha \bar\otimes 0)$ must be Fredholm and exact. It is easy to check the equality

$$(X \bar\otimes H, \alpha \bar\otimes 0) = (X^p \bar\otimes H, \bar\partial^p \bar\otimes 1_H)_{p=0}^n \quad .$$

As $N(\bar\partial^o_H) = N(\bar\partial^o) \bar\otimes H$ by Lemma 2.4 (which applies since $R(\bar\partial^o)$ is closed by (3.1)), we conclude that $(\Lambda^p[\Omega,H],\bar{\partial}_H^p)_{p=0}^n$ is semi--Fredholm and $R(\bar{\partial}_H^{p-1})=N(\bar{\partial}_H^p)$ for $p\geq 1$.

Let us remark that Proposition 3.7 improves the statement of [10, Theorem 2.7].

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