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by

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The theory of normed Köthe spaces is usually exposed within the framework of the space $M(X,\Sigma,\mu)$ of equivalence classes of measurable functions over the measure space (X,Σ,μ) (see A.C. Zaanen, [4], ch.15). However, most of the definitions and constructions make sense in a general Riesz space (for instance, the Fatou and Riesz-Fischer properties, the Lorentz pseudonorm). It is the purpose of this paper to give characterizations of Riesz spaces involving universal σ-completeness, the Egoroff property and weak σ-distributivity by using concepts taken from normed Köthe spaces theory.

1. Extended Riesz pseudonorms

All the vector lattices we shall consider will be real (all the results are also holding in the complex case).

If E is a vector lattice we shall use the notations

$$E_+ = \{x \in E \mid x \geq 0\}$$

$$x_+ = x \vee 0\ ,\quad x_- = (-x) \vee 0\ ,$$

$$|x| = x \vee (-x)\ .$$

If $x, x_n \in E$ $(n \in \mathbb{N})$ we shall write $x_n \uparrow_n x$ $(x_n \downarrow_n x)$ if x_n is increasing (decreasing) and $x = \sup_{n\in\mathbb{N}} x_n$ $(x = \inf_{n\in\mathbb{N}} x_n)$. When there is no doubt on the index we are refering to, we shall write $x_n \uparrow x$ $(x_n \downarrow x)$.

Definition 1.1. Let E be a vector lattice. An extended Riesz pseudonorm on E is a map $\rho : E \to \mathbb{R}_+ \cup \{\infty\}$ such that

i) $\rho(0) = 0$.

ii) $\rho(x+y) \leq \rho(x) + \rho(y)$ for $x, y \in E$.

iii) if $x, y \in E$ and $|x| \leq |y|$ then $\rho(x) \leq \rho(y)$.

The extended Riesz pseudonorm ρ is said to be an extended Riesz seminorm if $\rho(ax) = |a|\rho(x)$ for $a \in \mathbb{R}$, $x \in E$.

The extended Riesz pseudonorm ρ is said to be an **extended** Riesz quasinorm if $\rho(x) = 0$ implies x=0.

The extended Riesz pseudonorm ρ is said to be an extended Riesz norm if ρ is an extended Riesz seminorm and quasinorm.

The set of all extended Riesz pseudonorms on E is partially ordered by

$$\rho_1 \leq \rho_2 \iff \rho_1(x) \leq \rho_2(x), \ x \in E.$$

If ρ is an extended Riesz pseudonorm on E we shall denote by E_ρ the order ideal of elements $x \in E$ such that $\rho(x) < \infty$.

If F is an order ideal in the Riesz space E we shall denote by $\dot{I}_F$ the extended Riesz seminorm on E given by

$$\dot{I}_F(x) = 0 \quad \text{if } x \in F$$
$$\dot{I}_F(x) = \infty \quad \text{if } x \in E \setminus F.$$

If $x \in E$ we shall denote by $\| \ \|_x$ the extended Riesz seminorm on E given by

$$\|y\|_x = \inf \{a \mid a \in \mathbb{R}_+ , |y| \leq a|x|\}$$

Here we understand that $\inf \emptyset = \infty$.

Definition 1.2. Let E be a vector lattice. An extended Riesz pseudonorm ρ on E has the Riesz-Fischer property if for every $x, x_n \in E_+$ $(n \in \mathbb{N})$ such that $\sum_{i=1}^{n} x_i \uparrow_n x$ we have $\rho(x) \le \sum_{i=1}^{\infty} \rho(x_i)$.

Definition 1.3. Let E be a vector lattice. An extended Riesz pseudonorm on E has the weak Fatou property if there is a $k \in \mathbb{R}_+$ such that for every $x, x_n \in E_+$ $(n \in \mathbb{N})$ with $x_n \uparrow x$ we have $\rho(x) \le k \lim_{n\to\infty} \rho(x_n)$. The extended Riesz pseudonorm has the Fatou property if k=1.

Let ρ be an extended Riesz quasinorm on E. Then the map $(x,y) \mapsto \rho(|x-y|)$ defines a metric on E (the topology associated with this metric is not linear in the general case). We have the following result:

Theorem 1.1. E_ρ is complete as a metric space if and only if:

i) ρ has the Riesz-Fischer property;

ii) if $x_n \in E_+$ $(n \in \mathbb{N})$ is a sequence such that $\sum_{i=1}^{\infty} \rho(x_i) < \infty$ then the set $\left\{ \sum_{i=1}^{n} x_i \mid n \in \mathbb{N} \right\}$ has a least upper bound in E.

Proof. Suppose i) and ii) are satisfyied. We first prove that if $x_n \in E_+$ $(n \in \mathbb{N})$ and $\sum_{n=1}^{\infty} \rho(x_n) < \infty$ then the series $\sum_{n=1}^{\infty} x_n$ is convergent. Indeed, from ii) it follows that the set $\left\{ \sum_{i=1}^{n} x_i \mid n \in \mathbb{N} \right\}$ has a least upper bound x; from i) we have that $x \in E_\rho$.

Then
$$\sum_{i=1}^{n} x_{i+m} \uparrow_n x - \sum_{i=1}^{m} x_i$$

so i) implies

$$\rho\Big(x-\sum_{i=1}^{m} x_i\Big) \le \sum_{i=1}^{\infty} \rho(x_{i+m})$$

which shows that $\lim\limits_{m\to\infty} \rho\Big(x-\sum\limits_{i=1}^{m} x_i\Big)=0$.

Consider now a ρ-Cauchy sequence $x_n \in E\ (n\in\mathbb{N})$. It is sufficient to show that $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence, so we can assume that $\rho(x_{n+1}-x_n) \le \frac{1}{2^n}$. Put

$$u_n = (x_{n+1}-x_n)_+$$
$$v_n = (x_{n+1}-x_n)_-.$$

Then

$$\sum_{n=1}^{\infty} \rho(u_n) < \infty,$$
$$\sum_{n=1}^{\infty} \rho(v_n) < \infty,$$

so by the previous argument the series $\sum\limits_{n=1}^{\infty} u_n$ and $\sum\limits_{n=1}^{\infty} v_n$ are convergent. But then the series $\sum\limits_{n=1}^{\infty} (x_{n+1}-x_n)$ is convergent, and so is the sequence $(x_n)_{n\in\mathbb{N}}$.

Conversely, suppose E is complete as a metric space. We prove first that if $x_n \in (E_\rho)_+\ (n\in\mathbb{N})$ is an increasing sequence converging to $x \in E_\rho$ then $x_n \uparrow x$. Indeed, we have

$$x - x_n = \lim_{i\to\infty} x_{n+i} - x_n.$$

From

$$\rho(|u|-|v|) \le \rho(|u-v|),\quad u,v \in E$$

it follows that $(E_\rho)_+$ is closed, so $x-x_n \in (E_\rho)_+$.
Let $y \in E$ be such that $x_n \le y$, $n\in\mathbb{N}$. Then $x_n \le x\wedge y$, $n\in\mathbb{N}$;

as $(E_\rho)_+$ is closed, this implies that $x \le x \wedge y$ so $x \le y$. Thus $x_n \uparrow x$.

Now we prove i). Let $x, x_n \in E_+$ $(n \in \mathbb{N})$ be such that $\sum_{i=1}^{n} x_i \uparrow_n x$. We have to show that

$$\rho(x) \le \sum_{i=1}^{\infty} \rho(x_i). \tag{1}$$

If $\sum_{i=1}^{\infty} \rho(x_i) = \infty$ then (1) is satisfyied. So suppose $\sum_{i=1}^{\infty} \rho(x_i) < \infty$. Then $x_i \in E_\rho$ and the sequence (y_n) given by $y_n = \sum_{i=1}^{n} x_i$ is convergent to $z \in E_\rho$. But then $y_n \uparrow_n z$ so x=z. We have

$$\rho(x) \le \rho\left(x - \sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} \rho(x_i)$$

As $\lim_{n\to\infty} \rho\left(x - \sum_{i=1}^{n} x_i\right) = 0$ we have (1).

If $x_n \in E$ $(n \in \mathbb{N})$ is a sequence such that $\sum_{i=1}^{\infty} \rho(x_i) < \infty$ then the series $\sum_{i=1}^{\infty} x_i$ is convergent to a $x \in E_\rho$; as above, $\sum_{i=1}^{n} x_i \uparrow_n x$ so ii) is satisfyied.

Let ρ be an extended Riesz pseudonorm on E[1]. We associate with ρ the maps $\rho_T, \rho_L : E \to \mathbb{R}_+ \cup \{\infty\}$ defined as follows. For $x \in E$ let $\mathcal{T}(x)$ be the set of sequences $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in E$ and $\sum_{i=1}^{n} x_i \uparrow_n |x|$, $\mathcal{L}(x)$ the set of sequences $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in E_+$ and $x_n \uparrow_n |x|$. Then define

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1) For E=M(X,Σ,μ) with μ σ-finite, ρ_L was introduced by G.G. Lorentz (see A.C. Zaanen, [4], § 66). In the same case, ρ_T was introduced and thm. 1.2 was proved in D.Tomescu's paper, [2]. In this paper, ρ_T is denoted by ρ_M.

$$\rho_T(x) = \inf \left\{ \sum_{n=1}^{\infty} \rho(x_n) \;\middle|\; (x_n)_{n\in\mathbb{N}} \in \mathcal{T}(x) \right\}$$

$$\rho_L(x) = \inf \left\{ \sup_{n\in\mathbb{N}} \rho(x_n) \;\middle|\; (x_n)_{n\in\mathbb{N}} \in \mathcal{L}(x) \right\}.$$

Theorem 1.2. ρ_T is the greatest extended Riesz pseudonorm with the Riesz-Fischer property dominated by ρ. If ρ is an extended Riesz seminorm then ρ_T is an extended Riesz seminorm.

Proof. First we prove that ρ_T is an extended Riesz pseudonorm. Let $x, y \in E$ be such that $|x| \leq |y|$. If $(y_n)_{n\in\mathbb{N}} \in \mathcal{T}(y)$ then let

$$x_1 = y_1 \wedge |x|$$
$$x_n = \left(\sum_{i=1}^{n} y_i\right) \wedge |x| - \left(\sum_{i=1}^{n-1} y_i\right) \wedge |x| \;, \quad n > 1 .$$

Then $(x_n)_{n\in\mathbb{N}} \in \mathcal{T}(x)$ and $x_n \leq y_n$ so

$$\rho_T(x) \leq \sum_{n=1}^{\infty} \rho(x_n) \leq \sum_{n=1}^{\infty} \rho(y_n).$$

As $(y_n)_{n\in\mathbb{N}}$ is arbitrary in $\mathcal{T}$(y) we have that $\rho_T(x) \leq \rho_T(y)$.

Now let $x, y \in E$ and let $(x_n)_{n\in\mathbb{N}} \in \mathcal{T}(x)$, $(y_n)_{n\in\mathbb{N}} \in \mathcal{T}(y)$. Then $(x_n + y_n) \in \mathcal{T}(|x| + |y|)$ so

$$\rho_T(x+y) = \rho_T(|x+y|) \leq \rho_T(|x|+|y|) \leq \sum_{n=1}^{\infty} \rho(x_n + y_n) \leq$$
$$\leq \sum_{n=1}^{\infty} \rho(x_n) + \sum_{n=1}^{\infty} \rho(y_n)$$

which shows that $\rho_T(x+y) \leq \rho_T(x) + \rho_T(y)$.

Now we prove that ρ_T has the Riesz-Fischer property. We hate to show that if $x \in E_+$ and $(x_n)_{n\in\mathbb{N}} \in \mathcal{T}(x)$ then

$\rho_T(x) \le \sum_{n=1}^{\infty} \rho_T(x_n)$. We can assume $\sum_{n=1}^{\infty} \rho_T(x_n) < \infty$. Let $\varepsilon > 0$. For every n there is a sequence $(x_{nm})_{m \in \mathbb{N}} \in \mathcal{T}(x_n)$ such that

$$\sum_{m=1}^{\infty} \rho(x_{nm}) \le \rho_T(x_n) + \frac{\varepsilon}{2^n} .$$

Let $k \mapsto (n(k), m(k))$ be a bijection of $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$. Then the sequence $(y_k)_{k \in \mathbb{N}}$ given by

$$y_k = x_{n(k), m(k)}$$

belongs to $\mathcal{T}(x)$. So

$$\rho_T(x) \le \sum_{k=1}^{\infty} \rho(y_k) = \sum_{1 \le m, n < \infty} \rho(x_{nm}) \le \varepsilon + \sum_{n=1}^{\infty} \rho_T(x_n)$$

As ε is arbitrary we have the result.

If ρ_1 is an extended Riesz pseudonorm with the Riesz-Fischer property such that $\rho_1 \le \rho$ and if $(x_n)_{n \in \mathbb{N}} \in \mathcal{T}(x)$ then

$$\rho_1(x) \le \sum_{n=1}^{\infty} \rho_1(x_n) \le \sum_{n=1}^{\infty} \rho(x_n)$$

so $\rho_1(x) \le \rho_T(x)$ by the definition of ρ_T .

The last statement of the theorem is obvious.

Theorem 1.3.

i) ρ_L is and extended Riesz pseudonorm dominated by ρ.

ii) If ρ_1 is an extended Riesz pseudonorm with the Fatou property dominated by ρ then $\rho_1 \le \rho_L$.

iii) ρ_L has the Riesz-Fischer property. In particular $\rho_L \le \rho_T$.

iv) If ρ is an extended Riesz seminorm then ρ_L is an extended Riesz seminorm.

Proof.

i) Let $x, y \in E$ be such that $|x| \leq |y|$. If $(y_n)_{n\in\mathbb{N}} \in \mathcal{L}(y)$ then the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = |x| \wedge y_n$ belongs to $\mathcal{L}(x)$. Thus

$$\rho_L(x) \leq \sup_{n\in\mathbb{N}} \rho(x_n) \leq \sup_{n\in\mathbb{N}} \rho(y_n).$$

As $(y_n)_{n\in\mathbb{N}}$ is arbitrary in $\mathcal{L}(y)$ we have that $\rho_L(x) \leq \rho_L(y)$.

Now let $x, y \in E$ and let $(x_n)_{n\in\mathbb{N}} \in \mathcal{L}(x)$, $(y_n)_{n\in\mathbb{N}} \in \mathcal{L}(y)$. Then $(x_n + y_n)_{n\in\mathbb{N}} \in \mathcal{L}(|x|+|y|)$ so

$$\rho_L(x+y) \leq \rho_L(|x+y|) \leq \rho_L(|x|+|y|) \leq \sup_{n\in\mathbb{N}} \rho(x_n+y_n) \leq$$
$$\leq \sup_{n\in\mathbb{N}} \rho(x_n) + \sup_{n\in\mathbb{N}} \rho(y_n)$$

which shows that $\rho_L(x+y) \leq \rho_L(x) + \rho_L(y)$.

ii) Let ρ_1 be an extended Riesz pseudonorm with the Fatou property dominated by ρ and let $(x_n)_{n\in\mathbb{N}} \in \mathcal{L}(x)$. Then

$$\rho_1(x) \leq \sup_{n\in\mathbb{N}} \rho_1(x_n) \leq \sup_{n\in\mathbb{N}} \rho(x_n)$$

so $\rho_1(x) \leq \rho_L(x)$ by the definition of ρ_L.

iii) Let $x \in E_+$ and $(x_n)_{n\in\mathbb{N}} \in \mathcal{T}(x)$. We have to show that $\rho_L(x) \leq \sum_{n=1}^{\infty} \rho_L(x_n)$. We can assume $\sum_{n=1}^{\infty} \rho_L(x_n) < \infty$. Let $\varepsilon > 0$. For every n there is a sequence $(x_{nm})_{m\in\mathbb{N}} \in \mathcal{L}(x_n)$ such that

$$\sup_{m\in\mathbb{N}} \rho(x_{nm}) \le \rho_L(x_n) + \frac{\varepsilon}{2^n} .$$

Put
$$y_m = \sum_{n=1}^{m} x_{nm} .$$

Then $(y_m)_{m\in\mathbb{N}} \in \mathcal{L}(x)$ and

$$\rho_L(x) \le \sup_{m\in\mathbb{N}} \rho(y_m) \le \sup_{m\in\mathbb{N}} \sum_{n=1}^{m} \rho(x_{nm}) \le$$
$$\le \sup_{m\in\mathbb{N}} \sum_{n=1}^{m} \left(\rho_L(x_n) + \frac{\varepsilon}{2^n}\right) \le \varepsilon + \sum_{n=1}^{\infty} \rho_L(x_n).$$

As ε is arbitrary we have the result.

By thm. 1.2. it follows that $\rho_L \le \rho_T$.

iv) Obvious.

2. Universally σ-complete Riesz spaces

Definition 2.1. A Riesz space E is called universally σ-complete if every sequence $(x_n)_{n\in\mathbb{N}}$ of mutually disjoint elements in E_+ has a least upper bound.

In [1] the notion of an universal complete Riesz space is introduced:

A Riesz space E is called universally complete if every sistem $(x_\tau)_{\tau\in T}$ of mutually disjoint elements in E_+ has a least upper bound.

We observe that the two notions do not coincide. For-instance the space of all functions $f:\mathbb{R}\to\mathbb{R}$ such that $\{x \mid f(x)\neq 0\}$ is at most countable is super Dedekind complete, universally σ-complete but not universally complete.

Lemma 2.1. Let E be a Dedekind σ-complete Riesz space. Then E is universally σ-complete if and only if for every increasing unbounded sequence $(x_n)_{n\in\mathbb{N}}$ in E_+ there is an $u \in E_+ \setminus \{0\}$ such that $x_n \wedge mu \uparrow_n mu$ for every $m \in \mathbb{N}$.

Proof. Suppose E is universally σ-complete. We first prove the statement in the case when all x_n are contained in the band generated by an element $x \in E_+$.

For every $y \in E_+$ let P_y be the projection on the band generated by y, that is

$$P_y(z) = \sup_{n\in\mathbb{N}} z \wedge ny \quad , \quad z \in E_+ .$$

Put for $m, n \in \mathbb{N}$

$$e_{mn} = P_{(mx - x_n)_+}(x) \ ,$$

$$e_m = \inf_{n\in\mathbb{N}} e_{mn} .$$

By Cor. 31.2 of W.A.J. Luxemburg, A.C. Zaanen, [1],

$$P_{e_{mn}} = P_{(mx-x_n)_+} .$$

From

$$P_{(mx-x_n)_+}((mx-x_n)_-) = 0$$

it follows that

$$P_{e_{mn}}(x_n) \le m e_{mn} . \qquad (1)$$

For every $m, n \in \mathbb{N}$ we have

$$(x - e_{mn}) \wedge e_{mn} = 0$$

so, as $(mx - x_n)_+$ belongs to the band generated by e_{mn}, it follows that

$$(x - e_{mn}) \wedge (mx - x_n)_+ = 0 .$$

Thus

$$P_{(x-e_{mn})}((mx - x_n)_+) = 0$$

so

$$P_{(x-e_{mn})}(x_n) \geq m(x - e_{mn}) \tag{2}$$

Because $e_m \wedge (x - e_m) = 0$, the elements f_m given by $f_1 = e_1$, $f_m = e_m - e_{m-1}$ $(m > 1)$ are mutually disjoint; so there is $f \in E_+$ such that $m f_m \leq f$, $m \in \mathbb{N}$. From (1) applying P_{e_m} it follows that $P_{e_m}(x_n) \leq m e_m$.

Thus, applying P_{f_m} in the inequality from above,

$$P_{e_m}(x_n) = \sup_{1 \leq i \leq m} P_{f_i}(x_n) \leq \sup_{1 \leq i \leq m} i f_i \leq f . \tag{3}$$

Let $e = \sup_{m \in \mathbb{N}} e_m$. From (3) we have that

$$P_e(x_n) = \sup_{m \in \mathbb{N}} P_{e_m}(x_n) \leq f .$$

Since $P_x(x_n) = x_n$ we cannot have e=x. So if u=x-e we have u≠0; we shall prove that $x_n \wedge mu \uparrow_n mu$. Indeed, from (2) it follows that

$$x_n \geq P_{(x-e_m)}(x_n) \geq P_{(x-e_{mn})}(x_n) \geq m(x - e_{mn})$$

so

$$\sup_{n\in\mathbb{N}} m(x-e_m)\wedge x_n \geq \sup_{n\in\mathbb{N}} m\left[(x-e_m)\wedge(x-e_{mn})\right] = m(x-e_m).$$

This implies

$$\sup_{n\in\mathbb{N}} mu \wedge x_n = mu$$

for every $m\in\mathbb{N}$.

Consider now an unbounded increasing sequence $(x_n)_{n\in\mathbb{N}}$ in E_+. If there is a n_0 such that the sequence $(P_{x_{n_0}}(x_n))_{n\in\mathbb{N}}$ is unbounded then we are in the case considered above. So suppose that there is no such n_0. Let

$$v_n = \sup_{m\in\mathbb{N}} P_{x_n}(x_m)$$

Then $P_{v_n}(v_{n+1}) = v_n$ so the elements $w_n\ (n\in\mathbb{N})$ given by $w_1 = v_1$, $w_n = v_n - v_{n-1}\ (n>1)$ are mutually disjoint. If v is an upper bound for $\{w_n \mid n\in\mathbb{N}\}$ then v is an upper bound for $\{v_n \mid n\in\mathbb{N}\}$ so

$$x_n = P_{x_n}(x_n) \leq v_n \leq v$$

which is a contradiction.

Conversely, suppose that E satisfyies the condition in the statement of the lemma. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of mutually disjoint elements in E_+. If $\{x_n \mid n\in\mathbb{N}\}$ would not be order bounded, there would exist an $u\in E_+\setminus\{0\}$ such that $mu \wedge \sup_{1\leq i\leq n} x_i \uparrow_n mu$ for every $m\in\mathbb{N}$. Then

$$mu \wedge \sup_{1\leq i\leq n} x_i \wedge m x_k \uparrow_n mu \wedge m x_k.$$

But $m x_k \wedge \sup_{1\leq i\leq n} x_i \leq x_k$ so $m(u\wedge x_k) \leq x_k$ for every $m\in\mathbb{N}$.

As E is Archimedian it follows that $u\wedge x_k = 0$ for every

$k \in \mathbb{N}$. Then $u = \sup_{n \in \mathbb{N}} (u \wedge \sup_{1 \le i \le n} x_i) = 0$ which is a contradiction.

Theorem 2.1. Let E be a Dedekind σ-complete Riesz space. The following assertions are equivalent:

i) E is universally σ-complete.

ii) If ρ is an extended Riesz norm on E having the Riesz-Fischer property on E_ρ and $x_n \in E_+$ $(n \in \mathbb{N})$ are such that $\sum_{n=1}^{\infty} \rho(x_n) < \infty$ then the set $\{\sum_{i=1}^{n} x_i \mid n \in \mathbb{N}\}$ is order bounded.

iii) The same as ii) but with ρ having the Fatou property on E.

iv) If ρ is an extended Riesz norm on E having the weak Fatou property on E_ρ and $(x_n)_{n \in \mathbb{N}}$ is an increasing sequence of elements in E_+ such that $\sup_{n \in \mathbb{N}} \rho(x_n) < \infty$ then the set $\{x_n \mid n \in \mathbb{N}\}$ is order bounded.

v) The same as iv) but with ρ having the Fatou property on E.

Proof.

i) $\Longrightarrow$ ii) Suppose that the set $\{\sum_{i=1}^{n} x_i \mid n \in \mathbb{N}\}$ is unbounded. Then there is $u \in E_+ \setminus \{0\}$ such that

$mu \wedge \sum_{i=1}^{n} x_i \uparrow_n mu$ for every $m \in \mathbb{N}$. It follows that there is a n_o for which $u \wedge x_{n_o} \neq 0$; replacing u by $u \wedge x_{n_o}$ we can therefore assume that $u \in E_\rho$. Put

$$y_{m1} = mu \wedge x_1,$$
$$y_{mn} = mu \wedge \sum_{i=1}^{n} x_i - mu \wedge \sum_{i=1}^{n-1} x_i, \quad n > 1.$$

Then $0 \leq y_{mn} \leq x_n$ and $\sum_{i=1}^{n} y_{mi} \uparrow_n mu$.
It follows that

$$\rho(mu) \leq \sum_{n=1}^{\infty} \rho(y_{mn}) \leq \sum_{n=1}^{\infty} \rho(x_n)$$

so $\rho(u)=0$ and $u=0$, which is a contradiction.

i) $\Rightarrow$ iv). Suppose that the set $\{x_n \mid n \in \mathbb{N}\}$ is unbounded. Then there is $u \in E_+ \setminus \{0\}$ such that $mu \wedge x_n \uparrow_n mu$ for every $m \in \mathbb{N}$; as above we may assume $u \in E_\rho$. Then

$$\rho(mu) \leq k \sup_{n \in \mathbb{N}} \rho(mu \wedge x_n) \leq k \sup_{n \in \mathbb{N}} \rho(x_n).$$

It follows that $\rho(u)=0$ so $u=0$, which is a contradiction.

ii) $\Rightarrow$ iii), iv) $\Rightarrow$ v). Obvious.

iii) $\Rightarrow$ i). Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of mutually disjoint elements in E_+ and let B be the band generated by $\{x_n \mid n \in \mathbb{N}\}$. Define the extended Riesz norm ρ on E by

$$\rho(x) = \sup\left(\dot{I}_B(x), \sum_{n=1}^{\infty} \frac{1}{2^n} \| P_{x_n}(x) \|_{x_n}\right).$$

The norm ρ has the Fatou property and $\sum_{n=1}^{\infty} \rho(x_n) < \infty$. Thus the set $\{x_n \mid n \in \mathbb{N}\}$ is order bounded.

v) $\Rightarrow$ i). Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of mutually disjoint elements in E_+ and let B be the band generated by $\{x_n \mid n \in \mathbb{N}\}$. Define the extended Riesz norm ρ on E by

$$\rho(x) = \sup\left(\dot{I}_B(x), \sup_{n \in \mathbb{N}} \| P_{x_n}(x) \|_{x_n}\right).$$

The norm ρ has the Fatou property and $\sup_{n \in \mathbb{N}} \rho(\sup_{1 \leq i \leq n} x_i) < \infty$. Thus the set $\{x_n \mid n \in \mathbb{N}\}$ is order bounded.

Theorem 2.2. Let E be a Dedeking σ-complete Riesz space. The following assertions are equivalent:

i) E is universally σ-complete;

ii) For every extended Riesz norm ρ on E the space E_ρ is complete as a metric space if and only if ρ has the Riesz-Fischer property.

Proof. Follows from thm. 1.1. and thm. 2.1.

3. The Egoroff property

The element f in the Riesz space E has the Egoroff property if for every double sequence $(u_{nk})_{\substack{n\in\mathbb{N}\\ k\in\mathbb{N}}}$ in E_+ such that $u_{nk}\uparrow_k |f|$ there exists a sequence $(v_m)_{m\in\mathbb{N}}$ in E_+ which verifyies:

i) $v_m \uparrow |f|$.

ii) for every $m,n\in\mathbb{N}$ there is a $k(m,n)\in\mathbb{N}$ such that $v_m \le u_{n,k(m,n)}$.

The Riesz space E has the Egoroff property if every $f\in E$ has the Egoroff property (W.A.J. Luxemburg, A.C. Zaanen, [1], def. 67.2).

Theorem 3.1. Let E be a Riesz space with the Egoroff property. Then for every extended Riesz quasinorm ρ on E the extended Riesz pseudonorms ρ_L and ρ_T are quasinorms.

Proof. Let $x\in E_+$. Suppose that $\rho_L(x)=0$. Then for every $n\in\mathbb{N}$ there is a sequence $(x_{nk})_{k\in\mathbb{N}}$ in E_+ such that $x_{nk}\uparrow_k x$ and $\sup_{k\in\mathbb{N}}\rho(x_{nk})\le\frac{1}{n}$. The Egoroff property implies the existence of a sequence $(v_m)_{m\in\mathbb{N}}\in\mathcal{L}(x)$ such that for every $m\in\mathbb{N}$ there is a $k(m)\in\mathbb{N}$ with $v_m\le x_{m,k(m)}$. Then

$\rho(v_m) \le \rho(x_{m,k(m)}) \le \frac{1}{m}$, so

$$\rho(v_m) \le \sup_{k\in\mathbb{N}} \rho(v_k) = \lim_{k\to\infty} \rho(v_k) \le \lim_{k\to\infty} \frac{1}{k} = 0.$$

Because ρ is a quasinorm, $v_m = 0$ so $x = 0$.

From thm.1.3, $\rho_L \le \rho_T$ so ρ_T is also a quasinorm.

Theorem 3.2. Let E be a Riesz space with the Egoroff property. Then for every extended Riesz pseudonorm ρ the extended Riesz pseudonorm ρ_L has the Fatou property.

Proof. Let $x \in E_+$ and let $(x_n)_{n\in\mathbb{N}} \in \mathcal{L}(x)$. We have to prove that $\rho_L(x_n) \uparrow \rho_L(x)$. We can assume $\sup_{n\in\mathbb{N}} \rho(x_n) < \infty$. Let $\varepsilon > 0$. For every $n \in \mathbb{N}$ there is a sequence $(x_{nk})_{k\in\mathbb{N}} \in \mathcal{L}(x_n)$ such that $\sup_{k\in\mathbb{N}} \rho(x_{nk}) \le \rho_L(x_n) + \varepsilon$. The Eforoff property implies the existence of a sequence $(v_m)_{m\in\mathbb{N}} \in \mathcal{L}(x)$ such that for every $m \in \mathbb{N}$ there is $k(m)$ with $v_m \le x_{m,k(m)}$. Then $\rho(v_m) \le \rho(x_{m,k(m)}) \le$ $\le \rho_L(x_m) + \varepsilon$ so

$$\rho_L(x) \le \sup_{m\in\mathbb{N}} \rho(v_m) \le \varepsilon + \sup_{m\in\mathbb{N}} \rho_L(x_m)$$

As ε is arbitrary the result follows.

For $E = M(X, \Sigma, \mu)$ with μ σ-finite, theorems 3.1 and 3.2 can be found in A.C.Zaanen's book, [4] (§ 66, thm. 2 and thm. 4).

The rest of this section is devoted to the proof of the converse of thm. 3.2 in the case of a Dedekind σ-complete Riesz space.

Let E be a Riesz space. We shall call an element $f \in E$ regular if every order convergent sequence in the order ideal generated by f is relative uniformly convergent. This is equivalent to the following: for every sequence $(u_n)_{n\in\mathbb{N}}$ such that $0 \le u_n \le |f|$ and $u_n \downarrow 0$ and every $\varepsilon > 0$ there is a $n_\varepsilon \in \mathbb{N}$ such that $n \geqslant n_\varepsilon$ implies $u_n \le \varepsilon |f|$.

Lemma 3.1. Let E be an Archimedian Riesz space. If $f \in E$ is regular then f has the Egoroff property.

Proof. Let E(f) be the order ideal generated by f. Then relative uniform convergence on E(f) is defined by the norm $\| \ \|_f$ so f has the Egoroff property.

Lemma 3.2. Let E be a Dedekind σ-complete Riesz space. If $f \in E_+$ is not regular there is a sequence $(f_n)_{n\in\mathbb{N}}$ such that $0 \le f_n \uparrow f$ and $(\varepsilon f - f_n)_+ \neq 0$ for every $\varepsilon > 0$ and $n \in \mathbb{N}$.

Proof. There is a sequence $(g_n)_{n\in\mathbb{N}}$ and an $a \in (0,1)$ such that $0 \le g_n \uparrow f$ and $(af - g_n)_+ \neq 0$ for $n \in \mathbb{N}$. The sequence $(f_n)_{n\in\mathbb{N}}$ given by

$$f_n = P_{(g_n - af)_+}(f)$$

has the required properties.

Lemma 3.3. Let E be a Riesz space. Suppose that u, u_{nk}, u_n and v_n $(n, k \in \mathbb{N})$ are such that:

i) u_{nk} is increasing in n, decreasing in k and $u_{nk} \geqslant u_n$ for $n, k \in \mathbb{N}$.

ii) u_n is decreasing and for every $n \in \mathbb{N}$ we have $(u_n - u)_+ \neq 0$.

iii) $v_n \downarrow u$.

iv) for every $m \in \mathbb{N}$ there is a w_m in the convex hull of $\{u_{nk} \mid n, k \in \mathbb{N}\}$ such that $v_m \geqslant w_m$.

Then for every $m, n \in \mathbb{N}$ and $\varepsilon > 0$ there is a $k(m,n,\varepsilon) \in \mathbb{N}$ such that

$$(1+\varepsilon) v_m \geq u_{n,k(m,n,\varepsilon)} .$$

Proof. We shall use a slight refinement of the method in the proof of T.Chow's theorem (W.A.J. Luxemburg, A.C.Zaanen, [1], thm.67.7).

Let

$$w_m = \sum_{n,k} \lambda^m_{nk} u_{nk}$$

where $\lambda^m_{nk} \geq 0$, λ^m_{nk} are different from 0 for finitely many n and k and $\sum_{n,k} \lambda^m_{nk} = 1$. Set

$$\lambda^m_n = \sum_k \lambda^m_{nk}$$

and let $k(m,n) = \max\{\ell \mid \lambda^m_{n\ell} \neq 0\}$. We have

$$\sum_n \lambda^m_n = 1 ,$$

$$w_m \geq \sum_n \lambda^m_n u_{n,k(m,n)} .$$

Given $M, N \in \mathbb{N}$, let

$$\gamma = \sup\left\{ \sum_{n \geq N} \lambda^{m}_n \mid m \geq M \right\}$$

We prove that $\gamma=1$. Clearly $0 \leq \gamma \leq 1$. For $m \geq M$ we have

$$v_m - u \geq w_m - u \geq \sum_n \lambda^m_n (u_{n,k(m,n)} - u) \geq \sum_{n<N} \lambda^m_n (u_n - u) \geq$$

$$\geq \sum_{n<N} \lambda^m_n (u_N - u) = \left(\sum_n \lambda^m_n - \sum_{n \geq N} \lambda^m_n \right)(u_N - u) \geq (1-\gamma)(u_N - u) .$$

Hence

$$v_m - u \geq (1-\gamma)(u_N - u)_+ .$$

Since $v_m \downarrow u$ and $(u_N - u)_+ \neq 0$ it follows that $\gamma = 1$. Hence for $\varepsilon > 0$ there is a $p \geq M$ such that $\sum_{n \geq N} \lambda_n^p \geq \frac{1}{1+\varepsilon}$. Let

$$h(M,N,\varepsilon) = \max\{k(p,n) \mid \lambda_n^p \neq 0,\ n \geq N\}$$

Then

$$(1+\varepsilon)v_M \geq (1+\varepsilon)v_p \geq (1+\varepsilon)\sum_n \lambda_n^p u_{n,k(p,n)} \geq$$
$$\geq (1+\varepsilon)\sum_{n \geq N} \lambda_n^p u_{n,k(p,n)} \geq (1+\varepsilon)\sum_{n \geq N} \lambda_n^p u_{n,h(M,N,\varepsilon)} \geq$$
$$\geq (1+\varepsilon)\Big(\sum_{n \geq N} \lambda_n^p\Big) u_{N,h(M,N,\varepsilon)} \geq u_{N,h(M,N,\varepsilon)},$$

which completes the proof.

Lemma 3.4. Let E be a Dedekind σ-complete Riesz space. Suppose that for every $f \in E_+$ and every sequences $(f_{nk})_{n \in \mathbb{N}, k \in \mathbb{N}}$ such that $f_{nk} \uparrow_k f_n \uparrow_n f$ there is an $a > 0$ and a sequence $(v_n)_{n \in \mathbb{N}}$ such that:

i) $v_n \uparrow af$

ii) for every $m,n \in \mathbb{N}$ there is a $k(m,n) \in \mathbb{N}$ such that $v_m \leq f_{n,k(m,n)}$.

Then E has the Egoroff property.

Proof. By thm, 74.5 in W.A.J. Luxemburg's and A.C. Zaanen's book, [1] we have to show that every principal band has the Egoroff property in the Boolean algebra $\mathcal{P}$(E) of the projection bands.

So let B_u be the projection band determined by $u \in E_+$. Suppose that the projection bands A_{nk} satisfy $A_{nk} \uparrow B_u$.

Since the principal bands form an ideal in $\mathcal{P}(E)$ there are $u_{nk} \in E_+$ such that $A_{nk} = B_{u_{nk}}$. As $A_{nk} \subset B_u$ we can assume by lemma 74.1 in the quoted book that $u_{nk} \uparrow_k u$. Then there is a sequence $(v_n)_{n \in \mathbb{N}}$ in E_+ and an $a > 0$ such that

i) $v_n \uparrow au$;

ii) for every $m, n \in \mathbb{N}$ there is a $k(m,n) \in \mathbb{N}$ such that $v_m \leq u_{n,k(m,n)}$.

Then it follows that $B_{v_n} \uparrow B_u$ and $B_{v_m} \subset A_{n,k(m,n)}$ which completes the proof.

Theorem 3.3. Let E be a Dedekind σ-complete space such that for every extended Riesz norm ρ on E the extended Riesz seminorm ρ_L has the weak Fatou property. Then E has the Egoroff property.

Proof. We apply lemma 3.4. So let $f, f_{nk}, f_n \in E_+$ be such that $f_{nk} \uparrow_k f_n \uparrow_n f$. By lemma 3.1 we may assume that f is not regular. So by lemma 3.2. there is a sequence $(g_n)_{n \in \mathbb{N}}$ such that $0 \leq g_n \uparrow f$ and $(\varepsilon f - g_n)_+ \neq 0$ for every $\varepsilon > 0$ and $n \in \mathbb{N}$. Then replacing f_{nk} by $f_{nk} \wedge g_n$ and f_n by $f_n \wedge g_n$ we may assume that $(\varepsilon f - f_n)_+ \neq 0$ for every $\varepsilon > 0$ and $n \in \mathbb{N}$. Finally replacing f_{nk} by $\inf_{1 \leq m \leq n} f_{mk}$ we may assume that f_{nk} is decreasing in n.

Let K be the convex hull of $\{f_{nk} \mid n, k \in \mathbb{N}\}$ and let S be the set of $x \in E$ such that there is a $k \in K$ with $|x| \leq k$. S is an absolute convex set, so if

$$\rho(x) = \inf\{\lambda \in \mathbb{R}_+ \mid x \in \lambda S\}$$

(understanding that $\inf \emptyset = \infty$) then ρ is an extended Riesz seminorm on E. Because $|x| \leq f$ for every $x \in S$ and E is Archimedian it follows that ρ is an extended Riesz norm. We have

$$\rho_L(f_{nk}) \leq \rho(f_{nk}) \leq 1$$

so $\rho_L(f) = c < \infty$. There is a sequence $(s_m)_{m\in\mathbb{N}}$ such that $0 \leq s_m \uparrow f$ and $\sup_{m\in\mathbb{N}} \rho(s_m) \leq c+1$. Let $t_m = (c+1)^{-1} s_m$. It follows that $t_m \in S$ so there is $k_m \in K$ such that $t_m \leq k_m$. Put

$$u_n = f - f_n \qquad v_m = f - t_m$$

$$u_{nk} = f - f_{nk} \qquad u = c(c+1)^{-1} f.$$

Then u_{nk} is increasing in n and decreasing in k and we have $u_{nk} \geq u_n$. Also $v_m \downarrow u$ and $(u_n - u)_+ = ((c+1)^{-1} f - f_n)_+ \neq 0$. Finally $f - k_m$ is in the convex hull of $\{u_{nk} \mid n, k \in \mathbb{N}\}$ and $v_m \geq f - k_m$. Thus the hypothesis of lemma 3.3 are satisfyied. Let $\delta > 0$ be chosen such that $(1+\delta)(c+1)^{-1} - \delta > 0$; then by lemma 3.3. for every $m, n \in \mathbb{N}$ there is a $k(m,n) \in \mathbb{N}$ such that

$$(1+\delta) v_m \geq u_{n,k(m,n)}$$

that is

$$f_{n,k(m,n)} \geq (1+\delta) t_m - \delta f.$$

As $(1+\delta) t_m - \delta f \uparrow ((1+\delta)(c+1)^{-1} - \delta) f$ the hypothesis of lemma 3.4 are satisfyied. Thus E has the Egoroff property.

Corollary 3.1. Assume that the continuum hypothesis holds, and let μ be a locally finite and localizable [1] measure on X.

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[1] That is $M(X,\Sigma,\mu)$ is Dedekind complete and for every $A \in \Sigma$ with $\mu(A) > 0$ there is $B \in \Sigma$ with $B \subset A$ and $0 < \mu(B) < \infty$.

Let $M(X,\Sigma,\mu)$ be the Riesz space of equivalence classes of measurable functions on X. If for every extended Riesz norm ρ on $M(X,\Sigma,\mu)$ the extended Riesz seminorm ρ_L has the weak Fatou property then μ is σ- finite.

PROOF. It follows from thm.3.3. that $M(X,\Sigma,\mu)$ has the Egoroff property; then from W.A.J.Luxemburg, A.C.Zaanen, [1], thm.75.6, it follows that μ is σ- finite.

4. WEAKLY σ - DISTRIBUTIVE RIESZ SPACES

A Dedekind σ - complete Riesz-space is said to be weakly σ - distributive if for every order bounded sequence $(b_{mn})_{m\in\mathbb{N},\, n\in\mathbb{N}}$ decreasing in n we have

$$\sup_{m\in\mathbb{N}} \inf_{n\in\mathbb{N}} b_{mn} = \inf_{\varphi:\mathbb{N}\to\mathbb{N}} \sup_{m\in\mathbb{N}} b_{m,\varphi(m)}$$

(J.D.Maitland Wright,[3]).

A quasi-Stonian topological space is a compact space X such that for every open F_σ set G the closure $\overline{G}$ is open. A compact space X is quasi-Stonian if and only if the Riesz space C(X) of all real-valued continuous functions on X is Dedekind σ- complete.

A σ- meagre set in a topological space is a subset of a countable reunion of closed nowhere dense Baire sets.

Let X be a quasi-Stonian space . Then C(X) is weakly σ- distributive if and only if each σ - meagre subset of X is nowhere dense (J.D.Maitland Wright,[3], lemma L).

THEOREM 4.1. Let E be a Dedekind σ - complete weakly σ - distributive Riesz space. Then for every extended Riesz quasinorm ρ on E the extended Riesz pseudonorms ρ_L and ρ_T are quasinorms.

PROOF. Let $x \in E_+$ be such that $\rho_L(x)=0$. Then for every $m \in \mathbb{N}$ there is a sequence $(x_{mn})_{n\in\mathbb{N}}$ such that $0 \le x_{mn} \uparrow_n x$ and $\sup_{n\in\mathbb{N}} \rho(x_{mn}) \le \frac{1}{m}$. Let $y_{mn} = x - x_{mn}$. Then $y_{mn} \downarrow_n 0$ for $m \in \mathbb{N}$ so

$$\inf_{\varphi:\mathbb{N}\to\mathbb{N}} \sup_{m\in\mathbb{N}} y_{m,\varphi(m)} = 0$$

This implies

$$x = \sup_{\varphi:\mathbb{N}\to\mathbb{N}} \inf_{m\in\mathbb{N}} x_{m,\varphi(m)}.$$

For $\varphi: \mathbb{N} \to \mathbb{N}$ let $x_\varphi = \inf_{m\in\mathbb{N}} x_{m,\varphi(m)}$. Then

$$\rho(x_\varphi) \le \inf_{m\in\mathbb{N}} \rho(x_{m,\varphi(m)}) \le \inf_{m\in\mathbb{N}} \frac{1}{m} = 0 .$$

So $x_\varphi = 0$ and $x = \sup_{\varphi:\mathbb{N}\to\mathbb{N}} x_\varphi = 0$.

By thm. 1.3, iii), ρ_T is also a quasinorm.

Theorem 4.2. Let E be a Dedekind σ - complete space. Suppose that for every extended Riesz norm ρ on E the extended Riesz seminorm ρ_T is a norm. Then E is weakly σ - distributive.

PROOF. It is sufficient to show that every principal order ideal of E is weakly σ - distributive. So let $x \in E_+ \setminus \{0\}$ and let $E(x)$ be the principal ideal generated by x. There is a quasi-Stonian space X and a Riesz isomorphism T of $E(x)$ onto C(X) such that $T(x)=1$ (the function identic one). Suppose there is a σ- meagre set in X which is not nowhere dense. Then there is an open-closel set $\emptyset \neq Y \subset X$ and a sequence $(A_n)_{n\in\mathbb{N}}$ of closed Baire subsets of X such that $\mathring{A}_n = \emptyset$ and $\overline{\bigcup_{n\in\mathbb{N}} A_n} = Y$. We prove that there is a sequence $(D_n)_{n\in\mathbb{N}}$ of mutually disjoint nonvoid closed Baire subsets of X such that $\mathring{B}_n = \emptyset$ and $\overline{\bigcup_{n\in\mathbb{N}} B_n} = Y$

Indeed, let

$$Y_n = Y - \bigcup_{i=1}^{n} A_i.$$

Then Y_n is an open Baire set, so there is a sequence $(Y_{nm})_{m\in\mathbb{N}}$ of mutually disjoint open-closed sets such that $Y_n = \bigcup_{m\in\mathbb{N}} Y_{nm}$. Put $A_{mn} = A_{n+1} \cap Y_{nm}$. The sets A_{nm} are mutually disjoint and $\overline{\bigcup_{n,m\in\mathbb{N}} A_{nm}} = Y$. Finally range those non-void sets A_{nm} in a sequence $(B_n)_{n\in\mathbb{N}}$.

For a set $M \subset X$ let χ_M be the characteristic function of M.

Let $y = T^{-1}(\chi_Y)$. Define ρ by

$$\rho(z) = \infty, \quad z \in E \sim E(y)$$

$$\rho(z) = \sup_{n\in\mathbb{N}} \frac{1}{2^n} \sup_{t\in B_n} |T(z)(t)|, \quad z \in E(y).$$

As $\overline{\bigcup_{n\in\mathbb{N}} B_n} = Y$ it follows that ρ is an extended Riesz norm. Let $V_n = Y - \bigcup_{i=1}^{n} B_i$. V_n is an open Baire set so there is a sequence V_{nm} of mutually disjoint open-closed sets such that $V_n = \bigcup_{m\in\mathbb{N}} V_{nm}$. As B_{n+m} is compact and $B_{n+m} \subset V_n$ for $m \geq 1$ we may assume $B_{n+m} \subset \bigcup_{k=1}^{m} V_{nk}$. Let $y_{nm} = T^{-1}(\chi_{V_{nm}})$. Then $\rho(y_{nm}) \leq \frac{1}{2^{n+m}}$ and $\sum_{k=1}^{m} y_{nk} \uparrow_m y$. If follows that

$$\rho_T(y) \leq \sum_{m=1}^{\infty} \rho(y_{nm}) \leq \sum_{m=1}^{\infty} \frac{1}{2^{n+m}}.$$

Thus $\rho_T(y) = 0$, which is a contradiction.

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