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by

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N.Bourbaki ([1], § 2, exerc.5) uses the term "strongly lattice-ordered" ("fortement réticulé") for a ring A such that $|xy| = |x||y|$ for every $x, y \in A$. According to this, we define a strongly lattice-ordered module to be a lattice ordered module E over a lattice-ordered ring A such that $|ax| = |a||x|$ for every $a \in A$ and $x \in E$. In fact, we are interested in the case of modules over some function algebras.

After giving the definitions, we study in § 2 some first constructions which yield strongly lattice-ordered modules (completion, quotiens, inductive limits, duality).

In § 3 we consider the question of the extension of the extension of the ring of "scalars"; further facts related to the extension (e.g. the "regularity" in the sense of measure theory) and application of the results to obtain new proofs of some representation theorems known in measure theory will be given in a later work.

In § 4 we introduce the concept of principal module and in § 5 the construction of the spaces E_t; these will be used in § 6 which handles the AM-modules. For an AM-module E over $C_K(X)$ (the space of continuous functions on the compact X) a natural map T is defined from the dual E^* of E to the dual $C_K(X)^*$ of $C_K(X)$. Then we characterise those modules with the property that T is an isometric Riesz homomorphism.

In [2] N.Gretsky consider a Banach function space L_ρ over a measurable space Ω. The quotient space $N_\rho = \frac{L_\rho}{M_\rho}$ (M_ρ being the closure of the space of simple functions) is under certain assumptions an AM-space. By mean of a certain map he is able to represent the dual N^* as a space of bounded additive functions on the Boolean algebra of (equivalence classes of) measurable subsets of Ω.

We give in § 7 an abstract generalization of the above situation, namely an AM-space E and a representation of a Boolean algebra $\mathfrak{A}$ into the set of positive linear projections on E. We construct the map H from E^* to the space of bounded additive functions on $\mathfrak{A}$ and we apply the results of § 6 to identify the case when H is an isometric Riesz homomorphism.

§ 1. Preliminaries.

a) Vector lattices.

For a real vector lattice we shall use the standard notations:

$x_+ = x \vee 0$

$x_- = (-x) \vee 0$

$|x| = x \vee (-x)$.

We shall employ the concept of a complex vector lattice of Mittelmeyer-Wolff ([5]):

A complex vector lattice is a $\mathbb{C}$-vector space E together with a map $m: E \longrightarrow E$ (called modulus) satisfying:

i) $m(m(x)) = m(x), \ x \in E.$

ii) $m(ax) = |a| m(x), \ a \in \mathbb{C}, x \in E.$

iii) $m(m(m(x)+m(y)) - m(x+y)) = m(x)+m(y) - m(x+y),$
$x, y \in E.$

iv) E is spanned by $m(E)$.

Let $E_R = m(E) - m(E)$. E_R is a real vector subspace of E. Every $x \in E$ can be uniquely written as $x = x_1 + i x_2$ with $x_1, x_2 \in E_R$. We shall write $Re\, x = x_1$, $Im\, x = x_2$. The cone $m(E)$ defines on E_R a structure of real vector lattice; if we denote by $\leq$ its ordering we have

$$m(x) = \sup(x,-x),\ x \in E_R \tag{1}$$

$$m(x+y) \leq m(x) + m(y),\ x,y \in E \tag{2}$$

$$m(m(x)-m(y)) \leq m(x-y),\ x,y \in E \tag{3}$$

According to (1), we shall employ the notation $|x|$ instead of m(x) for every $x \in E$.

Throughout all the paper the letter K will denote one of the fields $\mathbb{R}$, $\mathbb{C}$. By a vector lattice we shall mean a real or complex vector lattice. To unify notation we shall write $\operatorname{Re} x = x,\ x \in E$ for a real vector lattice E.

Let E be a vector lattice. We define:

$$E_+ = \{x \in E \mid x \geq 0\}$$

$$Z(x) = \{y \in E \mid |y| \leq |x|\},\ x \in E$$

$$E(x) = \{y \in E \mid \exists a,\ a \in \mathbb{R},\ |y| \leq a|x|\},\ x \in E,$$

A set $A \subset E$ is order bounded if there is $x \in E$ such that $A \subset Z(x)$.

A set $A \subset E$ is solid if $Z(x) \subset A$ for every $x \in A$. A vector subspace $F \subset E$ is a vector sublattice (an order ideal) if $x \in F$ implies $\operatorname{Re} x \in F$ and $|x| \in F$ $(Z(x) \subset F)$. Every order ideal is a vector sublattice (for the complex case see [6]).

Various order properties required to E (e.g. Dedekind completeness) will in fact refer to E_R.

Let E,F be vector lattices. We shall use the notation $L(E,F)$ for the space of linear maps $U: E \to F$ and $L(E)$ for $L(E,E)$. A map $U \in L(E,F)$ is a Riesz homomorphism if $|U(x)| = U(|x|)$ for every $x \in E$. If E and F are real or if E and F are complex and F is Archimedian then $U \in L(E,F)$ is a Riesz homomorphism iff

$$x, y \in E_+ , x \wedge y = 0 \Rightarrow U(x) \wedge U(y) = 0 .$$

(see 6).

By "E and F are order isomorphic" we shall mean there is a bijective Riesz homomorphism $U \in L(E,F)$.

$L_r(E,F)$ will denote the space of order bounded maps $U \in L(E,F)$. If F is Dedekind complete then $L_r(E,F)$ is a Dedekind complete vector lattice. We have

$$U \geq 0 \iff U(E_+) \subset F_+ \tag{4}$$

$$|U|(x) = \sup_{y \in Z(x)} |U(y)| , \quad x \in E_+ \tag{5}$$

(for (5) in the case K= $\mathbb{C}$, see [6]).

Let E, F be vector lattices. A positive map $U \in L(E,F)$ is order continuous if for every downwards directed set $A \subset E_R$ with inf A=0 we have inf $U(A) = 0$. A positive map $U \in L(E,F)$ is σ - order continuous if for every decreasing sequence $(x_n)_{n \in N}$ with $x_n \in E_R$ and $\inf_{n \in \mathbb{N}} x_n = 0$ we have $\inf_{n \in \mathbb{N}} U(x_n) = 0$. If F is Dedekind complete, a map $U \in L_r(E,F)$ will be called (σ) order-continuous if $|U|$ is (σ) order-continuous.

Let E be a vector lattice, F⊂E an order ideal, $\frac{E}{F}$ the quotient vector space, $p : E \to \frac{E}{F}$ the quotient map. There is a unique structure of vector lattice on $\frac{E}{F}$ such that p becomes a Riesz homomorphism.

Let E be a vector lattice. A locally solid topology on E is a linear topology which admits a basis of neighborhoods of 0 consisting of solid sets. If τ is a locallysolid topology then (3) shows that the map $x \mapsto |x|$ is τ - uniformly continuous; so, if τ is Hausdorff and if F is the τ - completion of E then $x \mapsto |x| = m(x)$ extends uniquely to F. We can easily show that one obtains a structure of vector lattice on E (to check (iv) in the complex case, define the the involution $x \mapsto x^*$; $x \in E$, where $x^* = Re\, x - i\, Im\, x$. Since E is Archimedian we have

$$|x| = |x^*|$$

Thus $x \mapsto x^*$ is τ- continuous and can be extended to a involution on F. The maps $U_1, U_2 : F \to F$ given by

$$U_1(x) = \frac{1}{2}(|x| + x)$$

$$U_2(x) = \frac{1}{2}(|x| - x)$$

are continuous. We have

$$x = U_1\left(\tfrac{1}{2}(x+x^*)\right) - U_2\left(\tfrac{1}{2}(x+x^*)\right) + iU_1\left(\tfrac{1}{2i}(x-x^*)\right) - iU_2\left(\tfrac{1}{2i}(x-x^*)\right).$$

But if $x \in E$

$$U_1\left(\tfrac{1}{2}(x+x^*)\right) = (\mathrm{Re}\, x)_+ \in m(F)$$

$$U_2\left(\tfrac{1}{2}(x+x^*)\right) = (\mathrm{Re}\, x)_- \in m(F)$$

$$U_1\left(\tfrac{1}{2i}(x-x^*)\right) = (\mathrm{Im}\, x)_+ \in m(F)$$

$$U_2\left(\tfrac{1}{2i}(x-x^*)\right) = (\mathrm{Im}\, x)_- \in m(F).$$

Since $m(F)$ is τ- closed, the result follows).

A solid seminorm on a vector lattice E is a seminorm q with the property

$$|x| \leq |y| \Rightarrow q(x) \leq q(y),$$

A normed vector lattice is a vector lattice equipped with a solid norm.

The topological dual of a topological vector space E will be denoted by E^*.

b) Function spaces

Let X be a compact space. We shall use the following function spaces:

$C_K(X)$: the space of continuous K-valued functions on X.

$B_K(X)$: the space of bounded Borel K-valued functions on X.

$\mathcal{B}_K(X)$: the space of bounded Baire K-valued functions on X.

These spaces have canonical lattice-order, normed and multiplicative structures. The function identical equal to 1 will also be denoted by 1.

For a set $M\subset X$, χ_M will be its characteristic function.

It is an well known result (see W.A.J.Luxemburg, A.C., Zaanen,[3], exaple 27.7) that for any Riesz homomorphism $\varphi: C_K(X)\to K$ with $\varphi(1)=1$ there is an unique $t\in X$ such that $\varphi(f)=f(t)$, $f\in C_K(X)$. From this we can easily see that if $U: C_K(X)\to C_K(Y)$ is a Riesz homomorphism with $U(1)=1$ there is an unique continuous map $h: Y\to X$ such that $U(f)=f\circ h$, $f\in C_K(X)$.

c) Stonian spaces

A Stonian space is a compact space X such that for every open set $G\subset X$ its closure $\overline{G}$ is open.

A compact space X is Stonian iff $C_K(X)$ is Dedekind complete.

If X is a Stonian space there is a σ - order continuous Riesz homomorphism $r: B_K(X)\to C_K(X)$ with the properties:

i) for every $f\in B_K(X)$ the set

$$\{t\in X \mid f(t)\neq r(f)(t)\}$$

is meager

ii) if $(f_i)_{i\in I}$ is a family of elements in $C_K(X)$ and if $f\in B_K(X)$ is such that $f(t)=\sup_{i\in I} f_i(t)$ for every $t\in X$ then

$$r(f)=\sup_{i\in I} f_i .$$

d) σ- Stonian-spaces

A σ-Stonian space is a compact space X such that for every open F_σ set $G\subset X$ its closure $\overline{G}$ is open.

A compact space X is σ- Stonian iff $C_K(X)$ is Dedekind σ- complete.

If X is a σ- Stonian space there is a σ-order continuous Riesz homomorphism $\rho : \mathcal{B}_K(X) \to C_K(X)$ with the property: for every $f \in \mathcal{B}_K(X)$ the set

$$\{t \in X \mid f(t) \neq \rho(f)(t)\}$$

is meager (The existence of ρ and r is an well known result which can be easily proved by showing that the family of sets $M\subset X$ for which there is an open-closed set G such that $\{t \in X \mid \chi_M(t) \neq \chi_G(t)\}$ is meager is a σ - algebra $\mathfrak{U}$ containing every open-closed set if X is σ - Stonian and every open set if X is Stonian; then use the fact that the span of $\{\chi_M \mid M \in \mathfrak{U}\}$ is dense in $\mathcal{B}_K(X)(B_K(X))$.

e) Vector lattices with unit

A vector lattice with unit is a normed vector lattice E which has an element e (called the unit of E) such that

$$\|e\| = 1$$
$$|x| \leq \|x\| e, \quad x\in E.$$

A lattice-ordered algebra with unit is a vector lattice E with unit e equipped with a bilinear map $(x,y) \mapsto xy$ such that

$$xe = ex = x \quad , \quad x \in E$$

$$|xy| = |x||y| \quad , \quad x,y \in E.$$

A Riesz subalgebra of the lattice-ordered algebra E with unit e is a vector sublattice of E closed under multiplication and containing e.

For every vector lattice (lattice ordered algebra) E with unit e there is a compact space X and an (algebric) norm-preserving Riesz homomorphism $U: E \to C_K(X)$ such that $U(e)=1$ and $U(E)$ is dense in $C_K(X)$. We have $U(E)=C_K(X)$ iff E is norm complete.

If E is an Archimedian space and $x \in E \setminus \{0\}$ then $E(x)$ equipped with the norm $\| \; \|_x$ given by

$$\|y\|_x = \inf\{a \in \mathbb{R} \mid |y| \le a|x|\}$$

becomes a vector lattice with unit $|x|$. E is called uniformly complete if E(x) is norm complete for every $x \in E \setminus \{0\}$.

f) The monotone class theorem (P.A.Meyer,[4], ch.I, § 2, thm.19)

Let X be a set, $\mathcal{H}$ a vector space of real bounded functions on X containing 1, closed under uniform convergence and such that for every increasing uniformly bounded sequence $(f_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{H}$ the function $f=\lim_{n\to\infty} f_n$ belongs to $\mathcal{H}$. Let $\mathcal{C}$ a subset of $\mathcal{H}$ closed under multiplication. Then $\mathcal{H}$ contains every bounded function which is measurable with respect to the σ-algebra generated by the elements of $\mathcal{C}$.[1)]

§ 2. Strongly lattice-ordered modules

Definition 2.1. Let A be a vector lattice with unit e. A strongly lattice-ordered (s.l.o) module over A is a vector lattice E together with a bilinIar map $\phi: A \times E \to E$ with the properties:

i) $\phi(e,x) = x$, $x \in E$.

ii) $|\phi(f,x)| = \phi(|f|,|x|)$, $f \in A$, $x \in E$.

When the context will be clear we shall write fx instead of $\phi(f,x)$.

1) That is the σ-algebra generated by the sets $f^{-1}(A)$ with $f \in \mathcal{C}$ and $A \subset \mathbb{R}$ open.

A sub-A-module of E is a vector sublattice F of E such that $f\in A$, $x\in F$ implies $fx\in F$.

Let E_1, E_2 be s.l.o. modules over A. A linear mapp $U_1 : E_1 \to E_2$ is called A-linear if $U(fx) = fU(x)$ for $f\in A$ and $x \in E_1$.

Proposition 2.1. Let A be a vector lattice with unite and E a s.l.o module over A. Then:

a) $f\in A_+$ and $x\in E_+$ implies $fx\in E_+$.

b) $f\in A_R$ and $x\in E_R$ implies $fx\in E_R$.

c) for every $f\in A$ and $x\in E$ we have

$$|fx| \leq \|f\| \, |x|.$$

d) for every $f\in A_+$ the map $x \longmapsto fx$ is order continuous.

e) every order ideal of E is a sub-A-module.

f) if E is equipped with a locally solid topology the map $(f,x) \mapsto fx$ is continuous.

g) if E is a normed vector lattice then

$$\|fx\| \leq \|f\| \, \|x\|, \quad f\in A, x\in E.$$

PROOF.

a) $|fx| = |f|\,|x| = fx$, so $fx\in E_+$.

b) Follows from a).

c) We have $\|f\|e - |f| \in A_+$ so, according to a), $(\|f\|e - |f|)|x| \in E_+$. Thus

$$|fx| = |f|\,|x| \leq (\|f\|e)|x| = \|f\|\,|x|.$$

d),e),f),g). Follow from c).

Proposition 2.2. Let A be a vector lattice with unit, E a s.l.o.

module over A equipped with a locally solid Hausdorff topology τ and let $\overline{A}$ be the norm completion of A, $\overline{E}$ the τ- completion of E. Then there is a unique structure of s.l.o. module over $\overline{A}$ on $\overline{E}$ which extends the structure of E.

PROOF. According to prop.2.1 f), the map ϕ extends uniquely to a continuous biliniar $\overline{\Phi}: \overline{A} \times \overline{E} \to \overline{E}$. Since $x \mapsto |x|$ is τ - continuous, i) and ii) are preserved.

Proposition 2.3. Let A be a vector lattice with unit, E a s.l.o. module over A,F an order ideal of E. Then there is a unique structure of s.l.o. module over A on $\frac{E}{F}$ such that the quotient map $p: E \to \frac{E}{F}$ becomes A-linear.

PROOF. Obvious.

Proposition 2.4. Let A be a vector lattice with unit. Consider a vector lattice E and a family $(E_i)_{i \in I}$ of sublattices of E with the properties:

i) I is partially preordered and directed upwards;

ii) for every $i \in I$, E_i is a s.l.o. module over A.

iii) if $i_1, i_2 \in I$ and $i_1 \leq i_2$ then $E_{i_1} \subset E_{i_2}$ and the inclusion map $E_{i_1} \to E_{i_2}$ is A-linear.

iv) $\bigcup_{i \in I} E_i = E$,

Then there is a unique structure of s.l.o. modulo over A on E such that for every $i \in I$ the inclusion map $E_i \to E$ is A-linear.

PROOF. The proof is based on the well known algebraic construction of inductive limits.

This proposition will be applied in the following to the family $(E(x))_{x\in E\setminus\{0\}}$, $E\setminus\{0\}$ being partially preordered by

$$x \lesssim y \iff |x| \leq |y| .$$

Corollary 2.1. Let A a vector lattice with unit e, $\bar{A}$ its completion, E an uniformly complete s.l.o. module over A. Then there is an unique structure of s.l.o module over $\bar{A}$ on E which extends the structure of s.l.o. module over A.

PROOF: If we would have two structures of $\bar{A}$-module over E, then the identity map would be $\bar{A}$-linear by continuity so the two structure would coincide.

For the existence, apply prop.2.2 to A and the Banach space E(x), $x\in E\setminus\{0\}$. Because the inclusion map $E(x)\to E(y)$ (for $|x|\leq|y|$) is A-linear, it will also be $\bar{A}$-linear, by continuity. Finally apply prop.2.4.

Proposition 2.5. Let A be a vector lattice with unit e and let E be a real vector lattice or a complex Archimedian vector lattice. Suppose there is a bilinear map $(f,x)\mapsto fx\in E$, $f\in A$, $x\in E$ such that $ex=x$, $x\in E$. Then this map defines a structure of s.l.o. module over A on E iff:

i) For every $f\in A$ and every $x,y\in E$ such that $x\wedge y=0$ we have $|fx|\wedge|fy|=0$.

ii) For every $x\in E_+$ and every $f_1,f_2\in A_+$ such that $f_1\wedge f_2=0$ we have $f_1x\wedge f_2x=0$.

PROOF.

"$\Longrightarrow$" i) follows from prop.2.1 c) and ii) from def.2.1.

" $\Leftarrow$ " Suppose first E is real. From ii) it follows that the map $f \mapsto fx$ is a Riesz homomorphism for every $x \in E$. Let $f \in A$ and $x \in E$. Because $x_+ \wedge x_- = 0$ we have $|fx_+| \wedge |fx_-| = 0$ so

$$|fx| = |fx_+ - fx_-| = |fx_+| + |fx_-| = |f|x_+ + |f|x_- = |f||x|.$$

Now consider the case E complex. From ii) if follows that $f \in A_+$ and $x \in E_+$ implies $fx \in E_+$ so $f \in A_R$ and $x \in E_R$ implies $fx \in E_R$. The previous argument shows that E_R is a s.l.o. module over A_R. It follows from [6], prop.6 that E is a s.l.o. module over A.

Proposition 2.6. Let A be a vector lattice with unit and let Y be a compact space. Suppose that $C_K(Y)$ is a s.l.o. module over A. Then

$$fy = (f1)y \quad , \quad f \in A, \; y \in C_K(Y)$$

where by xy we mean the usual product in $C_K(Y)$.

PROOF. We show first that $t \in Y$, $y \in C_K(Y)$ and $y(t) = 0$ implies $(fy)(t) = 0$ for every $f \in A$. Indeed,

$$|(fy)(t)| = |fy|(t) \leq \|f\| \, |y|(t) = \|f\| \, |y(t)| = 0.$$

Now let $y \in C_K(Y)$. For $t \in Y$ we have $(y - y(t)1)(t) = 0$ so $(f(y - y(t)1))(t) = 0$, that is $(fy)(t) = (f1)(t)\, y(t)$.

Corollary 2.2. Let X, Y be compact spaces. Suppose that $C_K(Y)$ is s.l.o. module over $C_K(X)$. Then there is a continuous map $h : Y \to X$ such that

$$fy = (f \circ h)y \quad , \quad f \in C_K(X), \; y \in C_K(Y).$$

PROOF. The map $f \mapsto f1$, $f \in C_K(X)$ is a Riesz homomorphism, so there is a continuous map $h: Y \longrightarrow X$ such that $f1 = f \circ h$, $f \in C_K(X)$. According to prop.2.6. we have

$$fy = (f1)y = (f \circ h)y.$$

Proposition 2.7. Let A be a lattice-ordered algebra with unit and let E be a s.l.o. Archimedian module over a. Then

$$f(gx) = (fg)x, \quad f, g \in A, \; x \in E.$$

PROOF. We can assume that A is a dense Riesz subalgebra of $C_K(X)$ with X compact. Let $x \in E \setminus \{0\}$ and let F be the norm completion of $E(x)$. As E(x) is a s.l.o. module over A, F becomes a s.l.o. module over $C_K(X)$. But F is isomorphic to a space $C_K(Y)$ so by cor. 2.2. we have

$$f(gx) = (fg)x, \quad f, g \in A.$$

Proposition 2.8. Let X be a compact space, A a dense vector sublattice of $C_K(X)$ such that $1\in A$, E a s.l.o. module over A. Then for every $\varepsilon > 0$ and every $f,g,h \in A$ and $x \in E$ we have

$$|f(gx) - hx| \leq (\|fg-h\| + \varepsilon)|x|.$$

PROOF. Let $x \in E \setminus \{0\}$ and let

$$J(x) = \{y \in E \mid |y| \leq a|x| \; \forall a > 0\},$$

The space $F(x) = \dfrac{E(x)}{J(x)}$ is a vector lattice with unit $p_x(|x|)$ where $p_x : E(x) \longrightarrow F(x)$ is the quotient map. Let $\overline{F}(x)$ be the completion of F(x). As F(x) is a s.l.o. module over A, $\overline{F}(x)$ becomes a s.l.o. module over $C_K(X)$. By prop.2.7

$$p_x(|f(gx) - hx|) = |f(g p_x(x)) - h p_x(x)| = |(fg) p_x(x) - h p_x(x)| \leq$$
$$\leq \|fg - h\| p_x(|x|) = p_x(\|fg-h\| \, |x|).$$

Thus there is a $u \in J(x)$ such that

$$\|fg - h\| \, |x| - |f(gx) - hx| \geq u.$$

It follows that

$$|f(gx)-hx| \leq \|fg-h\|\,|x|+u \leq \|fg-h\|\,|x|+|u| \leq$$
$$\leq (\|fg-h\|+\varepsilon)|x| .$$

Proposition 2.9. Let A be a vector lattice with unit, E a vector lattice, F a Dedekind complete s.l.o. module over A. Then the map $(f,U)\mapsto fU$, $f\in A$, $U\in L_r(E,F)$ given by

$$(fU)(x) = fU(x)$$

defines a structure of s.l.o. module over A on $L_r(E,F)$.

PROOF. i) of def.2.1 is trivial. We check ii): if $x\in E_+$ then

$$|fU|(x) = \sup_{y\in Z(x)} |(fU)(y)| = \sup_{y\in Z(x)} |fU(y)| =$$
$$= \sup_{y\in Z(x)} |f|\,|U(y)| = |f| \sup_{y\in Z(x)} |U(y)| = |f|\,|U|(x) =$$
$$= (|f|\,|U|)(x) .$$

Proposition 2.10. Let A be a vector lattice with unit, E a s.l.o. module over A, F a Dedekind complete vector lattice. Then the map $(f,U)\mapsto fU$, $f\in A$, $U\in L_r(E,F)$ given by

$$(fU)(x) = U(fx)$$

defines a structures of s.l.o. module over A on $L_r(E,F)$.

PROOF. We apply prop.2.5. If $f\in A$ and $x\in E_+$ then

$$|fU|(x) = \sup_{y\in Z(x)} |(fU)(y)| = \sup_{y\in Z(x)} |U(fy)| \leq \sup_{y\in Z(x)} |U|(|fy|) \leq$$

$$\leq \|f\| \sup_{y\in Z(x)} |U|(|y|) = \|f\|\,|U|(x)$$

so

$$|fU| \leq \|f\|\,|U|\ , f\in A,\ U\in L_r(E,F)$$

which shows that i) of prop.2.5 is satisfied. Next we prove ii). We can assume that A is a dense vector sublattice of a space $C_K(X)$ with X compact such that $1\in A$. Let $U\in L_r(E,F)$, $U\geq 0$ and let $f_1, f_2\in A_+$ be such that $f_1\wedge f_2 = 0$. Put $V = (f_1U)\wedge(f_2U)$. Let $\varepsilon > 0$. There are $g_1, g_2\in C_K(X)_+$ such that $g_1+g_2 = 1$, $\|f_i g_i\| \leq \frac{\varepsilon}{8}$, i=1,2 (For instance take g_1 such that $0\leq g_1\leq 1$ $g_1(t) = 0$ on $\{t\in X \mid f_1(t)\geq \frac{\varepsilon}{8}\}$ and $g_1(t) = 1$ on $\{t\in X \mid f_2(t)\geq \frac{\varepsilon}{8}\}$. As A is dense in $C_K(X)$ and $1\in A$ there are $h_1, h_2\in A_+$ such that $h_1+h_2 = 1$, $\|f_i h_i\|\leq \frac{\varepsilon}{4}$, i=1,2. Let $x\in E$. By prop.2.8

$$f_i(h_i x) \leq \frac{\varepsilon}{2}x\ ,\ i=1,2.$$

We have

$$V(x) = V(h_1x + h_2x) \leq (f_1U)(h_1x) + (f_2U)(h_2x) =$$
$$= U(f_1(h_1x)) + U(f_2(h_2x)) \leq \varepsilon U(x).$$

As ε is arbitrary and F Archimedian, V(x)=0.

Definition 2.2. Let A,B be vector lattices with units, E a vector lattice which is s.l.o. module over A and over B. Then the two structures are called compatible if

$$f(gx) = g(fx), \quad x \in E, \ f \in A, \ g \in B.$$

Proposition 2.11. If in prop.2.9 (resp.prop.2.10) F (resp.E) has two compatible structures of s.l.o. modules, then the structures obtained on $L_r(E,F)$ are also compatible.

PROOF. Obvious

3. Extension theorems

Definition 3.1. Let A be a vector lattice with unit. A σ-continuous s.l.o. module over A is a s.l.o. module E over A such that for every $x \in E_+$ the map $f \mapsto fx$ is σ - order continuous.

A σ- order continuous s.l.o. module is Archimedian.

Proposition 3.1. Let X be a compact space, E, F σ- continuous s.l.o. modules over $\mathcal{B}_K(X)$, $U : E \longrightarrow F$ a positive σ - order continuous $C_K(X)$ - linear map. Then U is $\mathcal{B}_K(X)$- linear.

PROOF. It will be sufficient to show that for every $x \in E_+$ and every $f \in \mathcal{B}_{\mathbb{R}}(X)$ we have $U(fx) = fU(x)$. So let be $x \in E_+$. We put

$$\mathcal{H} = \{ f \in \mathcal{B}_{\mathbb{R}}(X) \mid U(fx) = fU(x) \}.$$

Clearly $C_{\mathbb{R}}(X) \subset \mathcal{H}$. $\mathcal{H}$ is closed under uniform convergence: indeed, if $f_n \in \mathcal{H}$, $\lim_{n\to\infty} \|f_n - f\| = 0$ then $f \in \mathcal{B}_{\mathbb{R}}(X)$ and

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$$|fx - f_n x| \le \|f - f_n\| x,$$
$$|f_n U(x) - f U(x)| \le \|f - f_n\| U(x),$$
$$|U(fx) - U(f_n x)| \le \|f - f_n\| U(x)$$

so

$$|U(fx) - fU(x)| \le 2\|f - f_n\| U(x).$$

As F is Archimedian, $U(fx) = fU(x)$.

Finally, let $f \in \mathcal{B}_{\mathbb{R}}(X)$, $f_n \in \mathcal{H}$ be such that $f_n \uparrow f$. As U is σ - order continuous and E,F are σ - continuous, we have

$$U(fx) = U((\sup_{n\in\mathbb{N}} f_n)x) = U(\sup_{n\in\mathbb{N}} f_n x) = \sup_{n\in\mathbb{N}} U(f_n x) =$$
$$= \sup_{n\in\mathbb{N}} f_n U(x) = fU(x).$$

By the monotone class theorem, $\mathcal{H} = \mathcal{B}_{\mathbb{R}}(X)$.

Definition 3.2. Let X be a compact space. A s.l.o. module E over $B_K(X)$ is called quasi-Radon if:

i) E is σ - continuous

ii) if $x \in E_+$, $f \in B_{\mathbb{R}}(X)$ and $(f_i)_{i\in I}$ is a family of continuous functions such that $f(t) = \sup_{i\in I} f_i(t)$, $t \in X$ then

$$fx = \sup_{i\in I} f_i x.$$

Proposition 3.2. Let X be a compact space E, F quasi-Radon s.l.o. modules over $B_K(X)$, $U: E \to F$ a positive order continuous $C_K(X)$-linear map. Then U is $B_K(X)$ - linear.

PROOF. It will be sufficient to show that for every $x \in E_+$ and every $f \in B_{\mathbb{R}}(X)$ we have $U(fx) = fU(x)$. So let $x \in E_+$. We put

$$\mathcal{H} = \{ f \in B_{\mathbb{R}}(X) \mid U(fx) = fU(x) \}$$

$$\mathcal{C} = \{ \chi_G \mid G \subset X,\ G \text{ open} \}$$

Clearly $1 \in \mathcal{H}$ and $\mathcal{C}$ is closed under multiplication. We have $\mathcal{C} \subset \mathcal{H}$. Indeed, if $G \subset X$ is open then let

$$\mathcal{F} = \{ fx \mid f \in C_{\mathbb{R}}(X),\ 0 \le f \le \chi_G \},$$

$\mathcal{F}$ is directed upwards and $\chi_G x = \sup \mathcal{F}$. Because U is order continuous we have

$$U(\chi_G x) = U(\sup \mathcal{F}) = \sup U(\mathcal{F}) = \sup \{ fU(x) \mid f \in C_{\mathbb{R}}(X),\ 0 \le f \le \chi_G \} = \chi_G U(x).$$

As in the proof of prop.3.1, $\mathcal{H}$ is closed under uniform congervence and monotone convergence. By the monotone class theorem, $\mathcal{H} = B_{\mathbb{R}}(X)$.

Theorem 3.1. Let X be a compact space, E a Dedekind σ-complete s.l.o. module over $C_K(X)$. There is an unique structure of σ-continuous s.l.o. module over $\mathcal{B}_K(X)$ on E which extends the structure of module over $C_K(X)$.

PROOF. The unicity follows from prop.3.1. For the existence, consider first the case $E = C_K(Y)$. As E is Dedekind σ- complete, Y is σ- Stonian. There is a continuous map $h : Y \to X$ such that

$$fy = (f \circ h)y \, , \; f \in C_K(X) \, ; \, y \in C_K(Y) \, .$$

Then the structure of s.l.o. module over $\mathcal{B}_K(X)$ we look for is given by

$$fy = \rho(f \circ h)y, \quad f \in \mathcal{B}_K(X) \, , \, y \in C_K(Y).$$

Now consider the general case. For every $x \in E \setminus \{0\}$ E(x) is isomorphic with a space $C_K(Y)$, so by the previous argument, E(x) becomes a s.l.o. module over $\mathcal{B}_K(X)$. If $|x| \leq |y|$, the inclusion map $E(x) \to E(y)$ is $\mathcal{B}_K(X)$-linear by prop.3.1. So by prop.2.4, E becomes a s.l.o. module over $\mathcal{B}_K(X)$.

Theorem 3.2. Let X be a compact space, E a Dedekind complete s.l.o. module over $C_K(X)$. There is an unique structure of quasi-Radon s.l.o. module over $B_K(X)$ which extends the structure of module over $C_K(X)$.

PROOF. The unicity follows from prop.3.2. For the existence, replace in the proof of thm.3.1 the σ - Stonian spaces by Stonian spaces, $\mathcal{B}_K(X)$ by $B_K(X)$ and ρ by r.

These theorems will be used to obtain, in a later paper, other extension theorems. They will find their application in giving new proofs to various representation theorems. In particular, Riesz's and Daniell's theorems will be derived as corollaries.

4. Principal modules

Proposition 4.1. Let A be a vector lattice with unit e and E a s.l.o. module over A equipped with a locally solid Hausdorff topology. The following assertions are equivalent:

i) For every $x \in E$ the set $\{fx \mid f \in A\}$ is dense in $E(x)$.

ii) For every $x \in E_+$ the set $\{fx \mid f \in A, 0 \leq f \leq e\}$ is dense in $\{y \in E \mid 0 \leq y \leq x\}$.

iii) For every $x \in E_+$ the set $\{fx \mid f \in A, |f| \leq e\}$ is dense in $Z(x)$.

iv) For every $x \in E$ the set $\{fx \mid f \in A, |f| \leq e\}$ is dense in $Z(x)$.

PROOF.

i) $\Rightarrow$ ii) Let $x \in E_+$ and $y \in E_+$, $y \leq x$. There is a net $(f_\delta)_\delta$ such that $f_\delta \in A$, $f_\delta x \to y$. Then if $g_\delta = (\mathrm{Re} f_\delta)_+ \wedge e$ we have $0 \leq g_\delta \leq e$ and

$$g_\delta x = (\mathrm{Re}\, f_\delta x)_+ \wedge x \longrightarrow y.$$

ii) $\Rightarrow$ iii) Let $x \in E_+$, $y \in Z(x)$ and let V be an open solid neighborhood of 0. There is a $\varepsilon > 0$ such that $\varepsilon |y| \in V$. We can find $y_1, \dots, y_n \in E_+$ and $a_1, \dots, a_n \in K$ such that

$$|a_i| = 1, \quad 1 \leq i \leq n$$

$$\left| y - \sum_{i=1}^{n} a_i y_i \right| \leq \varepsilon |y|,$$

$$\sum_{i=1}^{n} y_i \leq |y|$$

(if $K=\mathbb{R}$, put $y_1=y_+, y_2=y_-, a_1=1, a_2=-1$. If $K=\mathbb{C}$, see [6], prop.4).

There are nets $(f_{i\delta})_\delta$ such that $f_{i\delta}\in A$, $0\leq f_{i\delta}\leq e$, $f_{i\delta}x\to y_i$, $1\leq i\leq n$. Define inductively

$$g_{1\delta}=f_{1\delta},$$

$$g_{i\delta}=f_{i\delta}\wedge\Big(e-\sum_{j=1}^{i-1}g_{j\delta}\Big).$$

Then $g_{i\delta}\geq 0$, $\sum_{i=1}^n g_{i\delta}\leq e$, $g_{i\delta}x\to y_i$, $1\leq i\leq n$. Put

$$g_\delta=\sum_{i=1}^n a_i g_{i\delta}.$$

We have $|g_\delta|\leq e$ and $g_\delta x\to\sum_{i=1}^n a_i y_i$. It follows that there is δ_0 such that $y-g_{\delta_0}x\in V$.

iii) $\Rightarrow$ iv) We may assume A is a dense vector sublattice of $C_K(X)$ such that e=1.

Let $x\in E$ and let F be the closure of $\{fx\mid f\in A, |f|\leq e\}$. We show first that $|x|\in F$. Let V be a neighborhood of 0. There is a solid neighborhood W of 0 such that $W+W+W\subset V$. By hypothesis there is a $f\in A$ such that $|f|\leq e$ and $f|x|-x\in W$. We can find $\varepsilon>0$ such that $\varepsilon|x|\in W$ and $g\in A$ such that $|g|\leq e$, $\|gf-|f|\|\leq\varepsilon$ (There is $h\in C_K(X)$ such that $|h|\leq 1$ and $h(t)=\frac{|f(t)|}{f(t)}$ for $|f(t)|\geq\frac{\varepsilon}{2}$. As A is dense in $C_K(X)$ there is a $g\in A$ such that $|g|\leq e$ and $\|g-h\|\leq\varepsilon$; this g has the required properties). By prop.2.8

$$|g(f|x|)-|f||x||\leq\varepsilon|x|.$$

It follows that

$$gx - g(f|x|) \in W,$$
$$g(f|x|) - |f||x| \in W,$$
$$|f||x| - |x| \in W$$

so

$$gx - |x| \in V.$$

Let $y \in Z(x)$ and V a neighborhood of O. There is a solid neighborhood W of O such that $W+W+W \subset V$. By hypothesis there is $f \in A$ such that $|f| \leq e$ and $y - f|x| \in W$. By the previous argument there is $g \in A$ such that $|g| \leq e$ and $|x| - gx \in W$; consequently $f|x| - f(gx) \in W$. We can find $\varepsilon > 0$ such that $\varepsilon |x| \in W$; as A is dense in $C_K(X)$ there is $h \in A$ such that $|h| \leq \varepsilon$ and $\|fg - h\| \leq \varepsilon$. By prop. 2.8, $|f(gx) - hx| \leq \varepsilon x$. It follows $y - hx \in V$.

iv) $\Longrightarrow$ i) Obvious

Definition 4.1. Let A be a vector lattice with unit. A principal s.l.o. module over A is a s.l.o. module E over A equipped with a locally solid Hausdorff topology and satisfying i)-iv) of prop.4.1.

Theorem 4.1. Let A be a vector lattice with unit e and let E be a s.l.o. module over A equipped with a locally solid Hausdorff topology. Then E is principal iff for every $x_1, x_2 \in E_+$ such that $x_1 \wedge x_2 = 0$ and for every neighborhood V of O there are $f_1, f_2 \in A_+$ such that $f_1 \wedge f_2 = 0$ and

$$x_i - f_i x_i \in V, \quad i = 1, 2.$$

PROOF. Suppose E is principal. Let $x_1, x_2 \in E_+$ be such that $x_1 \wedge x_2 = 0$. Put $x = x_1 + x_2$. By hypothesis there are nets $(g_{i\delta})_\delta$ such that $g_{i\delta} \in A+$ and $g_{i\delta} x \to x_i$, $i=1,2$. Define

$$f_{i\delta} = g_{i\delta} - g_{1\delta} \wedge g_{2\delta}, \quad i=1,2.$$

We have $f_{1\delta} \wedge f_{2\delta} = 0$ and $f_{i\delta} x \to x_i$, $i=1,2$. From

$$|x_i - f_{i\delta} x_i| \leq |x_i - f_{i\delta} x|$$

we have that $f_{i\delta} x_i \to x_i$, $i=1,2$.

Conversely, suppose E satisfies the condition in the theorem. Replacing f_i by $f_i \wedge e$ we may assume that f_1 and f_2 also satisfy $f_i \leq e$. We shall prove that E has the property ii) of prop.4.1.

Let $x, y \in E_+$, $y \leq x$ and let V be a neighborhood of O. There is a solid neighborhood W of O such that $W+W+W \subset V$. We can find $n \in \mathbb{N}$ such that $\frac{2}{n} x \in W$. Define for $o \leq k \leq n-1$

$$u_k = \left(2y - \frac{2k+1}{n} x\right)_+ \wedge \frac{1}{n} x,$$

$$v_k = \left(2y - \frac{2k+1}{n} x\right)_- \wedge \frac{1}{n} x,$$

$$z_k = \left(2y - \frac{2k}{n} x\right)_+ \wedge \frac{1}{n} x.$$

We have

$$y - \frac{1}{n} x \leq \sum_{k=0}^{n-1} u_k \leq y \tag{6}$$

$$u_k \wedge v_k = 0, \quad 0 \leq k \leq n-1 \tag{7}$$

$$z_\ell \wedge \left(\frac{1}{n}x - u_k\right) = 0, \quad 0 \leq k < \ell \leq n-1 \tag{8}$$

$$v_k \vee z_k \geq \frac{1}{2n} x, \quad 0 \leq k \leq n-1 \tag{9}$$

It follows from (7) that there are nets $(f_{k\delta})_\delta$, $(g_{k\delta})_\delta$, $0 \leq k \leq n-1$, such that

$$f_{k\delta},\ g_{k\delta} \in A$$

$$0 \leq f_{k\delta} \leq e, \quad 0 \leq g_{k\delta} \leq e$$

$$f_{k\delta} \wedge g_{k\delta} = 0$$

$$f_{k\delta} u_k - u_k \to 0, \quad g_{k\delta} v_k - v_k \to 0.$$

Put

$$w_{k\delta} = u_k - \frac{1}{n} f_{k\delta} x.$$

(8) implies

$$z_\ell \wedge |w_{k\delta}| \to 0, \quad 0 \leq k < \ell \leq n-1 \tag{10}$$

We also have

$$(g_{k\delta} v_k) \wedge f_{k\delta} \left|u_k - \frac{1}{n}x\right| = 0$$

so

$$v_k \wedge |w_{k\delta}| \to 0, \quad 0 \leq k \leq n-1. \tag{11}$$

Let $0 \leq k < \ell \leq n-1$. From (9)

$$|w_{k\delta}| \wedge |w_{\ell\delta}| \wedge \frac{1}{2n} x \le |w_{k\delta}| \wedge |w_{\ell\delta}| \wedge (z_\ell \vee v_\ell) \le$$
$$\le (|w_{k\delta}| \wedge z_\ell) \vee (|w_{\ell\delta}| \wedge v_\ell)$$

so by (10) and (11)

$$|w_{k\delta}| \wedge |w_{\ell\delta}| = |w_{k\delta}| \wedge |w_{\ell\delta}| \wedge \frac{2}{n} x \le$$
$$\le 4\left(|w_{k\delta}| \wedge |w_{\ell\delta}| \wedge \frac{1}{2n} x\right) \longrightarrow 0, \qquad (12)$$
$$0 \le k < \ell \le n-1$$

(12) implies

$$\sum_{k=0}^{n-1} |w_{k\delta}| - \sup_{0 \le k \le n-1} |w_{k\delta}| \to 0$$

But

$$\sup_{0 \le k \le n-1} |w_{k\delta}| \le \frac{2}{n} x$$

so there is a δ_0 such that

$$\sum_{k=0}^{n-1} |w_{k\delta_0}| \in W + W .$$

Define

$$f = \frac{1}{n} \sum_{k=0}^{n-1} f_{k\delta_0} \, ;$$

We have

$$|y - fx| \le \left|y - \sum_{k=0}^{n-1} u_k\right| + \sum_{k=0}^{n-1} |w_{k\delta_0}|$$

so $y - fx \in V$ by (6).

Proposition 4.2. Let A be a vector lattice with unit, E a principal s.l.o. module over A, F a closed order ideal of E. Then $\frac{E}{F}$ is a principal s.l.o. module over A.

PROOF. Apply prop.4.1 ii)

Proposition 4.3. Let A be a vector lattice with unit e, E a Hausdorff locally convex locally solid s.l.o. module over A. Then E is principal iff for every $\varphi \in E^*$ and $x \in E_+$ we have

$$|\varphi|(x) = \sup_{\substack{f \in A \\ |f| \leq e}} |\varphi(fx)|.$$

PROOF. The necessity follows from prop.4.1 iii). For the sufficeny, let $x \in E_+$ and let $C = \{fx \mid f \in A, |f| \leq e\}$. Suppose $\overline{C} \neq Z(x)$. As $\overline{C}$ is convex, there is $\varphi \in E^*$, $y \in Z(x)$ and $a \in \mathbb{R}$ such that

$$\operatorname{Re} \varphi(fx) \leq a < \operatorname{Re} \varphi(y), \; f \in A, \; |f| \leq e.$$

It follows that

$$a < |\varphi(y)| \leq |\varphi|(|y|) \leq |\varphi|(x) \leq a$$

which is a contradiction.

5. The spaces E_t

Let X be a compact space and let E be a s.l.o. module over $C_K(X)$ equipped with a Hausdorff locally solid topology. We define for $t \in X$

$$\dot{I}_t = \{ f \in C_K(X) \mid f(t) = 0 \}$$

$$J_t = \{ f \in C_K(X) \mid f(s) = 0 \text{ for all } s \text{ in a neighborhood of } t \}$$

$$M_t = \left\{ \sum_{i=1}^{n} f_i x_i \;\middle|\; n \in \mathbb{N},\ f_i \in \dot{I}_t,\ x_i \in E \right\}$$

$$N_t = \left\{ \sum_{i=1}^{n} f_i x_i \;\middle|\; n \in \mathbb{N},\ f_i \in J_t,\ x_i \in E \right\}.$$

Obviously, $\overline{M_t} = \overline{N_t}$. Also, if $x \in N_t$ there is $f \in C_K(X)$ such that $0 \leq f \leq 1$, $1-f \in J_t$ and $fx = 0$.

Proposition 5.1. $\overline{N_t}$ is an order ideal of E.

PROOF. It is sufficient to show that N_t is an order ideal. Let $y \in N_t$ and $x \in Z(y)$. There is $f \in C_K(X)$ such that $1-f \in J_t$ and $fy = 0$. It follows

$$|fx| = |f||x| \leq |f||y| = |fy| = 0$$

so $x = (1-f)x \in N_t$.

For $t \in X$ let $E_t = \dfrac{E}{N_t}$ and let $p_t : E \rightarrow E_t$ be the quotient map.

If q is a solid continuous seminorm on E, we denote by q_t the quotient seminorm on E_t. For every $x \in E$ define the map $s_q(x) : X \rightarrow \mathbb{R}$ by

$$s_q(x)(t) = q_t(p_t(x)).$$

We also introduce the set

$$\mathcal{F}_q(x) = \left\{ \sup_{1 \leq i \leq n} q(f_i x) f_i \;\middle|\; n \in \mathbb{N},\ f_i \in (C_K(X))_+,\ \sup_{1 \leq i \leq n} f_i = 1 \right\}.$$

Proposition 5.2.

i) $s_q(fx) = |f|\, s_q(x)$ for $f \in C_K(X)$ and $x \in E$.

ii) $\mathcal{F}_q(x)$ is directed downwards and

$$s_q(x) = \inf \mathcal{F}_q(x)$$

In particular $s_q(x)$ is superior semicontinuous.

PROOF. i) Let $t \in X$. We have $(f - f(t)1)x \in M_t$

so

$$p_t(fx) = p_t\,(f(t)x) = f(t)p_t(x).$$

It follows

$$s_q(fx)(t) = q_t\,(p_t(fx)) = |f(t)|\, q_t(p_t(x)) = |f(t)|\, s_q(x)(t).$$

ii) Let $f_1, \dots, f_n, g_1, \dots, g_m \in (C_K(X))_+$ be such that

$$\sup_{1 \le i \le n} f_i = \sup_{1 \le j \le m} g_j = 1.$$

Put

$$h = \sup_{\substack{1 \le i \le n \\ 1 \le j \le m}} q((f_i \wedge g_j)x)\, f_i \wedge g_j .$$

We have

$$q((f_i \wedge g_j)x) = q((f_i \wedge g_j)|x|) \le \inf(q(f_i x), q(g_j x))$$

so

$$h \le \left(\sup_{1 \le i \le n} q(f_i x) f_i\right) \wedge \left(\sup_{1 \le j \le m} q(g_j x) g_j\right).$$

On the other hand

$$\sup_{\substack{1\le i\le n\\ 1\le j\le m}} f_i \wedge g_j = 1$$

so $h \in \mathcal{F}_q(x)$.

Let $f_1, \dots, f_n \in (C_K(X))_+$ be such that $\sup_{1\le i\le n} f_i = 1$ and let $t\in X$. There is j such that $f_j(t) = 1$. It follows that $x - f_j x \in M_t$ so

$$s_q(x)(t) = q_t(p_t(x)) = q_t(p_t(f_j x)) \le q(f_j x) = \\ = q(f_j x) f_j(t) \le \sup_{1\le i\le n} q(f_i x) f_i(t).$$

Thus

$$s(x) \le \inf \mathcal{F}_q(x)$$

Now let $t\in X$ and $\varepsilon > 0$. There is $z\in N_t$ such that

$$q(x+z) \le q_t(p_t(x)) + \varepsilon$$

There is $f \in C_K(X)$ such that $0 \le f \le 1$, $1-f\in J_t$ and $fz = 0$. We can find $g \in (C_K(X))_+$ such that $g \in \dot{I}_t$ and $f \vee g = 1$. We have

$$q(fx) = q(f(x+z)) = q(f|x+z|) \le q(x+z) \le q_t(p_t(x)) + \varepsilon$$

so

$$(\inf \mathcal{F}_q(x))(t) \le q(fx)f(t) \vee q(gx)g(t) = q(fx) \le s_q(x)(t) + \varepsilon.$$

As ε was arbitrary if follows that

$$\inf \mathcal{F}_q(x) \leq s(x).$$

Proposition 5.3. For every $x \in E$ we have

$$\sup_{t \in X} s_q(x)(t) \leq q(x).$$

If q is an AM-seminorm we have equality.

PROOF. We can assume $x \in E_+$. We have

$$s_q(x)(t) = q_t(p_t(x)) \leq q(x).$$

Suppose q is an AM-seminorm but $\sup\limits_{t \in X} s_q(x)(t) < q(x)$.
By prop.5.2. ii) there is $\varepsilon > 0$ and $f_1, \dots, f_n \in (C_K(X))_+$
such that $\sup\limits_{1 \leq i \leq n} f_i = 1$ and

$$\sup_{1 \leq i \leq n} q(f_i x) f_i \leq q(x) - e.$$

We can assume that for every i there is $t_i \in X$ such that $f_i(t_i) = 1$. It follows that

$$\sup_{1 \leq i \leq n} q(f_i x) \leq q(x) - e.$$

As $x = \sup\limits_{1 \leq i \leq n} f_i x$ we have

$$q(x) = \sup_{1 \leq i \leq n} q(f_i x) \leq q(x) - \varepsilon$$

which is a contradiction.

Proposition 5.4. Suppose E is principal. Then for every $t \in X$ we have $\dim E_t \leq 1$.

PROOF. As E_t is Archimedian, it is sufficient to show that $(E_t)_R$ is linearly ordered. To this end, we prove that if $z_1, z_2 \in (E_t)_+$ are such that $z_1 \wedge z_2 = 0$ then $z_1 = 0$ or $z_2 = 0$.

So let $z_1, z_2 \in (E_t)_+$, $z_1 \wedge z_2 = 0$. Suppose $z_1, z_2 \neq 0$. There are $x_1, x_2 \in E_+$ such that $x_1 \wedge x_2 = 0$ and $z_i = p_t(x_i)$, i=1,2. Also there is a neighborhood V of 0 in E_t such that $z_1, z_2 \notin V$.

By thm.4.1. there are $f_1, f_2 \in (C_K(X))_+$ such that $f_1 \wedge f_2 = 0$ and $p_t(x_i - f_i x_i) \in V$, $i = 1,2$. We have $f_1(t)$ or $f_2(t) = 0$; suppose for instance $f_1(t) = 0$. It follows that $p_t(f_1 x_1)$ so

$$z_1 = p_t(x_1) = p_t(x_1 - f_1 x_1) \in V$$

which is a contradiction.

The converse of the statement in prop.5.4 is false: take $A = C_{\mathbb{R}}([0,1])$, $E = L_1([0,1]) \times L_1([0,1])$ and define

$$f(x,y) = (fx, fy), \quad f \in A, \quad (x,y) \in E$$

where by fx we mean pointwise multiplication.

6. AM-modules

Let E be a normed vector lattice. We shall denote by c the injection of E into E^{**}. c is a Riesz homomorphism.

By E^{ω} we shall denote the space of order bounded norm and order continuous linear functionals on E; E^{ω} is an order ideal of E^*.

Let X be a compact space. We denote by $R_K(X)$ the vector normed lattice of K-valued bounded regular Borel measures on X. Riesz's theorem states there is a norm and lattice isomorphism $m: C_K(X)^* \to R_K(X)$ given by

$$\int_X f\,dm(\varphi) = \varphi(f), \quad f \in C_K(X), \ \varphi \in C_K(X)^*.$$

By prop.2.10, $B_K(X)^*$ is an s.l.o. module over $B_K(X)$. As $R_K(X)$ can be identified with an order ideal of $B_K(X)^*$ we obtain a structure of s.l.o. module over $B_K(X)$ on $R_K(X)$, which is transported by m on $C_K(X)^*$; in this sense we shall write $f\varphi$ for $f \in B_K(X)$ and $\varphi \in C_K(X)^*$. This structure can also be obtained by applying thm.3.2 (in fact this is the idea of deriving Riesz's theorem from thm.3.2 we have spoken of).

In the rest of the sequel we will consider a compact space X and an AM-space (not necessarily with unit) E which is s.l.o. module over $C_K(X)$. E^* is an AL-space and E^{**} is a Banach lattice with unit e. E^* is s.l.o. module over $C_K(X)$; by thm.3.2 E^* becomes a s.l.o. module over $B_K(X)$. E^{**} is a s.l.o. module over $C_K(X)$ and over itself, if we consider the unique product on E^{**} turning it into a vector lattice with unit e (the unit e of E^{**} is defined by $e(\varphi) = \|\varphi\|, \varphi \in E^*_+$). By prop.2.6, the two structures are compatible. It follows that E^{***} is s.l.o. module over $C_K(X)$ and E^{**} ; by prop.2.11 the two structures are compatible.

Let $c: E \longrightarrow E^{**}$, $d: E^{*} \longrightarrow E^{***}$ be the canonical maps; c and d are $C_K(X)$ - linear.

(,) will denote the duality between $C_K(X)^*$ and $C_K(X)$, E^{**} and E^* , E^{***} and E^{**}.

We define a map $T: E^* \longrightarrow C_K(X)^*$ by

$$(T\varphi, f) = (e, f\varphi) = (fe, \varphi), \ \varphi \in E^*, f \in C_K(X).$$

T is positive and $C_K(X)$ - linear.

Proposition 6.1.

i) T is order continuous , $B_K(X)$-linear and its range is an order ideal of $C_K(X)^*$.

ii) $\|T\varphi\| = \|\varphi\|$ for $\varphi \in E^*_+$.

iii) $\|T\varphi\| \leq \|\varphi\|$ for $\varphi \in E^*$.

PROOF.

i) Let $\mathcal{F} \subset E^*_+$ a downwards directed set with $\inf \mathcal{F} = 0$. As $e \in (E^*)^\omega$ we have for $f \in (C_K(X))_+$

$$\inf_{\psi \in \mathcal{F}} (T\psi, f) = \inf_{\psi \in \mathcal{F}} (e, f\psi) = (e, \inf_{\psi \in \mathcal{F}} f\psi) = (e, f \inf_{\psi \in \mathcal{F}} \psi) = 0.$$

So T is order continuous. By prop.3.2 it follows that T is $B_K(X)$ - linear; in particular $T(E^*)$ is a $B_K(X)$- submodule. Let $\mu \in T(E^*)$ and let $\lambda \in Z(\mu)$. By Radon-Nykodim's theorem there is $f \in B_K(X)$ such that $\lambda = f\mu$, so $\lambda \in T(E^*)$.

ii) $\|T\varphi\| = (T\varphi, 1) = (e, \varphi) = \|\varphi\|$.

iii) $\|T\varphi\| = \| |T\varphi| \| \leq \| T(|\varphi|)\| = \| |\varphi| \| = \|\varphi\|$.

Let $x \in E$. By s(x) we shall denote the function $s_q(x)$

constructed in § 5 taking for q the norm of E.

Proposition 6.2.

i) Let $x \in E_+$ and $f_1, \dots, f_n \in (C_K(X))_+$ such that $\sup_{1 \le i \le n} f_i = 1$. Then

$$c(x) \le \Big(\sup_{1 \le i \le n} \|f_i x\| f_i\Big) e .$$

ii) Let $x \in E_+$ and $\varphi \in E_+^*$. Then

$$\varphi(x) \le \int_X s(x)\, dm(T\varphi)$$

PROOF.

i) We have $\sup_{1 \le i \le n} f_i^2 = 1$ so

$$c(x) = \sup_{1 \le i \le n} f_i^2 c(x) .$$

But

$$f_i c(x) \le \|f_i c(x)\| e = \|c(f_i x)\| e = \|f_i x\| e$$

so $$f_i^2 c(x) \le \|f_i x\| f_i e .$$

Now i) follows.

ii) Let $f_1, \dots, f_n \in (C_K(X))_+$ be such that $\sup_{1 \le i \le n} f_i = 1$.

i) implies that

$$\varphi(x) = (c(x), \varphi) \leq \left(\left(\sup_{1\leq i\leq n} \|f_i x\| f_i \right) e, \varphi \right) =$$
$$= \left(T\varphi, \sup_{1\leq i\leq n} \|f_i x\| f_i \right).$$

Then we apply prop.5.2.

Let $t\in X$. If $\dim E_t \leq 1$ let $s_t : E_t \longrightarrow K$ be the unique K-linear injective map with the property $s_t(z) = \|z\|$ for $z \in (E_t)_+$. If $\dim E_t \leq 1$ for all $t\in X$ define $S : E \longrightarrow B_K(X)$ by

$$S(x)(t) = s_t(p_t(x)).$$

As $S(x) = s(x) \in B_K(X)$ for $x \in E_+$ we have that the range of S is indeed included in $B_K(X)$. Clearly S is a $C_K(X)$-linear Riesz homomorphism.

Theorem 6.1. The following assertions are equivalent:

i) T is isometric.

ii) T is one-to-one.

iii) T is a Riesz homomorphism,

iv) E is principal,

v) $\dim E_t \leq 1$ for all $t \in X$.

When i)-v) are fullfiled S is isometric and there is a positive linear map $R : C_K(X)^* \longrightarrow E^*$ such that

$$\|R(\lambda)\| \leq \|\lambda\|, \quad \lambda \in C_K(X)^*,$$
$$R(\lambda)(x) = \int_X S(x)\, dm(\lambda), \ \lambda \in C_K(X)^*, x \in E.$$
$$RT = 1_{E^*}.$$

PROOF.

i) $\Rightarrow$ ii). Obvious.

ii) $\Rightarrow$ iii) Let $\varphi_1, \varphi_2 \in E^*_+$ be such that $\varphi_1 \wedge \varphi_2 = 0$. Put $\lambda = (T\varphi_1) \wedge (T\varphi_2)$. As $\lambda \le T\varphi_i$ there are $f_i \in (B_K(X))_+$ such that $f_i \le 1$ and $\lambda = f_i T\varphi_i = T(f_i \varphi_i)$. If we denote $\psi_i = f_i \varphi_i$ we have $\psi_i \le \varphi_i$ and $\lambda = T\psi_1 = T\psi_2$. As T is one-to-one it follows $\psi_1 = \psi_2$ so $\psi_1 \le \varphi_1 \wedge \varphi_2 = 0$ and $\lambda = 0$.

iii) $\Rightarrow$ iv) We prove first that if $y \in E$, $\varphi_1, \varphi_2 \in E^*$ and $c(y)d(\varphi_1) = d(\varphi_2)$ then

$$(T\varphi_2, f) = \varphi_1(fy), \quad f \in C_K(X).$$

We have

$$c(y)d(f\varphi_1) = c(y)(fd(\varphi_1)) = f(c(y)d(\varphi_1)) = fd(\varphi_2) = d(f\varphi_2)$$
$$\varphi_1(fy) = (f\varphi_1)(y)$$
$$(T\varphi_2, f) = (fT\varphi_2, 1) = (Tf\varphi_2, 1)$$

so it is sufficient to make the proof when f=1. We have

$$(T\varphi_2, 1) = (e, \varphi_2) = (d(\varphi_2), e) = (c(y)d(\varphi_1), e) =$$
$$= (d(\varphi_1), c(y)e) = (d(\varphi_1), c(y)) = (c(y), \varphi_1) = \varphi_1(y).$$

Now let $\varphi \in E^*$ and $x \in E$. Because E^* is an AL-space the range of d is an order ideal of E^{***} so there is $\psi \in E^*$ such that $c(x)d(\varphi) = d(\psi)$. It follows

$$c(|x|)d(|\varphi|) = |c(x)||d(\varphi)| = |c(x)d(\varphi)| = |d(\varphi)| = d(|\varphi|).$$

By the preceding argument

$$(T\psi, f) = \varphi(fx), \quad f \in C_K(X),$$

$$(T|\psi|, 1) = |\varphi|(|x|).$$

Because T is a Riesz homomorphism $|T\psi| = T|\psi|$. So

$$|\varphi|(|x|) = (T|\psi|, 1) = (|T\psi|, 1) = \sup_{\substack{f \in C_K(X) \\ |f| \le 1}} |(T\psi, f)| =$$

$$= \sup_{\substack{f \in C_K(X) \\ |f| \le 1}} |\varphi(fx)|.$$

By prop.4.3, E is principal.

iv) $\Longrightarrow$ v) Apply prop.5.4.

v) $\Longrightarrow$ i) S is isometric: if $x \in E_+$ we have $\|S(x)\| = \|x\|$ by prop.5.3. It follows for $x \in E$

$$\|S(x)\| = \|\,|S(x)|\,\| = \|S(|x|)\| = \|\,|x|\,\| = \|x\|.$$

For $\lambda \in C_K(X)^*$ and $x \in E$ we have

$$\left| \int_X S(x)\, dm(\lambda) \right| \le \|S(x)\| \|\lambda\| = \|x\| \|\lambda\|$$

so $R(\lambda) \in E^*$ and $\|R(\lambda)\| \le \|\lambda\|$.

To prove $RT = 1_{E^*}$ it is sufficient to show that

$$(RT)(\varphi)=\varphi\,,\ \varphi\in E^*_+ .$$

Let $x\in E_+$. By prop.6.2. ii) we have

$$(RT)(\varphi)(x)\geq\varphi(x).$$

So $(RT)(\varphi)\geq\varphi$. On the other hand

$$\|(RT)\varphi\|\leq\|T\varphi\|\leq\|\varphi\|.$$

As E^* is an AL-space we have $(RT)(\varphi)=\varphi$.

Let $\varphi\in E^*$. We have

$$\|\varphi\|=\|R(T\varphi)\|\leq\|T\varphi\|\leq\|\varphi\|$$

so $\|T\varphi\|=\|\varphi\|$.

Proposition 6.3. Suppose i)-v) in thm.6.1 are fullfiled.

Then R is an order continuous $B_K(X)$ - linear Riesz homomorphism. If P=TR we have that $0\leq P\leq 1_{C_K(X)^*}$ and P is a projection onto the range of T.

PROOF. Clearly R is order continous and $C_K(X)$ -linear. By prop.3.2, it is $B_K(X)$ - linear . We have

$$P^2=TRTR=TR=P\,,$$

Let $\lambda\in(C_K(X)^*)_+$ and $f\in(C_K(X))_+$. We have

$$(T(R(\lambda)),f)=(e,fR(\lambda))=(e,R(f\lambda))=\|R(f\lambda)\|\leq$$
$$\leq\|f\lambda\|=(f\lambda,1)=(\lambda,f)$$

so $P \leq 1_{C_K(X)^*}$. It follows that P is a Riesz homomorphism.
Let $\lambda \in C_K(X)^*$; we have

$$T(R(|\lambda|)) = P(|\lambda|) = |P(\lambda)| = |T(R(\lambda))| = T|R(\lambda)|.$$

Because T is one-to-one it follows that $R(|\lambda|) = |R(\lambda)|$.

7 The s.l.o. module associated with a positive representation of a Boolean algebra

We shall need the following

LEMMA. Let E be a vector lattice and let $P_i \in L(E)$ be such that $0 \leq P_i \leq 1_E$ and $P_i^2 = P_i$, i=1,2. Then

$$P_1 P_2 x = (P_1 x) \wedge (P_2 x), \; x \in E_+.$$

PROOF. Put $z = (P_1 x) \wedge (P_2 x)$. We have $P_2 x \leq x$ so $P_1 P_2 x \leq P_1 x$. From the hypothesis $P_1 P_2 x \leq P_2 x$ so

$$P_1 P_2 x \leq z.$$

From $z \leq P_1 x$ it follows that

$$0 \leq (1_E - P_1) z \leq (1_E - P_1) P_1 x = 0$$

so $z = P_1 z$. As $z \leq P_2 x$ it follows that

$$z = P_1 z \leq P_1 P_2 x.$$

Let $\mathfrak{A}$ be a Boolean algebra, X its representation space. We shall denote by $S_K(\mathfrak{A})$ the space of continuous, finitely K-valued functions on X and by $\mathcal{M}_K(\mathfrak{A})$ the space of bounded additive K-valued function on $\mathfrak{A}$. There is an order and norm preserving isomorphism $V : C_K(X)^* \to \mathcal{M}_K(\mathfrak{U})$ given by

$$V(\varphi)(M) = \varphi(\chi_M), \quad \varphi \in C_K(X)^*, \; M \in \mathfrak{U}.$$

Let $\mathfrak{U}$ be a Boolean algebra and let E be a vector lattice. A positive representation of $\mathfrak{U}$ into L(E) is a map $h : \mathfrak{U} \to L(E)$ with the properties

i) $h(0) = 0$.

ii) $h(A) \geq 0$, $A \in \mathfrak{U}$.

iii) $h(A') = 1_E - h(A)$, $A \in \mathfrak{U}$,

A′ being the complement of A

iv) $h(A \wedge B) = h(A)h(B)$, $A, B \in \mathfrak{U}$.

It follows from the definition that h(A) is a projection such that $0 \leq h(A) \leq 1_E$ and that

$$h(A \vee B) = h(A) + h(B) - h(A)h(B).$$

Proposition 7.1. There is an unique structure of s.l.o. module over $S_K(\mathfrak{U})$ on E such that

$$\chi_A x = h(A)(x), \quad A \in \mathfrak{U}, \; x \in E.$$

PROOF. Let $f \in S_K(X)$ and $x \in E$; f can be uniquely written

$$f = \sum_{i=1}^{n} c_i \chi_{A_i}$$

with $c_i \in K$, $A_i \in \mathfrak{U}$, $A_i \wedge A_j = 0$ for $i \neq j$. We define

$$fx = \sum_{i=1}^{n} c_i h(A_i)(x).$$

It is easy to see that the map $(f,x) \longmapsto fx$ is bilinear. We prove that $|fx| = |f||x|$. We have $h(A_i)h(A_j) = 0$ for $i \neq j$ so

$$|h(A_i)(x)| \wedge |h(A_j)(x)| = h(A_i)(|x|) \wedge h(A_j)(|x|) = 0 .$$

It follows that

$$|fx| = \left|\sum_{i=1}^{n} c_i h(A_i)(x)\right| = \sum_{i=1}^{n} |c_i| h(A_i)(|x|) = |f||x| .$$

Proposition 7.2. Let $\mathfrak{A}$ be a Boolean algebra, X its representation space, E an uniformly complete vector lattice, $h : \mathfrak{A} \longrightarrow L(E)$ a positive representation. Then there is an unique structure of s.l.o. module over $C_K(X)$ on E which extends the structure of s.l.o. module over $S_K(\mathfrak{A})$.

PROOF. Follows from cor.2.1.

Proposition 7.3. Let $\mathfrak{A}$ be a Boolean algebra, X its representation space , E a vector lattice equipped with a locally solid Hausdorff topology, $h : \mathfrak{A} \longrightarrow L(E)$ a positive representation.

i) Suppose E uniformly complete. Then E is principal as a module over $S_K(\mathfrak{A})$ iff it is principal as a module over $C_K(X)$.

ii) E is principal as a module over $S_K(\mathfrak{A})$ iff for every neighborhood V of 0 and every $x_1, x_2 \in E_+$ such that $x_1 \wedge x_2 = 0$ there are $A_1, A_2 \in \mathfrak{A}$ such that $A_1 \wedge A_2 = 0$ and

$x_i - h(A_i)(x_i) \in V,\ i=1,2.$

PROOF.

i) Obvious.

ii) Suppose E is principal. Let $x_1, x_2 \in E_+$ be such that $x_1 \wedge x_2 = 0$ and let V be a neighborhood of O. There is a solid neighborhood W of O such that $W+W \subset V$. By thm.4.1. there are $f_1, f_2 \in (S_K(X))_+$ such that $f_1 \wedge f_2 = 0$ and $x_i - f_i x_i \in W,\ i=1,2$. There are $A_1, A_2 \in \mathfrak{A}$ such that $A_1 \wedge A_2 = 0$ and $f_i = \chi_{A_i} f_i$, i=1,2. It follows $\chi_{A_i} x_i - \chi_{A_i} f_i x_i \in W$ so

$$x_i - \chi_{A_i} x_i = x_i - f_i x_i + \chi_{A_i} f_i x_i - \chi_{A_i} x_i \in W + W \subset V.$$

Conversely, if E satisfies the condition of ii), it is principal by thm.4.1.

Let $\mathfrak{A}$ be a Boolean algebra, E an AM-space, $h: \mathfrak{A} \to L(E)$ a positive representation. Define a map $H: E^* \longrightarrow \mathcal{M}_K(\mathfrak{A})$ by

$$H(\varphi)(A) = \| h(A)^* \varphi \|,\ \varphi \in E^*_+,\ A \in \mathfrak{A}$$

where $h(A)^*$ is the transpose of h(A).

Theorem 7.1. The following assertions are equivalent:

i) H is isometric,

ii) H is one-to-one,

iii) H is a Riesz homomorphism,

iv) For every $x_1, x_2 \in E_+$ such that $x_1 \wedge x_2 = 0$ and every $\varepsilon > 0$ there are $A_1, A_2 \in \mathfrak{A}$ such that $A_1 \wedge A_2 = 0$ and $\| x_i - h(A_i)(x_i) \| \leq \varepsilon,\ i=1,2.$

PROOF. We can assume that E is Banach. Let X be the representation space of $\mathfrak{A}$. By prop.7.2, E is a s.l.o. module over $C_K(X)$. Let $T: E^* \longrightarrow C_K(X)^*$ be the map defined in § 6.

Observe that H=VT and apply thm.6.1 and prop.7.3.

R E F E R E N C E S

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