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Abstract

The problem we are considering is described by nonlinear Ito's equations in which the control variable is entering both drift and diffusion coefficients. In [1] assuming that the control range set is a convex one it is obtained the maximum principle in local form.

In the lack of the convexity assumption for drift coefficients one has to exploit so called relaxed controls. Proving that the convex cone of first variations in the relaxed control problem can be used as the first order approximation in the original problem and using a similar technique as in [1] one gets the maximum principle.

1. INTRODUCTION

We consider stochastic control differential equations

$$1)\ dx=f(t,x,u)dt+\sum_{i=1}^{k} g_i(t,x,u)dB_i(t),\ t\in[t_0,t_1],\quad x\in R^n,$$

with given initial condition $x(0)=x_0\in R^n$, where

$B(t)=(B_1(t),\dots,B_k(t))$ is a k-dimensional Brownian motion on the probability space $\{\Omega, F, P\}$.

As the functional to be minimized we consider

$$2)\ J(x,u)=E\{G(x(t_1))+\int_{t_0}^{t_1} L(t,x(t),u(t))dt\}$$

As admissible controls we allow any bounded measurable non-anticipative process with respect to σ- algebras F_t generated by $\{B(s),\ t_0\leq s\leq t\}$, $t_0\leq t\leq t_1$, taking values in the fixed set $U\subseteq R^m$; denote it by $\mathcal{U}$.

For any $u\in\mathcal{U}$ there exists an unique non-anticipative process $x^u(t)$ verifying (1) in integral form whose trajectories are continuous functions in $t\in[t_0,t_1]$ a.e. in $\omega\in\Omega$.

The dynamic programming approach in [2] and the convex analysis method in [3] are not suitable for our problem since the diffusion coefficients are depending on the control and we are not dealing with the convex case.

In the case the control range set U is a convex one then by using local variations in f and g_i we obtain in [1] the adjoint system and maximum principle via a general multiplier rule theorem.

It seems that the local variations are the most suitable ones for the control entering functions g_i. When the control range set U is not a convex one we are obliged somehow to consider that the control variable u splits into two parts $=(u^1,u^2)$, u^1 is entering in the functional and f only, and u^2 is entering in g_i only. In the case the problem contains a finite number of functional constraints $E\varphi_i(x(t_1))\leq 0$ $i=1,\ldots,m$, $E\varphi_i(x(t_1))=0$, $i=m+1,\ldots,m+m'$ then the maximum principle in [1] or in the present work (pointwise form) has the same formulation except the final value of the adjoint variable which will be $\psi(t_1)=\sum_{i=0}^{m+m'}\alpha_i \frac{\partial\varphi_i}{\partial x}(x_0(t_1))$, where $\alpha_i\geq 0$, $i=0,1,\ldots,m$, $\sum_{i=0}^{m+m'}|\alpha_i|\neq 0$, are same constants determined as in deterministic case applying a separation theorem in a finite dimensional space. The general procedure in [1] or in this work will not be affected by the prezence of these additional functional constraints

2. ASSUMPTIONS, DEFINITONS AND AUXILIARY RESULTS

From now on $u_0\in\mathcal{U}$ will be fixed.

The functions f, g_i, G and L in (1) and (2) are supposed to be continuous in $(t,x,u)\in[t_0,t_1]\times R^n\times R^m$, and they have first derivatives in x, $\frac{\partial h}{\partial x}(t,x,u)$, $h=f,g_i,G,L$, continuous in (x,u). In addition

a) $\| \frac{\partial h}{\partial x}(t,x,u) \| \leq M_\rho$, for $|u| \leq \rho$, $(t,x) \in [t_0,t_1] \times R^n$, $h=f,g_i$

$\| \frac{\partial g_i}{\partial u}(t,x,u) \| \leq M'_\rho (1+\|x\|)$ for $\|u\| \leq \rho$, and $\frac{\partial g_i}{\partial u}(t,x,u)$ is continuous in (x,u);

b) $\| h(t'',x,u) - h(t',x,u') \| \leq K_\rho \ (1+\|x\|^p)\, c(t',t'',u',u'')$, $\|u'\|, \|u''\| \leq \rho$

$h=f,L$ where $c: [t_0,t_1] \times [t_0,t_1] \times R^m \times R^m \to [0,\infty)$ is continuous and $C(t,t,u,u)=0$

c) $\| \frac{\partial h}{\partial x}(t,x,u) \|$, $\| h\ (t,x,u) \| \leq N_\rho\ (1+\|x\|^p)$ if $\|u\| \leq \rho$,

$h=G,L$.

Let $u_1(.),\ldots,u_\ell(.) \in \mathcal{U}$ be arbitrarily chosen. Let $\mathcal{P}_\ell$ be the set of all ℓ-dimensional bounded measurable functions $p:[t_0,t_1] \to R^\ell$ verifying $p_i(t) \geq 0$, $i=1,\ldots,\ell$. Denote $f^{r,p}(t,x) = f(t,x,u_0(t)) + r \sum_{i=1}^{\ell} p_i(t)(f(t,x,u_i(t)) - f(t,x,u_0(t)))$,

$$u^{r,q}(t) = u_0(t) + r \sum_{j=1}^{\ell} q_j(t)(u_j(t) - u_0(t))$$

for $r \in [0,1]$ $p,q \in \mathcal{P}_\ell$.

Of course $f^{r,p}(t,x)$, $u^{r,q}(t)$ are random functions.

Let $x_{r,p,q}$ be the Ito solution in

3) $dx = f^{r,p}(t,x)dt + \sum_{i=1}^{k} g_i(t,x,u^{r,q}(t))dB_i(t) \quad x(t_0)=x_0 \in R^n$

Using the smooth dependence of $x_{r,p,q}$ on $r \in [0,1]$ (see for example Lemma 3 in [1]) we get

4) $x_{r,p,q}(t) = x_0(t) + r\bar{x}_{p,q}(t) + o(t,r)$

where $\lim \frac{1}{r} \sup_{t \leq t_1} \{E|o(t,r)|^2\}^{1/2} = 0$, and $\bar{x}_{p,q}$ is the Ito solution in

5) $d\bar{x} = \left[F(t)\bar{x} + \sum_{i=1}^{\ell} p_i(t)(f_i(t) - f_0(t))\right]dt + \sum_{i=1}^{k}\Big(G_i(t)\bar{x}(t) +$

$+ g_{iu}(t) \sum_{j=1}^{\ell} q_j(t)(u_j(t) - u_0(t))\Big)\, dB_i(t)$

$\bar{x}(t_0) = 0$

$F(t)=\frac{\partial f}{\partial x}(t,x_0(t),u_0(t))$, $f_i(t)=f(t,x_0(t),u_i(t))$,

$f_0(t)=f(t,x_0(t),u_0(t))$, $G_i(t)=\frac{\partial g_i}{\partial x}(t,x_0(t),u_0(t))$

$g_{iu}(t) = \frac{\partial g_i}{\partial u}(t,x_0(t),u_0(t))$

In addition

6) $K_m \stackrel{\Delta}{=} E \sup_{t\le t_1} \|x_{r,p,q}(t)\|^m$, $m\geq 2$, uniformly in $r\in[0,1]$, $p,q\in \mathcal{O}\mathcal{P}_\ell$, a bounded set.

(see (4) in [1])

Since $f(t,x,u)$ is not linear in u and $\mathcal{U}$ is not convex it is obvious that the solution x_r doesn't correspond to an admissible control $u_r(t)\in\mathcal{U}$. Even if we suppose that $u^{r,q}(.)\in\mathcal{U}$, the solution x_r is still non-admissible since $f^{r,p}$ is not generated by an admissible control.

Our attention will be concentrated mainly on how to approximate f^r by random function corresponding to admissible controls such that the first approximation for the admissible solution to be the same with $\bar{x}(\cdot)$ in (5).

In deterministic problems the function $f^{r,p}$ is corresponding to so called "relaxed variations" of the fixed field $f_0(t,x)=f(t,x,u_0(t))$. The approximation of the $f^{r,p}(t,x)$ by admissible fields in deterministic problems is relied on the assumption that x belongs to a compact fix ~~set~~ in R^n which allow us to use the unity partition theorem. In stochastic case the compactness assumption is not a realistic one anymore and we have to use directly the properties of solutions in (3).

The defition of approximation for $f^{r,p}(t,x)$ is similar to that in deterministic case. We recall the definition of such approximations which were introduced in [4].

Let $r_0\in(0,1]$ be such that $r_0 \sum_{i=1}^{\ell} p_i(t) < 1$

$t\in[t_0,t_1]$. Fixe $r\in[0,r_0]$ arbitrarily and define I_s, $s=1,\dots,N$, a partition of $[t_0,t_1]$ in intervals whose measures are bounded by a number depending on r which will be specified later.

Each interval I_s is divided into $(\ell+1)$ subintervals $E_s^0,\dots,E_s^\ell$ such that meas $E_s^i=r\int_{I_s} p_i(t)dt$, $i=1,\dots,\ell$, meas $E_s^0=\int_{I_s}(1-r\sum_{i=1}^{\ell} p_i(t))dt$. The order in which we consider subintervals $E_s^0,\dots,E_s^\ell$ is not important. Denote π the partition of $[t_0,t_1]$ given by $E_s^0,\dots,E_s^\ell$, $s=1,\dots,N$, and define f^π as follows

7) $f^\pi(t,x)=f(t,x,u_i(t))$, $t\in E_s^i$, $i=0,1,\dots,\ell$, $s=1,\dots,N$, $x\in R^n$.

We call f^π the commutation function corresponding to π and $f_0(t,x),\dots,f_\ell(t,x)$, $f_i(t,x)=f(t,x,u_i(t))$.

In the same way one defines $L^\pi(t,x)$, $t\in[t_0,t_1]$, $x\in R^n$

7') $L^\pi(t,x)=L(t,x,u_i(t))$, $t\in E_s^i$, $i=0,1,\dots,\ell$, $s=1,\dots,N$, $x\in R^n$,

and $L^{r,p}(t,x)=L(t,x,u_0(t))+r\sum_{i=1}^{\ell} p_i(t)(L(t,x,u_i(t))-L(t,x,u_0(t)))$

where L is entering the functional to be minimized.

Let $\bar{\rho}$ and $\bar{M}$ be such that

8) $\|u_i(t,\omega)\|\le\bar{\rho}$, $(\forall)\ (t,\omega)\in[t_0,t_1]\times\Omega$, $i=0,1,\dots,\ell$,

$\|h(t,0,u)\|\le\bar{M}$ for $\|u\|\le\bar{\rho}$, $h=f,g_i$, $i=1,\dots,k$.

Denote by $\mathcal{O}\subseteq\mathcal{P}_\ell$ a bounded set.

<u>LEMMA 1</u>

Let (a), (b) and (c) be fulfiled. Let $u_0(.),u_1(.),\dots,u_\ell(.)\in\mathcal{U}$ be fixed. Let $r_0\in(0,1)$ and $\mathcal{O}\subseteq\mathcal{P}_\ell$ be such that $r_0\sum_{i=1}^{\ell} p_i(t)<1$ for all $p\in\mathcal{O}$. Then for each $\eta\in(0,1)$ and $r\in[0,r_0]$ there exists a partition π depending on η and r such that

$$\sup_{t',t''\in[t_0,t_1]}\left\{E\left|\int_{t'}^{t''}[h^{r,p}(t,y(t))-h^\pi(t,y(t))]\,dt\right|^2\right\}^{1/2}\le C_1\eta \quad \text{for } h=f,L,$$

uniformly with respect to y(.) solution in (3) corresponding to $p,q\in\mathcal{O}$ and $r\in[0,1]$, where the constant C_1 is not depending on p, η, r.

PROOF

Define $u_i^\delta(t) = \frac{1}{\delta}\int_{t-\delta}^{t} \bar{u}_i(s)ds$, where $\bar{u}_i(t) = \begin{cases} u_i(t), & t \in [t_0,t_1] \\ 0, & t \notin [t_0,t_1] \end{cases}$

By definition

9) $E\|u_i^\delta(t)\|^2 \le \sup_{t\in[t_0,t_1]} E\|u_i(t)\|^2$, $\lim_{\delta\to 0} u_i^\delta(t) = u_i(t)$ a.e. $(dt\times dP)$,

$i=0,1,\dots,\ell$.

Denote $f_i^\delta(t,x) = f(t,x,u_i^\delta(t))$, $f_i(t,x) = f(t,x,u_i(t))$.

Using (9) and (b) we get that there exists $\tilde{\delta} > 0$ sufficiently small such that

10) $\left\{\int_{t_0}^{t_1} E|f_i^{\tilde{\delta}}(t,y(t)) - f_i(t,x(t))|^2 dt\right\}^{1/2} \le \eta$, $\quad i=0,1,\dots,\ell$,

for all y solutions in (3) corresponding to $r\in[0,r_0]$, $p_i(t), q_j(t) \ge 0$

$r_0 \sum_{i=1}^{\ell} p_i(t) < 1$, $r_0 \sum_{j=1}^{\ell} q_j(t) < 1$.

Denote $\tilde{f}_i(t,x) = f_i^{\tilde{\delta}}(t,x)$, $i=0,1,\dots,\ell$. Using again (b) and continuity of $\varphi(t,s) = EC^2(t,s,u^{\tilde{\delta}}(t), u^{\tilde{\delta}}(s))$ on $[t_0,t_1]\times[t_0,t_1]$ since $\varphi(t,t)=0$, there exists $\Delta(\eta) > 0$ such that

11) $\{E|\tilde{f}_i(t'',y(t)) - \tilde{f}_i(t',y(t))|^2\}^{1/2} \le \eta$, $i=0,1,\dots,\ell$, if $|t'-t''| \le \Delta(\eta)$

uniformly with respect to $t\in[t_0,t_1]$ and y solutions in (3) corresponding to the parameters $r\in[0,1]$, $p(.), q(.)\in \mathcal{O}$. Similar properties will follow for the function $L(t,x,u)$ and we denote them by (10′) and (11′). Let the intervals I_s in partition be such that meas $I_s \le \Delta(\eta)$, meas $I_s \le \eta^2$.

Let $\tilde{f}^\pi$ be the commutation function corresponding to $\tilde{f}_0(t,x),\dots,\tilde{f}_\ell(t,x)$ and the partition π. Denote $\tilde{f}^{r,p}(t,x) = \tilde{f}_0(t,x) + r\sum_{i=1}^{\ell} p_i(t)(\tilde{f}_i(t,x) - \tilde{f}_0(t,x))$, where $p_i(t)$ are the same with those defining $f^{r,p}$. Computation gives

12) $\sup_{t',t''}\left\{E\left|\int_{t'}^{t''}[f^{r,p}(t,y(t)) - f^\pi(t,y(t))]dt\right|^2\right\}^{1/2} \le$

$\sqrt{t_1-t_0}\left(\int_{t_0}^{t_1} E|f^{r,p}(t,y(t)) - \tilde{f}^{r,p}(t,y(t))|^2 dt\right)^{1/2} +$

$$+\sqrt{t_1-t_0}\left(\int_{t_0}^{t_1} E\left|\tilde{f}^{\pi}(t,y(t))-f^{\pi}(t,y(t))\right|^2 dt\right)^{1/2}+$$

$$\sup_{t',t''}\left\{E\left|\int_{t'}^{t''}\left[\tilde{f}^{r,p}(t,y(t))-\tilde{f}^{\pi}(t,y(t))\right]dt\right|^2\right\}^{1/2}$$

Denote by I, II and III the terms in the right hand side in (12). Using (10) it follows

13) $I \leq \sqrt{t_1-t_0}\left\{\sum_{i=0}^{\ell}\left(\int_{t_0}^{t_1}\gamma_i(t)E\left|f^i(t,y(t))-\tilde{f}^i(t,y(t))\right|^2 dt\right\}^{1/2} \leq$

$(1+\ell)\sqrt{t_1-t_0}\,\eta$ where $\gamma_i \geq 0$ $\sum_{i=1}^{\ell}\gamma_i(t)=1,$

14) $II \leq \sqrt{t_1-t_0}\left\{\sum_{j=0}^{\ell}\int_{A_j}E\left|\tilde{f}_j(t,y(t))-f_j(t,y(t))\right|^2 dt\right\}^{1/2}(1+\ell)\sqrt{t_1-t_0}\,\eta$

where $A_j=\bigcup_{s=1}^{N}E_s^j$, and E_s^j are defining the partition π.

It remained to estimate III. Let $t_s \in I_s$ be arbitrarily chosen. On an arbitrary I_s we have

15) $\left\{E\left|\int_{I_s}\left[\tilde{f}^{r,p}(t,y(t))-\tilde{f}^{\pi}(t,y(t))\right]dt\right|^2\right\}^{1/2} \leq$

$\left\{E\left|\int_{I_s}\left[\tilde{f}^{r,p}(t,y(t))-\tilde{f}^{r,p}(t,y(t_s))\right]dt\right|^2\right\}^{1/2}+$

$\left\{E\left|\int_{I_s}\left[\tilde{f}^{\pi}(t,y(t)-\tilde{f}^{\pi}(t,y(t_s))\right]dt\right|^2\right\}^{1/2}+$

$\left\{E\left|\int_{I_s}\left[\tilde{f}^{r,p}(t,y(t_s))-\tilde{f}^{\pi}(t,y(t_s))\right. dt\right|^2\right\}^{1/2}$

The first term in the right hand side in (15) is majorized by

16) $I' \leq M_{\bar{f}}\sqrt{\text{meas } I_s}\left\{\int_{I_s}E\left|y(t)-y(t_s)\right|^2 dt\right\}^{1/2}$

Using the hypothesis (a) is follows $|g_i(t,x,\hat{u}(t))|, |f^{r,p}(t,x)| \leq M(1+|x|)$, where the constant M is depending on $M_{\bar{f}}$ and $\bar{M}$. Since $y(t)$ is an arbitrary solution in (3), we obtain

17) $E|y(t)-y(s)|^2 \leq C|t-s|$

where C is depending on M.

Hence,

18) $I' \leq CM_{\bar{\rho}} \sqrt{\text{meas} I_s} \left(\int_{I_s} |t-t_s| dt \right)^{1/2} \leq C \; M_{\bar{\rho}} (\text{meas} I_s)^{3/2}$

For the second term in (15) we get the same majorant

19) $\mathit{II}' \leq CM_{\bar{\rho}} \; (\text{meas} I_s)^{3/2}$

For the third therm in (15) we get

20) $\mathit{III}' \leq \left\{ E \left| \int_{I_s} \left[\tilde{f}^{r,p}(t, y(t_s)) - \left(1 - r \sum_{i=1}^{\ell} p_i(t)\right) f_0(t_s, y(t_s)) - r \sum_{i=1}^{\ell} p_i(t) f_i(t_s, y(t_s)) \right] dt \right|^2 \right\}^{1/2} + $

$+ \sum_{j=0}^{\ell} \left\{ E \int_{E_s^j} \left[\tilde{f}_j(t_s, y(t_s)) - \tilde{f}_j(t, y(t_s)) \right] dt \right|^2 \Big\}^{1/2}$

Using (11) it follows that any term in (20) is majorized by $\eta \; \text{meas} I_s$.

Hence

21) $\mathit{III}' \leq (1+2)\eta \; \text{meas} I_s$

and finally using (18), (19) and (21) in (15) we get

22) $\left\{E\left|\int_{I_s}\left[\tilde{f}^{r,p}(t,y(t))-\tilde{f}^{\pi}(t,y(t))\right]dt\right|^2\right\}^{1/2}\leq(2CM_{\bar{\rho}}\sqrt{\mathrm{meas}I_s}+(\ell+2)\eta)\mathrm{meas}I_s$

$$\leq(2CM_{\bar{\rho}}+(\ell+2))\eta\,\mathrm{meas}I_s$$

Regarding the function $L(t,x,u)$ we have similar inequalities to (12)-(15) which we shall denote by (12')-(15'). The inequation (16) has to be replaced by

16') $I'\leq N_{\bar{\rho}}\sqrt{\mathrm{meas}I_s}\left(\left(E\sup_{t\leq t_1}(1+|y(t)|^{p})^4\right)^{1/4}\left\{\int_{I_s}(E|y(t)-y(t_s)|^4)^{1/2}dt\right\}^{1/2}\right.$

Instead of (17) we use

17') $$E|y(t)-y(s)|^4\leq C_1|t-s|^2$$

for an arbitrary solution in (3), where the constant C_1 doesn't depend on the particular solution $y(\cdot)$ in (3) and t,s.

Since $C_2=E\sup_{t\leq t_1}(1+|y(t)|^{p})^4$ (see (4) in [1]) from (16') and (17') we get

18') $$I'\leq C'N_{\bar{\rho}}\sqrt{\mathrm{meas}I_s}\left(\int_{I_s}|t-t_s|dt\right)^{1/2}\leq C'N_{\bar{\rho}}(\mathrm{meas}I_s)^{3/2}$$

where $C'=C_1\cdot C_2$.

Similarly we get the same majorant for the second term in (15')

19') $$II\leq CN_{\bar{\rho}}(\mathrm{meas}I_s)^{3/2}$$

Writting (20') for the third term in (15'), (where f is replaced by L), and using (11') it follows that any term in (20') is majorized by $\eta\,\mathrm{meas}I_s$. Hence III' in (15') fulfils

21') $$III'\leq(\ell+2)\eta\,\mathrm{meas}I_s$$

and using (18'), (19') and (21') in (15') we get

22') $\left\{E\left|\int_{I_s}[\tilde{L}^{r,p}(t,y(t))-\tilde{L}^{\pi}(t,y(t))]dt\right|^2\right\}^{1/2} \leq (2C'N_{\bar{\rho}}\sqrt{\text{meas}I_s}+(\ell+2)\eta)\text{meas}I_s \leq$

$$\leq (2C'N_{\bar{\rho}}+(\ell+2))\eta\,\text{meas}I_s\ .$$

For the estimation of III in (12) or (12') follows noticing that any interval $[t',t'']$ can be covered by a finite number of intervals I_s and two other possibly subsets of some I_s.

Hence, the integral in III is estimated by a finite sum of integrals of the type in (15) ((15')) and another two integrals of the form

23) $\left\{E\left|\int_{\tilde{t}}^{t}[\tilde{f}^{r,p}(t,y(t))-\tilde{f}^{\pi}(t,y(t))]dt\right|^2\right\}^{1/2} \leq$

$$2\sqrt{\text{meas}\tilde{I}_s}\,M(E\sup_{t\leq t_1}(1+|y(t)|^2))^{1/2} \leq 4\eta M(\sqrt{K_2}+1)$$

23') $\left\{E\left|\int_{\tilde{t}}^{t}\tilde{L}^{r,p}(t,y(t))-\tilde{L}^{\pi}(t,y(t))\,dt\right|^2\right\}^{1/2} \leq$

$$2\sqrt{\text{meas}\tilde{I}_s}\,N_{\bar{\rho}}(E\sup_{t\leq t_1}(1+|y(t)|^p)^2)^{1/2} \leq 4\eta N_{\bar{\rho}}(\sqrt{K_{2p}}+1)$$

where $[t,\tilde{t}]\subseteq \tilde{I}_s$.

In conclusion, III in 22 and (22') is estimated by $\tilde{C}\eta \triangleq$ $[(2CM_{\bar{\rho}}+(\ell+2))(t_1-t_o)+4M(\sqrt{K_2}+1)]\eta$ and correspondingly by $[(2C'N_{\bar{\rho}}+(\ell+2))(t_1-t_o)+4N_{\bar{\rho}}(\sqrt{K_{2p}}+1)]\eta=\tilde{C}'\eta$. Since I and II in (22) ((22')) are majorized by $(\ell+1)\sqrt{t_1-t_o}\,\eta$ (see (13) and (14)) defining $\tilde{C}_1=\tilde{C}+2(\ell+1)\sqrt{t_1-t_o}$, $\tilde{C}'_1=C'+2(\ell+1)\sqrt{t_1-t_o}$ and $C_1=\max(\tilde{C}_1,\tilde{C}'_1)$ the proof is complete.

We are going to establish the connection between $x_{r,p,q}$ solution in (3) and the solution x_{π} obtained from the equation

24) $dx=f^{\pi}(t,x)dt+\sum_{i=1}^{k}g_i(t,x,u^{r,q}(t))dB_i(t)$,

$$x(t_o)=x_o\in R^n$$

which is similar to (3) except $f^{r,p}(t,x)$ is replaced by $f^{\pi}(t,x)$ corresponding to a fixed partition π .

Lemma 2

Let the hypotheses (a) and (b) be fulfiled for h=f. Let $u_o(.),\dots,u_\ell(.)\in\mathcal{U}$ be fixed and define $x_{r,p,q}$ the Ito solution in (3) corresponding to $r\in[0,1]$, $p,q\in\mathcal{P}_\ell$. Let π and f^{π} be given by Lemma 1 corresponding to $r\in(0,r_o]$ and $\eta=r^2$ where $r_o\sum_{i=1}^{\ell}p_i(t)<1$.

Define x_π the Ito solution in (24) corresponding to that f^{π} given in Lemma 1, and $q\in\mathcal{P}_\ell$ defining $x_{r,p,q}$.

Then

$$\limsup_{\substack{r\to 0\\ t\leq t_1}} \frac{1}{r}\left\{E/x_{r,p,q}(t)-x_\pi(t)/^2\right\}^{1/2}=0$$

If in addition, the hypothesis (b) and (c) are fulfiled for h=L, then $\limsup_{\substack{r\to 0\\ t\leq t_1}} \frac{1}{r}\left\{E/\int_{t_o}^{t}\left[L^{r,p}(t,x_{r,p,q}(t))-L^{\pi}(T,x_{r,p,q}(t))\right]dt/^2\right\}^{1/2}=0$

uniformly with respect to $p,q\in\mathcal{O}$ (bounded subset of $\mathcal{P}_\ell$).

Proof

By hypothesis, the conditions in Lemma 1 are fulfiled either for h=f, or h=L. Then for any $r\in(0,r_o]$ and $\eta=r^2$ there exists a partition π of $[t_o,t_1]$ depending on r and $p\in\mathcal{O}\subseteq\mathcal{P}_\ell$ such that the second statement follows and

$$25)\quad P\overset{\Delta}{=}\sup_{t\leq t_1} E/\int_{t_o}^{t}\left[f^{r,p}(s,x_{r,p,q}(s))-f^{\pi}(s,x_{r,p,q}(s))\right]ds/^2\leq C_1 r^4,\quad r\in[0,r_o]$$

By definition of $x_{r,p,q}$ and x_π we have

$$26)\quad E/x_{r,p,q}(t)-x_\pi(t)/^2\leq 3P+3E/\int_{t_o}^{t}\left[f^{\pi}(s,x_{r,p,q}(s))-f(s,x_\pi(s))\right]ds/^2+$$

$$+3\sum_{i=1}^{k}\int_{t_o}^{t}E/g_i(s,x_{r,p,q}(s),u^{r,q}(s))-g_i(s,x_\pi/s),u^{r,q}(s))/^2ds$$

The second term in the right hand side in (26) is majorized by

27) $$II \leq 3(M_{\bar{\xi}})^2(t_1-t_o)\int_{t_o}^{t} E|x_{r,p,q}(s)-x_{\pi}(s)|^2 ds$$

and the last term in (2.6) is bounded by $3k(M_{\bar{\xi}})^2 \int_{t_o}^{t} E|x_{r,p,q}(s)-x_{\pi}(s)|^2 ds$

In conclusion from (2.6) we get

28) $$\ell(t) \stackrel{\Delta}{=} E|x_{r,p,q}(t)-x_{\pi}(t)|^2 \leq 3\,P+3(M_{\bar{\xi}})^2\left[(t_1-t_o)+k\right]\int_{t_o}^{t}\ell(s)ds$$

Using (25) and Gronwall's lemma we obtain the first statement. The proof is complete.

3. Optimality principle

Since the set U is not supposed a convex one we need the following assumption

d) The control u splits into $u=(u_1,u_2)$, $u_1 \in R^{m_1}$, $u_2 \in R^{m_2}$, $m_1+m_2=m$, such that u_1 is entering in L and f only, and u_2 is entering g_i only.

Let $(x_o(.), u_o(.))$ be the optimal pair in the problem defined by (1) and (2). One defines two subsets of our admissible set of controls:

U_1 is the set consisting of all $u(.) \in \mathcal{U}$, $u(.)=(u_1(.),u_{o2}(.))$,

U_2 is the set consisting of all bounded measurable functions $u(.)=(u_{o1}(.), u_2(.))$ such that $u_2(t,.)$ is F_t-measurable and for $r \in (0,1)$ sufficiently small $(u_{01}(.), (1-r)u_{o2}(.)+ru_2(.)) \in \mathcal{U}$.

Remark 1

In the case $U=U_1 x U_2$, $U_1 \subseteq R^{m_1}$, $U_2 \subseteq R^{m_2}$ and U_2 is a convex set then $\mathcal{U}_i$ consists of all bounded measurable functions $u_i: [t_o,t_1] x \Omega \to U_i$ and $u_i(t,.)$ is F_t-measurable, i=1,2. The initial admissible set of controls $\mathcal{U}$ will be $\mathcal{U}=\mathcal{U}_1 x \mathcal{U}_2$.

The optimality principle has an integral form with respect to the sets $\mathcal{U}_1$ and $\mathcal{U}_2$ but in the case that the conditions in remark 1 hold then from integral form is getting a pointwise one with respect to $u_1 \in U_1$ and $u_2 \in U_2$.

Define

$$H(t,x,u,\psi,M)=(\psi,f(t,x,u))+L(t,x,u)+\sum_{i=1}^{k}(M_i,g_i(t,x,u))$$

where ψ, $M_i \in R^n$.

Theorem

Let the hypotheses (a)-(d) be fulfiled. Let $u_o(.)=(u_{o1}(.),u_{o2}(.))$ be the optimal control. Then there exists $M_i^o(t,\omega)$ measurable, nonanticipative, $\int_{t_o}^{t_1} E|M_i^o(t,\omega)|^2 dt < \infty$, and $\psi_o \in R^n$ unique such that the Ito solution of the adjoint equation

i) $$d\psi=-\frac{\partial H}{\partial x}(t,x_o(t),u_o(t),\psi,M^o(t))dt+\sum_{i=1}^{k} M_i^o(t)dB_i(t) \quad , \quad \psi(t_o)=\psi_o$$

verifies $\psi(t_1)=\frac{\partial G}{\partial x}(x_o(t_1))$ and the optimality conditions

ii) $$E\int_{t_o}^{t_1} H(t,x_o(t),(u_1(t),u_{o2}(t)),\psi(t),M^o(t))dt \langle E\int_{t_o}^{t_1} H(t,x_o(t),u_o(t),\psi(t),M^o(t))dt$$

for all $u_1(.) \in \mathcal{U}_1$,

iii) $$E\int_{t_o}^{t_1}\langle\frac{\partial H}{\partial u_2}(t,x_o(t),u_o(t),\psi(t),M^o(t)),\ u_2(t)-u_{o2}(t)\rangle dt \geq 0$$

for all $u_2(.) \in \mathcal{U}_2$.

In addition, if the conditions in the remark 1 hold then

ii') $$H(t,x_o(t),(u_1,u_{o2}(t)),\psi(t),M^o(t)) \geq H(t,x_o(t),u_o(t),\psi(t),M^o(t)) \quad (\forall)\ u_1 \in U_1$$

iii') $$\langle\frac{\partial H}{\partial u_2}(t_1 x_o(t),u_o(t),\psi(t),M^o(t)),u_2-u_{o2}(t)\rangle \geq 0 \quad (\forall)\ u_2 \in U_2$$

a.e. in $(t,\omega) \in [t_o,t_1] x \Omega$ with respect to the measure dtxdP.

Proof

Let $u(.)\in\mathcal{U}_1$, $p_1(t)\equiv 1$, $p_j(t)\equiv 0$, $j\neq 1$. Define $x_r(.)$ the solution in (3) corresponding to $u(.)\in\mathcal{U}_1$, $\tilde{p}(.)=(1,0,\dots,0)$. It follows (see (4))

$$x_r(t)=x_o(t)+r\bar{x}(t)+\theta(r,t), \quad \lim_{r\to 0}\frac{1}{r}\sup_{t\leq t_1}\left\{E|\theta(r,t)|^2\right\}^{1/2}=0$$

where $x(.)$ is the corresponding Ito solution in (5).

By hypothesis the conditions in Lemma 1 and 2 are fulfiled.

Using Lemma 1 for $\eta=r^2$ we get a partition π and $u_\pi(.)\in\mathcal{U}$ such that

29) $$\left\{E\left|\int_{t_o}^{t_1}\left[(1-r)L(t,x_r(t),u_o(t))+rL(t,x_r(t),u(t))-L(t,x_r(t),u_\pi(t))\right]dt\right|^2\right\}^{1/2}\leq r^2$$

Using Lemma 2 it follows that the Ito solution $x_\pi(.)$

30) $$dx=f(t,x,u_\pi(t))dt+\sum_{i=1}^{k}g_i(t,x,u_o(t))dB_i(t)$$

$$x(0)=x_o$$

has the structure

31) $$x_\pi(t)=x_o(t)+r\bar{x}(t)+\theta_1(r,t),$$

where $\lim_{r\to 0}\frac{1}{r}\sup_{t\leq t_1}\left\{E|\theta_1(r,t)|^2\right\}^{1/2}=0$

The functional is getting the form

32) $$J(x_\pi,u_\pi)=J(x_o,(u_o)+rE\left\langle\frac{\partial G}{\partial x}(x_o(t_1)),\bar{x}(t_1)\right\rangle+E\int_{t_o}^{t_1}\left[\langle L_x(t),\bar{x}(t)\rangle+L(t)-L_o(t)\right]dt$$
$$+\theta_1(r)$$

where $\lim_{r\to 0}\frac{\theta_1(r)}{r}=0$, $L_x(t)=\frac{\partial L}{\partial x}(t,x_o(t),u_o(t))$, $L(t)=L(t,x_o(t),u(t))$, $L_o(t)=L(t,x(t),u_o(t))$.

When we choose $u(.)\in\mathcal{U}_2$ the solution $x_r(.)$ defined in (4) corresponding to $p_i(t)\equiv 0$, $i=1,\dots,l$, $q_1(t)\equiv 1$, $q_j(t)\equiv 0$, $j\neq 1$,

$u_1(.)=u(.)$, is an admissible one and has the structure .

$$x_r(t)=x_o(t)+r\bar{x}(t)+\mathcal{O}(r,t)$$

where $\bar{x}(.)$ verifies (5) with p,q as defined.

This time the functional is getting the following form

33) $$J(x_r,u_r)=J(x_o,u_o)+rE\left\langle\frac{\partial G}{\partial x}(x_o(t_1)),\bar{x}(t_1)\right\rangle+E\int_{t_o}^{t_1}\langle L_x(t),\bar{x}(t)\rangle dt+\mathcal{O}_2(r)$$

where $\lim\limits_{r\to 0}\frac{1}{r}\mathcal{O}_2(r)=0$.

As the primal form of the first order necessary conditions we get

34) $$E\left\langle\frac{\partial G}{\partial x}(x_o(t_1)),\bar{x}(t_1)\right\rangle+E\int_{t_o}^{t_1}\left[\langle L_x(t),\bar{x}(t)\rangle+L(t)-L_o(t)\right]dt\geq 0$$

for all $\bar{x}(.)$ verifying (5) with $p_1\equiv 1$, $u_1(.)=u(.)$, $p_j\equiv 0$, $j\neq 1$, $q_j=0$, $j=1,\dots,\ell$ if $u(.)\in\mathcal{U}_1$, and

34') $$E\left\langle\frac{\partial G}{\partial x}(x_o(t_1)),\bar{x}(t_1)\right\rangle+E\int_{t_o}^{t_1}\langle L_x(t),\bar{x}(t)\rangle dt\geq 0$$

for all $\bar{x}(.)$ verifying (5) with $p_i\equiv 0$, $i=1,\dots l$, $q_1\equiv 1$, $u_1(.)=u(.)$, $q_j\equiv 0$, $j\neq 1$ if $u(.)\in\mathcal{U}_2$.

From now on the condition (34) or (34') is transformed into adjoint system and optimality principle using the same general scheme as in [1]. The proof is complete.

REMARK 2

Consider that a deterministic control system $\frac{dx}{dt}=f(t,x,u)$, $t\in[t_o,t_1]$ is perturbed by a noise described by $\sum_{i=1}^{k}g_i(t,x,v)dB_i(t)$ and we are trying to minimize the largest effect produced by the noise using controls u.

The problem can be stated as

$$1)\ \min_{u\in\mathcal{U}} \ \max_{v\in\mathcal{V}} \ E\left\{G(x^{u,v}(t_1)) + \int_{t_0}^{t_1} L(t,x^{u,v}(t),\ u(t))dt\right\}$$

under the constraints

$$2)\ dx=f(t,x,u)dt + \sum_{i=1}^{k} g_i(t,x,v)dB_i(t), \quad t\in[t_0,t_1].$$

$$x(t_0)=x_0$$

where the sets $\mathcal{U}$ and $\mathcal{V}$ consist of all nonanticipative $u:[t_0,t_1]\times\Omega\to U(U\subseteq R^{m_1})$, $v:[t_0,t_1]\times\Omega\longrightarrow V(V\subseteq R^{m_2})$ with respect to σ- algebras $\{\mathcal{F}_t\}$, $t\in[t_0,t_1]$, generated by the k-dimensional standard Brownian motion $(B_1(.),\ldots,B_k(.))$ Let $(u_0(.),\ v_0(.))$ be the optimal pair for the problem (1) and (2). Then, under the hypothesis (a)-(c) in theorem and V a convex set we get the optimality condition in pointwise form given in theorem except the sign "≥ 0" in (iii′) which will be replaced by "≤ 0 for all $v\in V$".

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