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DERIVATIONS OF C*-ALGEBRAS WHICH ARE INVARIANT UNDER AN AUTOMORPHISM GROUP

by

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1.INTRODUCTION

Let A be an algebra. By a *derivation* of A we mean a linear mapping $\delta:D(\delta)\to A$ ($D(\delta)$ being a subalgebra of A) such that $\delta(ab)=a\delta(b)+\delta(a)b$ for all $a,b\in D(\delta)$. A derivation δ of A is said to be *inner* if there exists $a_o\in A$ such that $\delta(a)=a_oa-aa_o$ $(=ad(a_o)(a))$ for all $a\in D(\delta)$. If A is a *-algebra, a derivation $\delta:D(\delta)\to A$ is called *symmetric* if a) $D(\delta)$ is a *- subalgebra of A, and b) $\delta(a^*)=\delta(a)^*$ for all $a\in D(\delta)$.

Derivations of operator algebras were studied by several authors. We mention here only the famous theorem of Sakai [6] which asserts that if A is a simple C*- algebra with unit, then all bounded derivations (i.e. $D(\delta)=A$) of A are inner. In what follows we present a more general framework for this theorem. We also study unbounded derivations of operator algebras which commute with some group of *-automorphisms. A C*-dynamical system (or covariant system) is a triple (B,G,β) consisting of a C*-algebra B, a locally compact group G and a continuous homomorphism β of G into the group Aut(B) of *-automorphisms of B, equipped with the topology of pointwise convergence.

For the bounded derivations we shall discuss the following problem:

(*) Let (B,G,β) be a C*-dynamical system, where B is a C*-algebra with unit. Suppose the fixed point algebra $B^\beta=\{b\in B \mid \beta_g(b)=b\ (\forall)g\in G\}$ is simple. If these conditions are fulfilled, find conditions on the dynamical system such that every derivation of B which commutes with β, be implemented by an element in B^β.

Of course, in the unbounded case we give conditions such that a derivation δ be a generator of a strongly continuous one-parameter group of *-automorphisms.

2. ERGODIC GROUPS OF AUTOMORPHISMS

Let (B,G,β) be a C*-dynamical system. Assume that B has a unit, that G is compact and acts ergodically on B. For every $b\in B$ consider the following integration:

$$\varphi(b)=\int_G \beta_g(b)\,dg$$

where dg is the normalised Haar measure on G. Then $\varphi(b)\in B^\beta=\mathbb{C}.1$ for all $b\in B$. Moreover, φ is positive, faithful and G-invariant. It is easy to see that φ is the unique G-invariant state on B.

2.1. PROPOSITION. *Let (B,G,β) be as above. If $\delta:B\to B$ is a bounded derivation which commutes with β, then $\delta\equiv 0$.*

PROOF. Representing B in the GNS representation defined by the G-invariant state φ we may assume $\varphi(b)=(b\xi_o,\xi_o)$ for some cyclic vector ξ_o (for B) in the Hilbert space H_φ. Furthermore, there exists a strongly continuous unitary representation $g\to u_g$ of G on H_φ such that $\beta_g(b)=u_g b u_g^{-1}$. Since φ is the unique G-invariant state on B, it follows that φ is the unique G-invariant normal state on the weak closure $\bar{B}$ of B. On the other hand, since $\beta_g(b)=u_g b u_g^{-1}$ and $g\to u_g$ is strongly continuous, $(\bar{B},G,\beta)$ is a W*-dynamical system. Therefore we can consider the following weak *-integration in $\bar{B}$:

$$\bar{\varphi}(b)=\int_G \beta_g(b)\,dg, \qquad b\in\bar{B}.$$

Obviously $\bar{\varphi}(\bar{B})=\bar{B}^\beta$ and $\bar{\varphi}$ is normal. If $\bar{B}^\beta\neq\mathbb{C}.1$, then for every normal state ψ on $\bar{B}^\beta$, we have that $\psi\circ\bar{\varphi}$ is a normal G-invariant state on $\bar{B}$. Since φ is the unique normal G-invariant state on $\bar{B}$ it follows that $\bar{B}^\beta=\mathbb{C}.1$ and $\bar{\varphi}=\varphi$. Hence G acts ergodically on $\bar{B}$. Obviously, $\delta\beta_g=\beta_g\delta$ on $\bar{B}$ for all $g\in G$. We may assume that δ is a symmetric derivation. Then by [5, Corollary 8.6.6] we have $\delta\equiv 0$.

2.2. REMARK. If G is not compact, then the conclusion of Proposition 2.1 does not hold. Indeed, let $B=K(H)+\mathbb{C}.1$ where $K(H)$ is the C*-algebra of compact operators on some Hilbert space H.

Consider an unitary $U\varepsilon\mathcal{B}(H)$ which does not commute with any compact operator. Then $\beta(b)=UbU^*$ $(b\varepsilon B)$ defines an ergodic $*$-automorphism of B. The derivation $\delta=Ad(U)$ commutes with β and is not equal to zero.

For the study of unbounded derivations we need some preliminary results.

For $f\varepsilon L^1(G)$ we denote by $\beta(f)$ the operator on B defined by:

$$\beta(f)(b)=\int_G f(g)\beta_g(b)d_g.$$

Let Π be an irreducible unitary representation of G, $\hat{\Pi}\varepsilon\hat{G}$ its unitary equivalence class and $\chi_{\hat{\Pi}}$ its normalised character $\chi_{\hat{\Pi}}=(\dim\Pi)Tr(\Pi_g^{-1})$, where Tr is the usual trace on the Hilbert space of dimension $\dim\Pi$. Then $\beta(\chi_{\hat{\Pi}})$ is a bounded projection of B onto a norm-closed subspace $B(\hat{\Pi})$ of B called the *spectral subspace* of $\hat{\Pi}$ in B. We denote $\beta(\chi_{\hat{\Pi}})=\beta(\hat{\Pi})$.

2.3.LEMMA. *Let* (B,G,β) *be a* C^**-dynamical system. Assume that* B *has a unit and* G *is compact. If* $T:D(T)\to B(D(T)\subset B)$ *is a linear, closed, densely defined operator which commutes with* β, *then* $B(\hat{\Pi})\cap D(T)$ *is dense in* $B(\hat{\Pi})$ *for all* $\hat{\Pi}\varepsilon\hat{G}$.

PROOF. We consider the following norm on D(T):

$$|||b|||=||b||+||Tb||, \quad b\varepsilon D(T).$$

Endowed with this norm, D(T) becomes a Banach space. Since T commutes with β_g, $g\varepsilon G$ it follows that D(T) is G-invariant. On the other hand, if $g\varepsilon G$ and $b\varepsilon B$, we have:

$$|||\beta_g(b)|||=||\beta_g(b)||+||T\beta_g(b)||=||\beta_g(b)||+||\beta_g(Tb)||=|||b|||.$$

Therefore β_g, $g\varepsilon G$ are continuous on $(D(T), |||\cdot|||)$.

Also, it is obvious that for all $b\varepsilon D(T)$, the map $g\to\beta_g(b)$ is continuous from G into $(D(T), |||.|||)$. Then, we consider the following integration in $(D(T), |||.|||)$:

(1) $\tilde{\beta}(\hat{\Pi})(b)=\int_G \chi_{\hat{\Pi}}(g)\beta_g(b)dg, \quad b\varepsilon D(T).$

Obviously $\tilde{\beta}(\hat{\Pi})(b)=\beta(\hat{\Pi})(b)$, $b\varepsilon D(T)$. Since G is compact, A is generated by the spectral subspaces $B(\hat{\Pi})$, $\Pi\varepsilon\hat{G}$ [7, Theorem 2]. Therefore, since D(T) is dense in B and $\beta(\hat{\Pi})$ continuous we have that $\beta(\hat{\Pi})D(T)$ is dense in $B(\hat{\Pi})$.

2.4. LEMMA. *Let* (B,G,β) *and* T *be as in the preceding lemma. Suppose moreover that* G *acts ergodically on* B. *Then we have:*

i) $\bigcup_{\hat{\Pi}\in\hat{G}} B(\hat{\Pi})\subset D(T)$ *and* $\beta(\hat{\Pi})T\subset T\beta(\hat{\Pi})$ *for all* $\hat{\Pi}\in\hat{G}$.

ii) *If we denote by* S *the linear subspace generated by* $\bigcup_{\hat{\Pi}\in\hat{G}} B(\hat{\Pi})$ *then* S *is a dense subspace of analytic elements for* T *(see for exemple* [1, Definition 3.1.17]) *and* $T\subset S$.

iii) S *is a core for* T *(that is the closure of* $T|_S$ *is* T).

iv) *If* T′ *is a closed linear operator which commutes with* β *and* $T\subset T'$, *then* $T=T'$.

PROOF. (i) By [3, Proposition 2.1], $B(\hat{\Pi})$ is finite dimensional for all $\hat{\Pi}\in\hat{G}$. By Lemma 2.3. $B(\hat{\Pi})\cap D(T)$ is dense in $B(\hat{\Pi})$, $\hat{\Pi}\in\hat{G}$. Therefore, $B(\hat{\Pi})\subset D(T)$ for all $\hat{\Pi}\in\hat{G}$.

Now, let $b\in D(T)$. By (1), the integral

$$\beta(\hat{\Pi})(b)=\int_G \chi_{\hat{\Pi}}(g)\beta_g(b)\,dg$$

converges in $(D(T),|||.|||)$. Since T is continuous from $(D(T),|||.|||)$ into B with the original norm we have

$$T\beta(\hat{\Pi})(b)=\beta(\hat{\Pi})T(b).$$

(ii) By (i) $T\subset S$. According to [7, Theorem 2], S is dense in B. Since $B(\hat{\Pi})$ is finite dimensional and $T(B(\hat{\Pi}))\subset B(\hat{\Pi})$, $\hat{\Pi}\in\hat{G}$, it follows that all elements of S are analytic for T.

(iii) Since β_g is a (strongly) continuous representation of G in $(D(T),|||.|||)$, by [7, Theorem 2] it follows that S is dense in $(D(T),|||.|||)$.

(iv) follows from (iii).

2.5. THEOREM. *Let* (B,G,β) *be a* C*- *dynamical system. Suppose* B *has a unit,* G *is compact and acts ergodically on* B. *Let* δ *be a closed, densely defined symmetric derivation on* B *which commutes with* β. *Then we have:*

(i) δ *is the generator of a strongly continuous one-parameter group of* *-*automorphisms of* B, $\{\alpha_t\}_{t\in\mathbb{R}}$, *and* $\alpha_t\beta_g=\beta_g\alpha_t$ *for all* $t\in\mathbb{R}$, $g\in G$.

(ii) *If* T *is a linear, closed operator which commutes with* β *and* $\delta\subset T$ *then* $\delta=T$.

PROOF. (ii) follows from Lemma 2.4 (iv).

(i) We prove firstly that δ is a generator. In order to prove this fact, we shall verify the hypothesis of [1, Corollary 3.2.57]. Let φ be the unique G-invariant state on B (which is

faithful). We show that $\varphi(\delta(D(\delta)))=\{0\}$. Obviously $\varphi(\delta(1))=0$. By Lemma 2.4 (i) and (iii) it is sufficient to prove that $\varphi(B(\hat{\Pi}))=\{0\}$ for all nontrivial $\hat{\Pi}\varepsilon\hat{G}$. Let $\hat{\Pi}\varepsilon\hat{G}$, $\hat{\Pi}$ nontrivial, and $b\varepsilon B$. Then we have:

$$\begin{aligned}\varphi(\beta(\hat{\Pi})b)&=\varphi\left(\int \chi_{\hat{\Pi}}(g)\beta_g(b)\,dg\right)=\\&=\iint \chi_{\hat{\Pi}}(g)\beta_{hg}(b)\,dg\,dh=\\&=\int dg\,\chi_{\hat{\Pi}}(g)\int \beta_{hg}(b)\,dh=\\&=\varphi(b)\int \chi_{\hat{\Pi}}(g)\,dg=\\&=0.\end{aligned}$$

By Lemma 2.5 (ii), δ possesses a dense set of analytic elements. By [1, Corollary 3.2.57], δ is the generator of a strongly continuous one-parameter group $\{\alpha_t\}_{t\varepsilon R}$ of *-automorphisms of B. Applying the Hille-Yosida Theorem [1, Theorem 3.1.10 (2)] it follows that $\alpha_t\beta_g=\beta_g\alpha_t$ for all $t\varepsilon R$, $g\varepsilon G$.

2.6. REMARK. The condition that δ be closed in Theorem 2.5 can be replaced by the following weaker one: Suppose there exist two closed operators T_1, T_2 on B such that T_1 and T_2 commute with β, T_1T_2 is densely defined and $\delta=T_1T_2$. Then it results that δ is closable and the conclusion of Theorem 2.5 holds with $\bar{\delta}$ instead of δ.

3. CROSSED PRODUCTS

Let (A,Γ,α) be a C*-dynamical system. Suppose $A\subset\mathcal{B}(H)$ for some Hilbert space H. Recall some notations and results from the theory of crossed products (see for exemple [4]). Let $C_{oo}(\Gamma,A)$ be the algebra of all continuous A-valued functions on Γ with compact support. The reduced (or regular) crossed product of A and Γ denoted $C_r^*(A,\Gamma,\alpha)$ is defined as the C*-algebra generated by the operators $\tilde{\rho}(\varphi)$ on $L^2(\Gamma,H)$:

$$\tilde{\rho}(\varphi)f(\gamma)=\int_\Gamma \alpha_{\bar{\gamma}^1}(\varphi(p))f(\bar{p}^1\gamma)\,dp\ ,$$

for $\varphi\varepsilon C_{oo}(\Gamma,A)$, $f\varepsilon L^2(\Gamma,H)$ and $\gamma\varepsilon\Gamma$.

If $\bar{\lambda}$ is the continuous representation of Γ on $L^2(\Gamma,H)$ given by:

$$(\bar{\lambda}_p f)(\gamma)=f(\bar{p}^1\gamma)$$

for $f\varepsilon L^2(\Gamma,H)$ and $p,\gamma\varepsilon\Gamma$ it is easy to see that

$$\bar{\lambda}_p C_r^*(A,\Gamma,\alpha) \subset C_r^*(A,\Gamma,\alpha)$$

and

$$C_r^*(A,\Gamma,\alpha)\bar{\lambda}_p \subset C_r^*(A,\Gamma,\alpha)$$

for all $p\varepsilon\Gamma$. Therefore $\bar{\lambda}_p \varepsilon M(C_r^*(A,\Gamma,\alpha))$ (the multiplier algebra of $C_r^*(A,\Gamma,\alpha)$).

We can also define a faithful *-representation of A on $L^2(\Gamma,H)$ by:

$$\tilde{\rho}(a)f(\gamma)=\alpha_{\gamma^{-1}}(a)f(\gamma),$$

$a\varepsilon A$, $f\varepsilon L^2(\Gamma,H)$, $\gamma\varepsilon\Gamma$. We will have $\tilde{\rho}(a)\varepsilon M(C_r^*(A,\Gamma,\alpha))$.

Now, we define the dual action β of α on $C_r^*(A,\Gamma,\alpha)$. Let W be the following unitary operator on $L^2(\Gamma\times\Gamma,H)$:

$$Wf(\gamma,p)=f(\gamma,\gamma p) \text{ for } f\varepsilon L^2(\Gamma\times\Gamma,H) \quad \gamma,p\varepsilon\Gamma.$$

Let also $C_r^*(\Gamma)$ denote the C*-algebra generated by the left regular representation of $C_{oo}(\Gamma)$ on $L^2(\Gamma)$. Define the map β from $M(C_r^*(A,\Gamma,\alpha))$ into $B(L^2(\Gamma\times\Gamma,H))$ by:

$$\beta(a)=W^*(a\otimes 1)W.$$

We have:

$$\beta(a)=a\otimes 1 \quad \text{for } \tilde{a}\varepsilon\rho(A),$$

$$\beta(\bar{\lambda}_\gamma)=\bar{\lambda}_\gamma\otimes\lambda_\gamma \quad \text{for } \gamma\varepsilon\Gamma.$$

(Here $\lambda_\gamma\varepsilon B(L^2(\Gamma))$ is defined by $(\lambda_\gamma f)(p)=f(\gamma^{-1}p)$). The following proposition is a consequence of a remarkable result of M. B. Landstad [4, Theorem 3]:

3.1. PROPOSITION. $\rho(A)$ *consists of all elements* $a\varepsilon M(C_r^*(A,\Gamma,\alpha))$ *satisfying:*

(i) $\beta(a)=a\otimes 1$,

(ii) $a\cdot\lambda(\varphi)$ *and* $\lambda(\varphi)a\varepsilon C_r^*(A,\Gamma,\alpha)$ *for all* $\varphi\varepsilon C_{oo}(\Gamma)$,

(iii) $\gamma\to\bar{\lambda}_\gamma a\bar{\lambda}_{\gamma^{-1}}$ *is norm continuous and* $\alpha_\gamma(a)=\bar{\lambda}_\gamma a\bar{\lambda}_{\gamma^{-1}}$

for all $\gamma\varepsilon\Gamma$.

The next proposition deals with norm-continuous (also called uniformly continuous) group actions on a C*-algebra. Clearly the main applications will be in the case of discrete groups.

3.2. PROPOSITION. *Let A be a simple C*-algebra with unit, and let* α *be a norm continuous action of the locally compact group*

Γ *on* A. *Then, for every derivation* $\delta: C_r^*(A,\Gamma,\alpha)\to C_r^*(A,\Gamma,\alpha)$ *which commutes with* β *in the sense that* $(\delta\otimes i)\beta=\beta\delta$, *there exists* $a_o\in\tilde{\rho}(A)\subset M(C_r^*(A,\Gamma,\alpha))$ *such that* $\delta=ad(a_o)$.

PROOF. Let $\tilde{\delta}$ and $\tilde{\beta}$ be the extensions of δ and respectively β to the weak closure of $C_r^*(A,\Gamma,\alpha)$. Obviously $(\tilde{\delta}\otimes i)\tilde{\beta}=\tilde{\beta}\tilde{\delta}$. (Since δ is inner, it follows that $\tilde{\delta}\otimes i$ has a sense). The proof of the proposition will be given in several steps:

I. (A'',Γ,α'') is a W^*-dynamical system and the W^*-crossed product $W^*(A'',\Gamma,\alpha'')$ is equal with the weak closure of $C_r^*(A,\Gamma,\alpha)$. These facts follow easily from the norm continuity of $\gamma\to\alpha_\gamma$.

II. $\tilde{\delta}(\tilde{\rho}(A))\subset\tilde{\rho}(A)$.

Let $a\in\tilde{\rho}(A)$. We shall show that $\tilde{\delta}(a)$ satisfies Landstad's conditions (in Proposition 3.1):

(i) $\tilde{\beta}(\tilde{\delta}(a))=(\tilde{\delta}\otimes i)\tilde{\beta}(a)=\tilde{\delta}(a)\otimes 1$.

Therefore $\tilde{\beta}(\tilde{\delta}(a))=\tilde{\delta}(a)\otimes 1$.

(ii) Since A is unital, we have $\lambda(\varphi)\in C_r^*(A,\Gamma,\alpha)$ for all $\varphi\in C_{oo}(\Gamma)$. Since obviously $\tilde{\delta}(a)\in M(C_r^*(A,\Gamma,\alpha))$ we have that $\lambda(\varphi)a$ and $a\cdot\lambda(\varphi)\in C_r^*(A,\Gamma,\alpha)$.

(iii) From (i) and [4, Theorem 1] it follows that $\tilde{\delta}(a)\in A''$. Since α'' is uniformly continuous, it follows that $\gamma\to\bar{\lambda}_\gamma\tilde{\delta}(a)\bar{\lambda}_{\gamma^{-1}}=$ $=\alpha''_\gamma(\tilde{\delta}(a))$ is norm continuous. Therefore, by Proposition 3.1., $\tilde{\delta}(a)\in\tilde{\rho}(A)$.

III. $\tilde{\delta}(\tilde{\rho}(A)\bar{\lambda}_\gamma)\subset\tilde{\rho}(A)\bar{\lambda}_\gamma$ for all $\gamma\in\Gamma$.

Indeed let $a\in\tilde{\rho}(A)$ and $\gamma\in\Gamma$. Then as in II it can be proved that $\tilde{\delta}(a\bar{\lambda}_\gamma)\bar{\lambda}_{\gamma^{-1}}\in\tilde{\rho}(A)$. Hence $\tilde{\delta}(a\bar{\lambda}_\gamma)\in\tilde{\rho}(A)\bar{\lambda}_\gamma$.

IV. The end of the proof.

Since $\tilde{\delta}(\tilde{\rho}(A))\subset\tilde{\rho}(A)$ and A is simple with unit, then by the Theorem of Sakai, there exists $a_o\in\tilde{\rho}(A)$ such that $\tilde{\delta}|_{\tilde{\rho}(A)}=ad(a_o)|_{\tilde{\rho}(A)}$. Let $\delta_o=\tilde{\delta}-ad(a_o)$ on $W^*(A'',\Gamma,\alpha'')$. We shall show that $\delta_o\equiv 0$. There exists $b_o\in W^*(A'',\Gamma,\alpha'')$ such that $\delta_o=ad(b_o)$. Further, since $\delta_o(\tilde{\rho}(A))=\{0\}$ we have $b_o\in\tilde{\rho}(A)'$. On the other hand, for $\gamma\in\Gamma$ we have

$$\delta_o(\bar{\lambda}_\gamma)=b_o\bar{\lambda}_\gamma-\bar{\lambda}_\gamma b_o=(b_o-\bar{\lambda}_\gamma b_o\bar{\lambda}_{\gamma^{-1}})\bar{\lambda}_\gamma .$$

By III, $b_o-\bar{\lambda}_\gamma b_o\bar{\lambda}_{\gamma^{-1}}\in\rho(\tilde{A})$. Obviously $\bar{\lambda}_\gamma b_o\bar{\lambda}_{\gamma^{-1}}\in\tilde{\rho}(A)'$, and therefore

$b_o - \bar{\lambda}_\gamma b_o \bar{\lambda}_{\gamma}^{-1}$ belongs to the center of $\tilde{\rho}(A)$. Since A is simple, $b_o - \bar{\lambda}_\gamma b_o \bar{\lambda}_{\gamma}^{-1} = \mu \cdot 1$ for some $\mu \in \mathbb{C}$. Hence $\delta_o(\bar{\lambda}_\gamma) = \mu \cdot \bar{\lambda}_\gamma$. From this, it results that $\delta_o(\bar{\lambda}_\gamma^n) = n\mu\bar{\lambda}_\gamma^n$ for all $n \in \mathbb{N}$. The boundedness of δ_o implies $\mu = 0$. Therefore $\delta_o \equiv 0$, whence $\delta = ad(a_o)$.

Now we shall study the unbounded derivations which commute with the dual action on crossed products. For this we shall assume in addition that A is separable and Γ is discrete and commutative. Let $G = \hat{\Gamma}$, $B = C^*(A, \Gamma, \alpha)$ and $\beta = \hat{\alpha}$.

3.3. LEMMA. *Let (A, Γ, α) be a C^*-dynamical system. Assume that A is a simple C^*-algebra with unit and Γ is discrete and commutative. Let (B, G, β) be its dual system. If $T: D(T) \to B$ $(D(T) \subset B)$ is a linear, closed, densely defined operator such that:*

1) *$D(T)$ is a subalgebra,*
2) *$A \subset D(T)$,*
3) *$\beta_g T \subset T \beta_g$ $(\forall)$ $g \in G$,*

then $\bigcup_{\gamma \in \Gamma} A\bar{\lambda}_\gamma \subset D(T)$ and $T(A\bar{\lambda}_\gamma) \subset A\bar{\lambda}_\gamma$ for all $\gamma \in \Gamma$.

PROOF. By Lemma 2.3, $A\bar{\lambda}_\gamma \cap D(T)$ is dense in $A\bar{\lambda}_\gamma$. Let

$$A_o^\gamma = \{a \in A \mid a\bar{\lambda}_\gamma \in D(T)\}.$$

Then, since $A \subset D(T)$ and $D(T)$ is a subalgebra it follows that A_o^γ is a two-sided ideal of A. Since $A\bar{\lambda}_\gamma \cap D(T)$ is dense in $A\bar{\lambda}_\gamma$ and A is simple, we have $A_o^\gamma = A$. Therefore $A\bar{\lambda}_\gamma \subset D(T)$. The fact that $T(A\bar{\lambda}_\gamma) \subset A\bar{\lambda}_\gamma$ may be proved similarly with the analogous fact in Lemma 2.4 (i).

3.4. LEMMA. *Let (A, Γ, α), (B, G, β) and T be as in the preceding lemma. Then T possesses a dense subspace $S \subset D(T)$ of analytic elements such that $TS \subset S$. Moreover S is a core for T.*

PROOF. The proof is an obvious adaptation of the proof of Lemma 2.4. (ii) and (iii) with $S = \text{lin span}(\bigcup_{\gamma \in \Gamma} A\bar{\lambda}_\gamma)$.

3.5. THEOREM. *Let (A, Γ, α) be a C^*-dynamical system. Assume that A is a separable, simple C^*-algebra with unit and Γ is discrete and commutative. Denote by (B, G, β) its dual system. If $\delta: D(\delta) \to B$ is a symmetric, closed, densely defined derivation such*

that:

1) $A \subset D(\delta)$,

2) $\beta_g \delta \subset \delta \beta_g$ $(\forall)$ $g \in G$,

then we have

(i) *There exists* $h_o \in A$, $h_o = h_o^*$ *such that* $\delta_1 = \delta - ad(h_o)$ *is the generator of a strongly continuous one parameter group of* $*$*-automorphisms of* B, $\{\tau_t\}_{t \in \mathbb{R}}$, *and* $\tau_t \beta_g = \beta_g \tau_t$ $(\forall)$ $t \in \mathbb{R}$, $g \in G$.

(ii) *If* T *is a closed, densely defined operator on* B *which commutes with* β *such that* $\delta \subset T$, *then* $\delta = T$.

PROOF. (i) By Lemma 3.3 $\delta(A) \subset A$. Since A is simple with unit, by the Theorem of Sakai there exists $h_o \in A$, $h_o = h_o^*$ such that $\delta|_A = ad(h_o)|_A$. Let $\delta_1 = \delta - ad(a_o)$ (on $D(\delta)$). We prove that δ_1 satisfies the hypothesis of [1, Corollary 3.2.57]. For $b \in B$, we consider the following integration:

$$\varepsilon_o(b) = \int_G \beta_g(b)\,dg.$$

Then $\varepsilon_o(b) \in A$ for all $b \in B$. Since A is separable there exists a faithful state ψ_o on A. Then $\varphi_o = \psi_o \circ \varepsilon_o$ is a faithful G-invariant state on B. Obviously $\varphi_o(A\bar{\lambda}_\gamma) = 0$ for all $\gamma \in \Gamma$ with $\alpha_\gamma \neq \iota$. Then $\varphi_o(\delta_1(\bigcup_{\gamma \in \Gamma} A\bar{\lambda}_\gamma)) = 0$. Since by Lemma 3.4, $S = \text{lin span} \bigcup_{\gamma \in \Gamma} A\bar{\lambda}_\gamma$ is a core for δ_1 we have $\varphi_o(\delta_1(D(\delta_1)) = 0$. (Here $D(\delta_1) = D(\delta)$.) The first condition is thus verified.

By Lemma 3.4, δ_1 possesses a dense set of analytic elements. Therefore, by [1, Corollary 3.2.57], δ_1 is a generator. The fact that $\tau_t \beta_g = \beta_g \tau_t$ $(t \in \mathbb{R}, g \in G)$ is a consequence of the Hille-Yosida Theorem.

(ii) follows from Lemma 3.4.

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