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SOME CRITERIA FOR JAPANESE RINGS

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1. INTRODUCTION AND DEFINITIONS

In this paper we shall prove two theorems concerning the finiteness of the integral closure of a Noetherian domain in its field of quotients or in a finite extension of it.

This paper is divided into two sections. In the first section we shall deal with the integral closure of a Noetherian domain A in its field of quotients K. We shall prove a theorem which gives a sufficient criterion for the lifting of the finiteness of the integral closure from the factorizations A/P, where P are the minimal primes containing an element $0 \neq x \in A$, to the ring A itself. Our theorem was also proved in ([11] Th.1.8) but in more restrictive conditions.

In the second section , we shall prove a theorem which generalizes the well- known theorem of J.Tate ([13] Lemma6.3 , or [3]) concerning the lifting of the property of being a Japanese ring : " If A is a Noetherian domain and $0 \neq x \in A$ such that for every $P \in \mathrm{Ass}_A(A/xA)$ the ring A/P is Japanese and A is xA-adically complete, then A is Japanese." This theorem also generalizes some results obtained by H.Seydi ([12] Th.1.1) and J.Marot ([5] Prop.2.5). Using this theorem they may also get an easy proof for Marot's theorem ([4]) concerning the stability of the Nagata rings at the adic completion.

In this paper we shall use the following:

DEFINITIONS i) A Noetherian domain A will be called a Mori ring , iff the integral closure of A in its field of quotients is finite over A.

ii) A Noetherian domain A will be called a Japanese ring , iff for any finite extension K' of the field of quotients of A, the integral closure A' of A in K' is finite over A.

iii) A Noetherian ring A will be called a Nagata ring iff for every $P \in \mathrm{Spec}(A)$ the ring A/P is Japanese.

REMARKS: The Mori rings are called N-1 rings in ([6]) and the Japanese rings are called N-2 rings in ([6]) The Nagata rings are the same with the pseudo-geometric rings in ([7] , Ch.VI) and with the Universally Japanese rings in ([3]). If $P \in \mathrm{Spec}(A)$ we denote by $k(P)$ the quotient field of A/P. If I is an ideal in A we denote by r(I) the radical of I , $r(I)=\bigcap_{I \subseteq P \in \mathrm{Spec}(A)} P$

In this paper we shall essentially use the following

THEOREM 1.1 (Mori-Nagata) Let A be a Noetherian domain and let A' be the integral closure of A in its field of quotients . Then:

i) A' is a Krull ring.

ii) If $P \in Spec(A)$ then there are only a finite number of $P' \in Spec(A')$ such that $P = P' \cap A$, and , for any such P' , $[k(P'):k(P)] < \infty$

For the proof see ([7] (33.10))

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2. MORI RINGS

THEOREM2.1 Let A be a Noetherian domain and x a nonzero element in A. Suppose that:

i) A is xA-adically complete.

ii) For all minimal prime $P \in Ass_A(A/xA)$ the ring A/P is Mori and $xA_P = PA_P$.

iii) If A' is the integral closure of A in its field of quotients , then for any minimal prime P' over xA' the ideal $P = P' \cap A$ is minimal in $Ass_A(A/xA)$

Then A is Mori.

Proof: Let $\{P_1, \ldots P_n\}$ be the set of the minimal primes in $Ass_A(A/xA)$. If P' is a prime ideal associated to xA' (which is minimal by Th.1.1 i)) we deduce from iii) that there exists an i , $1\leq i\leq n$, such that $P'\cap A = P_i$. Obviously $A_{P_i}\subseteq A'_{P_i}$; since A_{P_i} is normal by ii) it follows that $A_{P_i} = A'_{P_i}$. So A'_{P_i} is local and its maximal ideal is $P'A'_{P_i}$. Hence $A_{P_i} = A'_{P_i} = A'_{P'}$, and $P_iA_{P_i} = P'A'_{P'}$. It follows that P' is unique in Spec(A') such that $P'\cap A = P_i$. Put $\bar{P}'_i$ instead of P'.

The homomorphism $A/P_i \longrightarrow A'/\bar{P}'_i$ is injective and integral. On the other hand one gets the equalities:

$$k(\bar{P}'_i) = A'_{\bar{P}'_i}/\bar{P}'_iA'_{\bar{P}'_i} = A_{P_i}/P_iA_{P_i} = k(P_i)$$

and since A/P_i is Mori it followst that $A'/\bar{P}'_i$ is finite over A/P_i , $1\leq i\leq n$. From ii) we get that $xA'_{\bar{P}'_i} = \bar{P}'_iA'_{\bar{P}'_i}$

On the other hand since A' is a Krull domain (Th.1.1) we get that $A' = \bigcap_{ht(Q)=1} A'_Q = A'_{\bar{P}'_1}\cap\ldots\cap A'_{\bar{P}'_n}\cap\Big(\bigcap_{\substack{ht(Q)=1\\ Q\neq \bar{P}'_i}} A'_Q\Big)$

So $xA' = xA'_{\bar{P}'_1}\cap\ldots\cap xA'_{\bar{P}'_n}\cap\Big(\bigcap_{\substack{ht(Q)=1\\ Q\neq \bar{P}'_i}} A'_Q\Big) = \bigcap_{i=1}^{n} xA'_{\bar{P}'_i}\cap A' = \bigcap_{i=1}^{n} \bar{P}'_i$. Hence $r(xA')=xA'$ and so $xA'\cap A = r(xA) = \bigcap_{i=1}^{n} P_i$

Since $A'/\bar{P}'_i$ is finite over A/P_i we get the following injective homomorphism $A/r(xA) = A/\bigcap_{i=1}^{n} P_i \longrightarrow \prod_{i=1}^{n}(A/P_i) \longrightarrow \prod_{i=1}^{n}(A'/\bar{P}'_i)$. Since the homomorphism $A'/xA' = A'/\bigcap_{i=1}^{n}\bar{P}'_i \longrightarrow \prod_{i=1}^{n}(A'/\bar{P}'_i)$ is injective , we deduce that A'/xA' is finite over $A/r(xA)$, hence it is finite over A/xA. Since A is xA-adically complete and separated and A' is xA'-adically separated we get by a known lemma ([6]) that A' is finite over A.

REMARKS The second part of ii) is verified for instance when $xA=r(xA)$. The condition iii) is verified for instance when A satisfies the condition (S_2) of Serre ([4] 17.I)

Now let us give an aplication of the theorem above. Let us remind the following well-known :

LEMMA 2.2 Let A be a NOetherian ring and $x_1,\dots,x_n$ an A-sequence contained in the Jacobson radical of A , such that the ideal $I = \sum_{i=1}^{n} x_i A$ is pime. Then A is a domain and , moreover any subset of $\{x_1,\dots,x_n\}$ generates a prime ideal.

Proof: We may suppose that I is principal ideal , and let $x\in A$ such that $I=xA$. Since x is non-zero-divizor it follows that $ht(xA)=1$. Let $P\subsetneq xA$ be another prime ideal. Take $a\in P$; we may form by induction a sequence (a_n) ; $a_n\in A$, $\forall\, n\in \mathbb{N}$,such that $a=a_n x^n, \forall n\in\mathbb{N}$. Hence $a\in \bigcap_{n\geq 1} x^n A=0$, and so $P=0$.

PROPOSITION 2.3 Let A be a Noetherian ring and $x_1,\dots x_n$ an A-sequence contained in the Jacobson radical of A such that the ideal $I= \sum_{i=1}^{n} x_i A$ is prime. If A/I is Mori then A is Mori.

Proof: Using lemma 2.2 we may suppose that I is principal ideal. Put $I=xA$; let $\hat{A}$ be the xA-adic completion of A. Then $A/xA = \hat{A}/x\hat{A}$ hence $\hat{A}/x\hat{A}$ is Mori. Since the property of being Mori descendes by faithfully flat homomorphisms we may suppose that A is xA-adically complete. Since $\bigcap_{n\geq 1} x^nA = 0$ and xA is prime , the conditions of the theorem 2.1 are verified . Hence A is Mori.

COROLLARY 2.4 If A is a local ring which satisfies the conditions of the prop. 2.3 and $Gr_I(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ is Mori then A is Mori.

Proof: By a well-known theorem we have $Gr_I(A)=(A/I)[Y_1,\ldots,Y_n]$ where Y_i are indeterminates. Hence $Gr_I(A)$ is Mori iff A/I is Mori. By the prop. 2.3 it follows that A is Mori.

REMARK: Since normality is a local property the following criterion holds: ". Let A be a Noetherian domain and I a prime ideal generated by an A-sequence contained in the Jacobson radical of A. If A/I is normal then A is normal." As we did in corollary 2.4 $Gr_{I_M}(A_M)$ is normal for every M maximal ; then A_M is normal and hence A is normal . This remark have been also proved by Seydi in ([11] Prop.I.7.4) in the case when I is principal, but the proof is longer then ours.

3. JAPANESE RINGS

LEMMA 3.1 Let A be a Noetherian domain and B a free A-algebra of rank $= n$. Take $0 \neq x \in A$; then we have:

i) For every $Q \in \mathrm{Ass}_B(B/xB)$ we get $Q \cap A = P \in \mathrm{Ass}_A(A/xA)$

ii) Moreover, if $\mathrm{Ass}_A(A/xA)$ has no embedded primes then $\mathrm{Ass}_B(B/xB)$ has the same property.

Proof: i) We may suppose that (A, P) is local. Let $B = \bigoplus_{i=1}^{n} Au_i$ and take $Q \in \mathrm{Ass}_B(B/xB)$. Then there exists $b \in B$ such that $Q = (xB : b)$ and $b = \sum_{i=1}^{n} b_i u_i$, with $b_i \in A$. Let $P = Q \cap A$ and take $a \in P$. It follows that $ab = xz$, where $z = \sum_{i=1}^{n} z_i u_i$, $z_i \in A$. Hence $\sum_{i=1}^{n} (ab_i) u_i = \sum_{i=1}^{n} (xz_i) u_i$ and so $ab_i = xz_i \in xA$, for $i = \overline{1,n}$

We claim that there exist an i, $1 \leq i \leq n$, such that $b_i \notin xA$. Indeed if $b_i \in xA$ for $i = \overline{1,n}$, it follows that $b_i = xd_i$, $d_i \in A$, hence $b \in xB$ so $Q = B$, which is a contradiction. We deduce that $P \subset \bigcup_{P_i \in \mathrm{Ass}_A(A/xA)} P_i$; by the maximality of P we deduce that there exist $P_i \in \mathrm{Ass}_A(A/xA)$ such that $P = P_i$.

ii) If $\mathrm{ht}(P_i) = 1$, for $i = \overline{1,n}$, it follows that $\mathrm{ht}(Q) = 1$ (B being integral over A)

LEMMA 3.2 Let A be an integral domain and K its field of quotients. Let K' be a finite extension of K. Then there exist a free A-algebra B of rank$=[K':K]$ such that

K' is the field of quotients of B.

Proof: Using induction on the minimal number of generators of K' as a K-algebra we may suppose that $K'=K(u)$ If $P_u(T)$ is the minimal polynomial of u over K , then multiplying this polynomial with a suitable element of A we may suppose that $P_u(T) \in A[T]$ and is monic. Hence the A-algebra $B=A[u]$ is free over A and $\mathrm{rank}_A(B)=[K':K]$

The following lemma lightly extends a classical lemma due to Nagata ([7](33.11))

LEMMA3.3 Let A be a Noetherian domain , K its field of quotients and $0 \neq x \in A$. Let K' a finite extension of K and A' the integral closure of A in K'. Then for every minimal prime ideal Q over xA' , it follows that $Q \cap A \in \mathrm{Ass}_A(A/xA)$.

Proof: Let B be the free A-algebra built in lemma 3.2 By Nagata's lemma we have $P'=Q \cap B \in \mathrm{Ass}_B(B/xB)$ and by lemma 3.1 it follows that $P=Q \cap A = P' \cap A \in \mathrm{Ass}_A(A/xA)$

The following lemma is another form of a result due to Nishimura ([8]) and is contained in its proof:

LEMMA3.4 Let A be a Krull domain and let $0 \neq x \in A$ such that , A/P is Noetherian , for every minimal prime P over xA. Then A/xA is Noetherian.

We are now ready to prove the main result of this section

THEOREM 3.5 Let A be a Notherian domain and x an element in A. Suppose that ;

i) For all prime $P \in \mathrm{Ass}_A(A/xA)$, A/P is Japanese.

ii) $\bar{A}$ is xA-adically complete.

Then A is Japanese.

Proof: Let K' be a finite extension of the field of quotients K of A. Let A' be the integral closure of A in K'. We want to show that A' is finite over A.

Let $\{Q_1, \ldots Q_n\}$ be the set of all the minimal primes associated to xA'. Let $P_i = Q_i \cap A$. By the theorem 1.1 and using the free A-algebra built in lemma 3.2 it follows that $[k(Q_i):k(P_i)] < \infty$ and by ii) we deduce that A'/Q_i is finite over A/P_i . In particular A'/Q_i is Noetherian ; by lemma 3.4 it follows that A'/xA' is Noetherian.

We claim that A'/r(xA') is finite over $\bar{A}/x\bar{A}$. Indeed the canonical homomorphism $A'/r(xA') = A'/\bigcap_{i=1}^{n} Q_i \longrightarrow \prod_{i=1}^{n}(A'/\bar{Q_i})$ is injective. On the other hand $\prod_{i=1}^{n}(A'/\bar{Q_i})$ is finite over $\prod_{i=1}^{n}(A/P_i)$ so it is finite over A/xA.

Hence $\bar{A'}/r(xA')$ is finite over $\bar{A}/x\bar{A}$.

Put $J=r(xA')$; since A'/xA' is Noetherian there exists $n \in N$ such that $J^n \subseteq xA'$. Applying the induction in the following exact sequence :

$$0 \longrightarrow J^n/J^{n+1} \longrightarrow A'/J^{n+1} \longrightarrow A'/J^n \longrightarrow 0$$

we deduce that A'/xA' is finite over A/xA. Since A is xA-adically complete and A' is xA'-adically separated a well known lemma leads us to the fact that A' is finite over A.

COROLLARY 3.6 (Tate) Let A be a Noetherian normal domain and x an element in A. Suppose that :

i) xA is a prime ideal and A/xA is Japanese.

ii) A is xA-adically complete.

Then A is Japanese.

REMARK: The proof of this theorem in ([13]) depends on the characteristic of A. The method of Tate cannot be used to prove our theorem.

The following theorem which has been proved for the first time by J.Marot in ([4] Prop.1) , is a positive answer to a conjecture of A.Grothendieck ([3] n^{o}(7.4.8))

COROLLARY3.7 (Marot) Let A be a Noetherian ring and I an ideal in A. Suppose that

i) (A,I) is Zariski complete.

ii) A/I is Nagata.

Then A is Nagata.

Proof: Using the induction on the number of generators of I we may suppose that I is principal : I=xA. Take $Q \in \mathrm{Spec}(A)$: if $xA \subseteq Q$ then A/Q is Japanese by ii); if $xA \nsubseteq Q$ let B=A/Q. Take $P \in \mathrm{Ass}_B(B/xB)$. Then there exists

$P' \in Spec(A)$, such that $xA \subseteq P'$ and $A/P' \cong B/P$. By ii) we deduce that B/P is Japanese and by theorem 3.5 we deduce that B is Japanese, hence A is Nagata ring.

OPEN QUESTION

The statement of the theorem of J.Marot and theorem 3.5 leads to the following question:

Suppose A is a Noetherian domain and I is an ideal in A such that:

i) A is I-adically complete .

ii) A/P is Japanese for every $P \in Ass_A(A/I)$.

Does it follow that A is Japanese?

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