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The governing equations of the macroscopic phenomenon are obtained in §1 with one of the methods exposed in [1] and taking advantage of accepted approximations. The existence of the steady solutions and a regularity result are proved in §2, where we also give a uniqueness "criterion". Because of the general non-uniqueness of the solutions, we study in §3 the asimptotic stability in the mean.

## 1. The fundamental equations

The description of the phenomena which appear when a fluid pass through a porous medium have always come up against peculiar difficulties. Thus the problem of the heat transfer between the fluid and the solid parts of the porous medium is still opened [2]; we shall use the hypothesis which suppose that the two component parts have equal mean temperatures.

In order to formulate the problem, we need a method to cross the microscopic level and to obtain the macroscopic behaviour (see [1] and [3]). So, let us introduce a probability density ($\alpha$) and define the macroscopic value ($a_m$) of a microscopic magnitude (a) by means of the following convolution:

(1) $a_m(\bar{x},t) = (a * \check{\alpha})(\bar{x},t)$ where $\check{\alpha}(\bar{x},t) = \alpha(-\bar{x},-t)$.

However, we have to take care because not every macroscopic value obtained as before has a physical signification in filtration theory.

As the velocity of the fluid in the porous medium is far smaller than the acoustic velocity, the motion induces little changes of the pressure. That's why we neglect the variations of the thermodynamic quantities owing to pressure changes. Moreover, we assume that the temperature differences are small enough so as the Boussinesq approximation is applicable to our problem. Thus the governing equations at the microscopic level are:

(2) $\operatorname{div} \bar{u} = 0$

(3) $\rho_0 \frac{\partial \bar{u}}{\partial t} + \rho_0 (\bar{u}\cdot\nabla)\bar{u} = -\nabla p + \mu \Delta \bar{u} + \rho_0 \left[1-\beta(T_f - T_0)\right]\bar{g}$

(4) $c_f \rho_0 \left( \frac{\partial T_f}{\partial t} + \bar{u}\cdot\nabla T \right) = -\operatorname{div} \bar{Q}_f$

(5) $C_S \rho_S \frac{\partial T_S}{\partial t} = -\operatorname{div} \bar{Q}_S$

Using the index f for the fluid and the index S for the solid, we have noted with C, T and $\bar{Q}$, respectively, the specific heat at constant volume, the temperature and the heat flux transmitted by conduction. In the fluid part, $\bar{u}$, p, $\mu$, $\beta$, $\rho_0$ and $T_0$ stand for the velocity, the pressure, the vîscosity, the volumetric coefficient of thermal expansion, the density and the temperature of a reference state, respectively. Our solid is immobile and not deformable, the density $\rho_S$ being constant. As usual $\bar{g}$ stand for the gravitational acceleration.

On the fluid-solid interface the following conditions are imposed:

(6) $\bar{u} = \bar{0}$

(7) $T_f = T_s$

Now, the Darcy's velocity, the temperature and the pressure of the "filtration" fluid can be defined by:

$$\bar{v} = \bar{u}_m$$

$$T = \frac{1}{m}\left(T_f\right)_m = \frac{1}{1-m}\left(T_s\right)_m$$

$$p = \frac{1}{m}\, p_m$$

where m is the porosity [1]. If the macroscopic motion is considered to be uniformly and slow, the equations (2)-(5) together with the conditions (6)-(7) give:

(8) $\operatorname{div} \bar{v} = 0$

(9) $\dfrac{\rho_0}{m}\dfrac{\partial \bar{v}}{\partial t} + \dfrac{\mu}{K}\,\bar{v} = -\nabla p + \rho_0\left[1-\beta(T-T_0)\right]\bar{g}$

(11) $c_f \rho_0\left[m\dfrac{\partial T}{\partial t} + \operatorname{div}\left(\bar{u}T_f\right)_m\right] = -\operatorname{div}\left(\bar{Q}_f\right)_m + \phi_{fs}$

(12) $(1-m)\, c_s \rho_s \dfrac{\partial T}{\partial t} = -\operatorname{div}\left(\bar{Q}_s\right)_m - \phi_{fs}$

where $\phi_{fs} = (\bar{Q}_f - \bar{Q}_s)\,\bar{n}\,\delta_{fs} * \check{\alpha}$ is the heat flux on the fluid-solid interface ($\bar{n}$-the normal towrds the fluid, $\delta_{fs}$ -the unit impulse of the interface) and K is the permeability [1].

The thermal conductivity $\lambda^*$ [2] can be introduced in the classical manner, so that by adding (11) and (12) to get:

$$(13) \qquad (\rho c)^* \frac{\partial T}{\partial t} + c_f \rho_o \, div(\bar{u} T_f)_m = \lambda^* \Delta T$$

where $(\rho c)^* = m c_f \rho_o + (1-m) c_S \rho_S$

We surpass the last difficulty by neglecting $div\{(u-\bar{u}_m)[T_f-(T_f)_m]\}_m$ in comparison with the other terms of (13). Thus, the heat equation becomes:

$$(10) \qquad \gamma \frac{\partial T}{\partial t} + m \bar{v} \nabla T = k \Delta T$$

where the number $\gamma = (\rho_o c_f)^{-1} (\rho c)^*$ and the thermal diffusion coefficient $k = (\rho_o c_f)^{-1} \lambda^*$ are characteristic to every porous medium.

Further on the fundamental equations which describe the present phenomenon at the macroscopic level are (8), (9) and (10).

## 2. The steady solutions

In this section we prove the existence of the weak solutions of the system (8)-(10) in the stationary case, with prescribed temperature on the boundary. We also study the uniqueness and the regularity of these solutions.

Let $\Omega$ be a bounded open set of class $\mathcal{C}^2$ in $\mathbb{R}^n$ ($n \geq 2$) with boundary $\Gamma$, let $\bar{g} \in \underline{L}^\infty(\Omega)$ be a given vector function and $\theta \in W_m^{(1)}(\Omega)$ ($m > n$) be a given scalar function. (Without loss of generality we can assume that $\Omega$ is included in a cube of edge length 1). We are looking for a vector function $\bar{v} = (v_1, \ldots, v_n)$

and for two scalar functions p and T, which are defined in $\Omega$, satisfying the following equations and boundary conditions:

(14) $$\operatorname{div} \bar{v} = 0 \quad \text{in } \Omega$$

(15) $$\frac{\mu}{K} \bar{v} = -\nabla p + \rho_0 \left[1-\beta(T-T_0)\right] \bar{g} \quad \text{in } \Omega$$

(16) $$m \bar{v} \nabla T = k \Delta T \quad \text{in } \Omega$$

(17) $$T = \theta \quad \text{on } \Gamma$$

(18) $$\bar{v} \cdot \bar{n} = 0 \quad \text{on } \Gamma$$ ($\bar{n}$ - the normal of $\Gamma$)

Let $\psi$ be any scalar function such that

(19) $$\psi \in W_m^{(1)}(\Omega) \quad \text{and} \quad \psi = \sigma \quad \text{on } \Gamma$$

where $\sigma = \mu^{-1} k^{-1} K m \rho_0 \|\bar{g}\|_\infty \left[1-\beta(\theta-T_0)\right]$. Later on we shall specify (more) the way in which $\psi$ must be chosen. Using the notations (20)-(23):

(20) $$\bar{\varphi} = -\|\bar{g}\|_\infty^{-1} \bar{g} \quad \left(\|\bar{\varphi}\|_\infty = 1\right)$$

(21) $$\bar{u} = k^{-1} m \bar{v}$$

(22) $$\Pi = \mu^{-1} k^{-1} K m p$$

(23) $$S = \mu^{-1} k^{-1} K m \rho_0 \|\bar{g}\|_\infty \left[1-\beta(T-T_0)\right] - \psi$$ )

the system (14)-(18) becomes:

(24) $\operatorname{div} \bar{u} = 0$ in $\mathcal{Q}$

(25) $\bar{u} = -\nabla\pi - S\bar{\varphi} - \Psi\bar{\varphi}$ in $\mathcal{Q}$

(26) $\bar{u}\nabla S + \bar{u}\nabla\Psi = \Delta S + \Delta\Psi$ in $\mathcal{Q}$

(27) $S = 0$ on $\Gamma$

(28) $\bar{u}\cdot\bar{n} = 0$ on $\Gamma$.

In order to prove the existence of the weak solutions of the problem (24)-(28) we use the following lemma [4]:

<u>Lemma 1</u>. Let B be a reflexive Banach space, A a dense subspace (with his own separated locally convex topology) continously embedded in B, and $A', B'$ the adequate dual spaces. Then $B' \subseteq G(B)$ if G is an weakly continuous operator $G: B \to A'$ such that

$$\lim_{\substack{\|x\|\to\infty \\ (x\in A)}} \frac{\langle Gx, x\rangle}{\|x\|} = \infty$$

Let us make the correspondence with our problem. Taking in account that the space $H = \{\bar{v}\in \underline{L}^2(\mathcal{Q}) \mid \operatorname{div}\bar{v} = 0,\ \bar{v}\cdot\bar{n}|_\Gamma = 0\}$ is the closure in $\underline{L}^2(\mathcal{Q})$ of the space $\mathcal{V} = \{\bar{v}\in\underline{\mathcal{D}}(\mathcal{Q}) \mid \operatorname{div}\bar{v} = 0\}$ [5], then $B = H\times H_0^1(\mathcal{Q})$ - Hilbert space with the scalar product $[(\bar{u},S),(\bar{v},T)] = (\bar{u},\bar{v})_2 + (S,T)_{H_0^1}$, $(\bar{u},S),(\bar{v},T)\in H\times H_0^1(\mathcal{Q})$ where $(S,T)_{H_0^1} = \int_{\mathcal{Q}} \nabla S\,\nabla T\,d\mathcal{Q}$ and $A = \mathcal{V}\times\mathcal{D}(\mathcal{Q})$. The operator $G: B\to A'$ is defined by:

$$\langle G(\bar{u},S),(\bar{v},T)\rangle = [(\bar{u},S),(\bar{v},T)] + (S\bar{\varphi},\bar{v})_2 + \tag{29}$$
$$+\, b(\bar{u}, S+\Psi, T), \quad (\forall)\ (\bar{v},T)\in A;$$

where $b(\bar{u},s,T) = \int_{\Omega} (\bar{u}\nabla s) T \, d\Omega$ .

Now we consider the following variational formulation of the problem (24)-(28): find $(\bar{u},s) \in \mathcal{B}$ such that

(30) $\langle G(\bar{u},s),(\bar{v},T)\rangle = -[(\psi\bar{\varphi},\psi),(\bar{v},T)]$ , $(\forall)(\bar{v},T) \in \mathcal{A}$

We have made this association because if $\bar{u}$, S and $\pi$ are smooth functions satisfying (24)-(28) then $(\bar{u},S)$ is a solution of (30). Conversely, if $(\bar{u},s) \in \mathcal{B}$ satisfies (30) then choosing the test functions in a proper manner we get:

(31) $(\bar{u} + s\bar{\varphi} + \psi\bar{\varphi}, \bar{v})_2 = 0 \qquad (\forall)\ \bar{v} \in \mathcal{V}$

(32) $(-\Delta s + \bar{u}\nabla s + \bar{u}\nabla\psi - \Delta\psi, T) = 0 \qquad (\forall)\ T \in \mathcal{D}(\Omega)$

As $\bar{u} + s\bar{\varphi} + \psi\bar{\varphi} \in \underline{L}^2(\Omega)$ and the orthogonal complement of H in $\underline{L}^2(\Omega)$ is $H^{\perp} = \{\bar{w} \in \underline{L}^2(\Omega) \mid \bar{w} = \nabla p, p \in H^1(\Omega)\}$ [5] it follows from (31) that $(\exists)\ \pi \in H^1(\Omega)$ such that (25) it satisfied in $\underline{L}^2(\Omega)$ . Since $-\Delta s + \bar{u}\nabla s + \bar{u}\nabla\psi - \Delta\psi \in H^{-1}(\Omega)$ from (32) results (26) in the distribution sense. Finally, because $(\bar{u},s) \in \mathcal{B}$ the relations (24), (27) and (28) are satisfied in the distribution and the trace senses, respectively.

Here it is the existence result:

<u>Theorem 1</u>. The problem (30) has at least one solution $(\bar{u},s) \in \mathcal{B}$ , and there exists $\pi \in H^1(\Omega)$ such that (24)-(28) are satisfied.

**Proof.** Let us notice that $(-\psi\bar{\varphi}, \Delta\psi) \in \mathcal{B}'$ because $-\psi\bar{\varphi} \in \underline{L}^\infty(\Omega)$ and $\Delta\psi \in H^{-1}(\Omega)$. Hence if we prove that $\psi$ from (19) can be chosen such that G satisfy the hypothesis of Lemma 1, then $(\exists)(\bar{u}, S) \in \mathcal{B}$ solution of the problem (30).

First of all, G is weakly continuous:

Let a sequence $(\bar{u}_n, S_n) \longrightarrow (\bar{u}, S)$ weakly in B; then for any $(\bar{v}, T) \in \mathcal{A}$ we have:

$$\left| \langle G(\bar{u}_n, S_n), (\bar{v}, T) \rangle - \langle G(\bar{u}, S), (\bar{v}, T) \rangle \right| \leqslant$$

$$\leqslant \left| [(\bar{u}_n, S_n), (\bar{v}, T)] - [(\bar{u}, S), (\bar{v}, T)] \right| + \left| \left( \bar{u}_n - \bar{u}, T\nabla\psi + T\nabla S \right)_2 \right| +$$

$$+ \left| (S_n - S, \bar{\varphi}\bar{v})_2 \right| + \left| b(\bar{u}_n, S_n - S, T) \right| \leqslant \|\bar{u}_n\|_2 \cdot \|\nabla T\|_\infty \|S_n - S\|_2 +$$

+ (terms which tends to zero). But also $\|S_n - S\|_2 \to 0$ as the embedding $H_0^1(\Omega) \subseteq L^2(\Omega)$ is compact, and hence $G(\bar{u}_n, S_n) \rightharpoonup G(\bar{u}, S)$ weakly in A'.

Before showing the last property of G, we will prove the following lemma:

**Lemma 2.** For any $\gamma > 0$ there exists some $\psi_\gamma$ satisfying (19) with : $\quad |b(\bar{u}, \psi_\gamma, S)| \leqslant \gamma \|(\bar{u}, S)\|_{\mathcal{B}}^2 \quad , \quad (\forall)(\bar{u}, S) \in \mathcal{A}$

**Proof of Lemma 2.** Let $\rho(x) = d(x, \Gamma)$ =the distance from x to $\Gamma$. For any $\varepsilon > 0$ we consider $\Omega_\varepsilon = \{ x \in \Omega \mid \rho(x) < 2\exp(-\varepsilon^{-1}) \}$ and $\theta_\varepsilon \in \mathcal{C}^2(\bar{\Omega})$ the function that has the following properties [5]:

$\theta_\varepsilon$ =1 in some neighbourhood of $\Gamma$ (which depends on $\varepsilon$)

$\theta_\varepsilon = 0$ in $\Omega \setminus \Omega_\varepsilon$

$$\left| \frac{\partial \theta_\varepsilon}{\partial x_k} \right| \leqslant \frac{\varepsilon}{\rho(x)} \quad \text{in } \bar{\Omega}_\varepsilon \quad , \quad (\forall) k \in \{1, 2, \ldots, n\}$$

Let us put $\psi_\varepsilon = \theta_\varepsilon \cdot \tau$ ; obviously $\psi_\varepsilon$ satisfy (19) and also $(\forall)\, s \in \mathcal{D}(\Omega)$ :

$$(33) \qquad \| s \nabla \psi_\varepsilon \|_2 \leq \left[ 2 \int_\Omega s^2 \left( \tau^2 |\nabla \theta_\varepsilon|^2 + \theta_\varepsilon^2 |\nabla \tau|^2 \right) d\Omega \right]^{\frac{1}{2}} \leq$$

$$\leq \sqrt{2} \left( \varepsilon \sqrt{n} \left\| \tau \frac{s}{\rho} \right\|_{L^2(\Omega_\varepsilon)} + \| s |\nabla \tau| \|_{L^2(\Omega_\varepsilon)} \right) \leq \sqrt{2} \left[ \varepsilon \sqrt{n} \|\tau\|_\infty \left\| \frac{s}{\rho} \right\|_2 + \eta(\varepsilon) \|s\|_{\frac{2m}{m-2}} \right]$$

where $\eta(\varepsilon) = \left( \int_{\Omega_\varepsilon} |\nabla \tau|^m d\Omega_\varepsilon \right)^{\frac{1}{m}}$ tends to zero as $\varepsilon \to 0$ because $\tau \in W_m^{(1)}(\Omega)$ ; also as $m > n$ it follows $\|\tau\|_\infty < \infty$.

Using the well-known inequalities:

$$\left\| \frac{s}{\rho} \right\|_2 \leq C_1 \|s\|_{H_0^1}$$

$$\|s\|_{\frac{2m}{m-2}} \leq C_2 \|s\|_{H_0^1} \qquad (\forall)\, s \in H_0^1(\Omega)$$

from (33) we get:

$$\| s \nabla \psi_\varepsilon \|_2 \leq C_3 \max\{\varepsilon, \eta(\varepsilon)\} \|s\|_{H_0^1}$$

Therefore $(\forall)\, (\bar{u}, s) \in A$ the following relation is true:

$$|b(\bar{u}, \psi_\varepsilon, s)| \leq C_4 \max\{\varepsilon, \eta(\varepsilon)\} \|(\bar{u}, s)\|_B^2$$

Then Lemma 2 is proved if $\varepsilon$ is sufficiently small such that :

$$C_4 \max\{\varepsilon, \eta(\varepsilon)\} \leq \gamma$$

Now, let us return to the main proof. Taking advantage of the following inequality [6]:

$$(34) \qquad \|s\|_2 \leq \frac{1}{\sqrt{\alpha}} \|s\|_{H_0^1} \qquad (\forall)\, s \in H_0^1(\Omega) \qquad \left( \alpha = \frac{3+\sqrt{13}}{2} \pi^2 \right)$$

we have for any $(\bar{u}, s) \in A$ :

$$\langle G(\bar{u},s),(\bar{u},s)\rangle = \|(\bar{u},s)\|_B^2 + (s\bar{\varphi},\bar{u})_2 + b(\bar{u},\psi'_\gamma,s) \geq$$

$$\geq \|(\bar{u},s)\|_B^2 - \|s\|_2 \|\bar{\varphi}\|_\infty \|\bar{u}\|_2 - |b(\bar{u},\psi'_\gamma,s)| \geq \left(1-\frac{1}{2\sqrt{\alpha}}-\gamma\right)\|(\bar{u},s)\|_B^2$$

Choosing

(35) $$\gamma < 1-\frac{1}{2\sqrt{\alpha}} \qquad \text{(for example } \gamma = 0.9\text{)}$$

the last property of G is demonstrated and also Theorem 1.

For uniqueness we have only a poor result because of some specific features of the problem.

Since the expression $b(\bar{u},s,T)$ does not necessarily make sense for $(\bar{u},s)\in B$, $T\in H_0^1(\Omega)$, we introduce the Banach space $\tilde{B} = H\times[H_0^1(\Omega)\cap L^\infty(\Omega)]$ with the norm $\|(\bar{u},s)\|_{\tilde{B}} = \|(\bar{u},s)\|_B + \|s\|_\infty$. Some properties of b are given:

(36) $$b(\bar{u},T,s) = -b(\bar{u},s,T)\ ;\ b(\bar{u},s,s)=0\ ;\ (\forall)(\bar{u},s)\in\tilde{B},\ T\in H^1(\Omega)$$

(37) b is trilinear and continuous, defined on $H\times H_0^1(\Omega)\times[H_0^1(\Omega)\cap L^\infty(\Omega)]$

The properties (36) and (37) are consequences of the two relations:

$$\int_\Omega(\bar{u}\nabla s)T\,d\Omega = \int_\Gamma T(s\bar{u})\bar{n}\,d\Gamma - \int_\Omega \nabla T(s\bar{u})\,d\Omega$$

$$\left|\int_\Omega(\bar{u}\nabla s)T\,d\Omega\right| \leq \|\bar{u}\|_2\|s\|_{H_0^1}\|T\|_\infty$$

Moreover, since A is dense in $\tilde{B}$, a continuity argument ensured by (37) shows that every $(\bar{u},s)\in B$, solution of (30), verify:

(38) $$\langle G(\bar{u},S),(\bar{v},T)\rangle = -\left[(\psi\bar{\varphi},\psi),(\bar{v},T)\right] \quad (\forall)(\bar{v},T)\in\tilde{B}$$

Now we can state our uniqueness "criterion":

Proposition 1. Let $(\bar{u},S)$ be a solution of the problem (30) with $\|S\|_\infty < 2-\alpha^{-\frac{1}{2}}$ ($\alpha$ given at (34)). Then $(\bar{u},S)$ is unique in B among the vector functions essentially bounded on the last component.

Proof. As $\|S\|_\infty < 2-\frac{1}{\sqrt{\alpha}}$ there exists $\gamma_0>0$ such that $\|S\|_\infty + 2\gamma_0 < 2-\frac{1}{\sqrt{\alpha}}$; let's keep in mind that $\gamma_0$ satisfy also (35).

If $(\bar{u}_1,S_1)\in\tilde{B}$ is another solution of (30), then puting $\bar{v}=\bar{u}_1-\bar{u}$ and $T=S_1-S$ (obviously $(\bar{v},T)\in\tilde{B}$) from (38) we get:

$$\begin{aligned}
0 &= \langle G(\bar{u}_1,S_1),(\bar{v},T)\rangle - \langle G(\bar{u},S),(\bar{v},T)\rangle = \\
&= \|(\bar{v},T)\|_B^2 + (T\bar{\varphi},\bar{v})_2 + b(\bar{v},\psi_{\gamma_0},T) + b(\bar{v},S,T) \geqslant \\
&\geqslant \|(\bar{v},T)\|_B^2 - \|T\|_2\|\bar{v}\|_2 - \gamma_0\|(\bar{v},T)\|_B^2 - \|S\|_\infty\|\bar{v}\|_2\|\nabla T\|_2 \geqslant \\
&\geqslant \left(1-\frac{1}{2\sqrt{\alpha}}-\gamma_0-\frac{1}{2}\|S\|_\infty\right)\|(\bar{v},T)\|_B^2
\end{aligned}$$

It follows $\|(\bar{v},T)\|_B^2 \leqslant 0$ that is $(\bar{u},S)=(\bar{u}_1,S_1)$ in B, and Proposition 1 is proved.

As in general $\tilde{B}$ is different from B, we cannot treat a less restricted case even if we impose supplementary conditions on $\sigma$ (respective $\psi_\gamma$). Finally we notice the following "a priori" estimation of the solutions $(\bar{u},S)\in\tilde{B}$ of (30):

$$\|(\bar{u},S)\|_B \leqslant \frac{\sqrt{1+\alpha^{-1}}}{1-\gamma-(2\sqrt{\alpha})^{-1}}\|\nabla\psi_\gamma\|_2 \quad (\forall)\,\gamma<1-\frac{1}{2\sqrt{\alpha}}$$

About the regularity of the solutions we can obtain some informations if the dimension of the space is $n\leqslant 3$. For example we'll show the following result:

Proposition 2. If $\Omega \subseteq \mathbb{R}^3$ is of class $\mathscr{C}^\infty$ and $\overline{\varphi}, \theta$ are functions of class $\mathscr{C}^\infty$ in $\overline{\Omega}$, then any solution of (30) is of class $\mathscr{C}^\infty$ in $\Omega$.

Proof. Eliminating $\overline{u}$ and definning $\sigma = s + \psi$, the system (24)-(28) becomes:

(39) $\Delta \pi = - \operatorname{div}(\sigma \overline{\varphi})$ in $\Omega$

(40) $\frac{\partial \pi}{\partial n} = - \tau \overline{\varphi} \cdot \overline{n}$ on $\Gamma$ ($\overline{n}$-the exterior normal on $\Gamma$)

(41) $\Delta \sigma = -(\nabla \pi + \sigma \overline{\varphi}) \nabla \sigma$ in $\Omega$

(42) $\sigma = \tau$ on $\Gamma$

Now a simple technique can be used. As $s \in H_0^1(\Omega)$ and $\psi \in W_m^{(1)}(\Omega)$, then $\sigma \in W_2^{(1)}(\Omega)$ and $\operatorname{div}(\sigma \overline{\varphi}) \in L^2(\Omega)$; since the following condition is satisfied

$$-\int_{\Omega} \operatorname{div}(\sigma \overline{\varphi})\, d\Omega = -\int_{\Gamma} \tau \overline{\varphi}\, \overline{n}\, d\Gamma$$

then $(\exists)\, \pi \in W_2^{(2)}(\Omega)$ satisfying (39)-(40) [5]. It implies that $(\nabla \pi + \sigma \overline{\varphi}) \in \underline{W}_2^{(1)}(\Omega) \subseteq \underline{L}^6(\Omega)$, and because $\nabla \sigma \in \underline{L}^2(\Omega)$ it follows that $(\nabla \pi + \sigma \overline{\varphi}) \nabla \sigma \in L^{3/2}(\Omega)$. From the same classical results [5], there exists $\sigma \in W_{3/2}^{(2)}(\Omega)$, which verify (41)-(42); from one of Sobolev embedding theorems [5] $\left(p - \frac{n}{k} = \frac{2}{3} - \frac{2}{3} = 0\right)$ results $\sigma \in L^\alpha(\Omega)$ $(\forall)\, \alpha \in [1,\infty)$. As $\nabla \sigma \in \underline{W}_{3/2}^{(1)}(\Omega)$, recalling the regularity result for the problem (39)-(40) we get $\pi \in W_{3/2}^{(3)}(\Omega)$ and then $\nabla \pi \in \underline{L}^\alpha(\Omega)$ $(\forall)\, \alpha \in [1,\infty)$. As $\nabla \sigma \in \underline{W}_{3/2}^{(1)}(\Omega) \subseteq \underline{L}^3(\Omega)$ then $(\nabla \pi + \sigma \overline{\varphi}) \nabla \sigma \in L^\beta(\Omega) \left(\frac{3}{4} \leq \beta < 3\right)$. It follows $\sigma \in W_\beta^{(2)}(\Omega)$ and $\nabla \sigma \in \underline{W}_\beta^{(1)}(\Omega)$ $(\forall)\, \beta \in [\frac{3}{4}, 3)$. But $\frac{1}{\beta} - \frac{1}{3} = \frac{3-\beta}{3\beta}$ and hence $\nabla \sigma \in \underline{L}^\alpha(\Omega)$ $(\forall)\, \alpha \geq 1$. Therefore we obtain again an improvement: $\pi \in W_\alpha^{(2)}(\Omega)\ (\forall)\alpha \in [1,\infty)$ , that is $\nabla\pi \in \underline{W}_\alpha^{(1)}(\Omega)\ (\forall)\alpha \in [1,\infty)$

Now the following proposition can be easily proved by induction on $K \in \mathbb{N}\ (K \geq 1)$ :

$$v \in W_\alpha^{(K)}(\Omega) \text{ and } \nabla\pi \in \underline{W}_\alpha^{(K)}(\Omega) \quad (\forall)\alpha \in [1,\infty)$$

This proposition can be proved repeating, on the step K, the last part of the previous demonstration.

Finally, recalling again the embedding theorems, we get that $v$ and $\pi$ , together with any derivative are continous.

3. <u>The stability in the mean</u>

In this section we revert to the unsteady case and determine under which conditions the quadratic mean of a perturbation tends to zero when the time increases.

We consider a basic phenomenon $(\bar{v}, T, p)$ , solution of (8)-(10) with the boundary conditions (17)-(18), which can take place in $\Omega(t)$ , an open set of class $\mathscr{C}^2$ in $\mathbb{R}^n\ (n \geq 2)$ with boundary $\Gamma$, included in a cube of edge length $d$ . Together with any other altered state $(\bar{v}^*, T^*, p^*)$ , which is also solution of (8)-(10) and (17)-(18), we introduce the perturbation quantities $\bar{w} = \bar{v}^* - \bar{v}$ , $\varrho = T^* - T$ and their quadratic means:

$$W = \left[\int_\Omega \frac{1}{2} w^2 d\Omega\right]^{\frac{1}{2}} , \quad Q = \left[\int_\Omega \frac{1}{2}\varrho^2 d\Omega\right]^{\frac{1}{2}}$$

which are obviously in connection with the energy of the disturbance brought about by ( $\bar{v}^*, T^*, p^*$ ). Thus, a basic state is said to be stable in the mean [6] if the energy of any disturbance tends to zero as the time increases, that is:

(44) $$\lim_{t \to \infty} W = \lim_{t \to \infty} Q = 0$$

Now, here it is our stability creterion:

Theorem 2. The basic state $(\bar{v}, T, p)$ is stable in the mean if:

(45) $$Ra < \frac{\alpha}{m} \qquad \left( \alpha = \frac{3+\sqrt{13}}{2}\, \pi^2 \right)$$

where $Ra = \mu^{-1} k^{-1} \|\bar{g}\|_\infty d^2 \rho_0 K \beta \|\nabla T\|_\infty$ is the Rayleigh number associated to the basic state.

Proof. Let $(\bar{v}^*, T^*, p^*)$ be an altered state. With any initial conditions, the perturbations quantities $\bar{w}$ and $q$ satisfy the following system:

(46) $$\operatorname{div} \bar{w} = 0 \quad \text{in } \Omega$$

(47) $$\frac{\rho_0}{m} \frac{\partial \bar{w}}{\partial t} + \frac{\mu}{K} \bar{w} = -\nabla (p^* - p) - \rho_0 \beta q \bar{g} \quad \text{in } \Omega$$

(48) $$\gamma \frac{\partial q}{\partial t} + m \bar{w} \nabla q + m \bar{w} \nabla T + m \bar{v} \nabla q = k \Delta q \quad \text{in } \Omega$$

(49) $$q = 0 \quad \text{on } \Gamma$$

(50) $$\bar{w} \cdot \bar{n} = 0 \quad \text{on } \Gamma$$

Multiplying (47) and (48) with $\bar{w}$, respectively $q$, and using the well-known formula of the material derivative for $W^2$ and $Q^2$ ( $\Omega(t)$ being a material domain) we obtain:

(51) $$\frac{\rho_0}{m} \frac{d W^2}{dt} = -\frac{2\mu}{K} W^2 - \rho_0 \beta \left( q \bar{g}, \bar{w} \right)_2$$

(52) $$\gamma \frac{dQ^2}{dt} = - m\, b(\overline{w}, T, \vartheta) - k \|\vartheta\|^2_{H_0^1}$$

If $\overline{g}$ and $\nabla T$ are supposed to be essentially bounded, then:

(53) $$|b(\overline{w}, T, \vartheta)| \leqslant 2 \|\nabla T\|_\infty W \cdot Q$$

(54) $$|(\vartheta \overline{g}, \overline{w})_2| \leqslant 2 \|\overline{g}\|_\infty W \cdot Q$$

As the inequality (34) implies:

(55) $$\|\vartheta\|^2_{H_0^1} \geqslant \frac{2\alpha}{d^2} Q^2$$

then from (51)-(52), via (53)-(54), we get:

(56) $$\frac{\rho_0}{m} \frac{dW^2}{dt} \leqslant - \frac{2\mu}{K} W + 2\rho_0 \beta \|\overline{g}\|_\infty W \cdot Q$$

(57) $$\gamma \frac{dQ^2}{dt} \leqslant 2m \|\nabla T\|_\infty W \cdot Q - \frac{2\alpha k}{d^2} Q^2$$

Now (56) and (57) can be put in only one equation:

(58) $$\frac{d}{dt}\begin{pmatrix} W \\ Q \end{pmatrix} + \overline{\overline{A}} \begin{pmatrix} W \\ Q \end{pmatrix} \leqslant 0$$

where $\overline{\overline{A}}$ is the following matrix

$$\overline{\overline{A}} = \begin{pmatrix} m\mu\rho_0^{-1} K^{-1} & - m\beta \|\overline{g}\|_\infty \\ - m\gamma^{-1} \|\nabla T\|_\infty & \alpha k \gamma^{-1} d^{-2} \end{pmatrix}$$

Integrating (58) from 0 to t we are lead to

$$\begin{pmatrix} W(t) \\ Q(t) \end{pmatrix} \leq \exp\left(-\bar{\bar{A}}t\right) \begin{pmatrix} W(0) \\ Q(0) \end{pmatrix} \tag{59}$$

Let us study the characteristic equation of the matrix $\bar{\bar{A}}$:

$$\lambda^2 - \left(\frac{m\mu}{\rho_0 K} + \frac{\alpha k}{\gamma d^2}\right)\lambda + \frac{m}{\gamma}\left(\frac{\alpha k \mu}{d^2 \rho_0 K} - m\beta \|\nabla T\|_\infty \|\bar{g}\|_\infty\right) = 0 \tag{60}$$

Obviously $\bar{\bar{A}}$ has distinct eigenvalues which are also strictly positive because of (45). Thus Theorem 2 is proved, using classical results on spectral decomposition.

Theorem 2, for the steady case, becomes another uniqueness "criterion":

<u>Proposition 3</u>. Let $(\bar{v}, T)$ be a solution of (14)-(18) with $\|\nabla T\|_\infty < \dfrac{\alpha \mu k}{m d^2 \rho_0 \beta K \|\bar{g}\|_\infty}$. Then $(\bar{v}, T)$ is unique in $\underline{L}^2(\Omega) \times L^2(\Omega)$.

<u>Proof</u>. From the hypothesis it follows that $(\bar{v}, T)$ is a stable state. Since W and Q are constant, from (44) results $\|\bar{w}\|_2 = \|\theta\|_2 = 0$ , which is the desired conclusion.

Finally we remark that the condition (45), in which the porosity has its part, suits to certain situation revealed by experiments [2].

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