INSTITUTUL DE MATEMATICA

INSTITUTUL NATIONAL PENTRU CREATIE STIINTIFICA SI TEHNICA

ISSN 0250-3638

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by

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PREPRINT SERIES IN MATHEMATICS

No.70/1980

BUCURESTI

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December 1980

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Alexandru Buium

1. Introduction. Consider a sequence $Y_o \subset Y_1 \subset Y_2 \subset \dots$ of projective varieties over a field K such that Y_o is a projective space of dimension at least 3 and Y_{i-1} is an ample Cartier divisor on Y_i for each $i \geqslant 1$. If all Y_i's are assumed to be smooth, itfollows by [10] that every Y_i must be isomorphic to a projective space and each Y_{i-1} is contained in Y_i as a hyperplane. Now if we allow Y_i to have normal singularities, it follows by [1] that Y_1 must be isomorphic with the projective cone over some Veronese embedding of Y_o and that Y_o is contained in Y_1 as the hyperplane section at infinity in that cone. By [3] such a Y_1 may be identified with the weighted projective space $P(1,..,1,q)$ for some natural q. The aim of this paper is to prove that in the above context, every Y_i must be isomorphic to a weighted projective space, more precisely that there exists a sequence of natural numbers $q_1,q_2,..$ such that each Y_i is isomorphic to $P(1,..,1,q_1,..,q_i)$ and each Y_{i-1} is contained in Y_i in a natural way (see Corollary 4.2). There are two facts which lead to this : first a certain property of the singularities of weighted projective spaces

(see Corollary 2.5) and second a Grothendieck-Lefschetz-type result (see Proposition 3.2) where such singularities appear. In fact, the results we prove in §2 about the divisor class group of weighted projective varieties are stronger than we need for our purpose; however they have some interest in themselves.

Throughout this paper we will use the following notations for a noetherian scheme W. $Coh(W)$ will mean the class of coherent sheaves on W, $LF(W)$ will mean the class of coherent locally free sheaves on W, $P(W)$ will mean the class of invertible sheaves on W and if W is integral and normal, $C(W)$ will mean the class of coherent reflective sheaves of rank one on W. Recall that $Pic(W)$ and $Cl(W)$ are the groups of isomorphism classes of $P(W)$ and $C(W)$ respectively.

Given a natural k we will say that $F \in Coh(W)$ has the property (S_k) if for any $x \in W$ we have

$$\text{prof } F_x \geqslant \inf(k, \dim F_x) \quad .$$

F will be called Cohen-Macaulay if it has (S_k) for any natural k.

All varieties will be supposed to be irreducible.

2. Reflective sheaves and property (S_3). Let W be a noetherian integral normal scheme. It is known by [9] that every member of $C(W)$ has (S_2). Let us make the following definition which will reveal itself in §3 :

DEFINITION 2.1. W is said to be strongly (S_3) if any member of $C(W)$ has (S_3). A noetherian normal domain A is said to be strongly (S_3) if $\operatorname{Spec} A$ is strongly (S_3).

REMARKS : 2.2.1. If W is strongly (S_3) it has (S_3). Conversely, if W has (S_3) and it is locally factorial then W is strongly (S_3).

2.2.2. W is strongly (S_3) if and only if for any $x \in W$, the local ring $\mathcal{O}_{x,W}$ is strongly (S_3). Indeed if we suppose W is strongly (S_3) and if $\underline{a}$ is a divisorial ideal in $\mathcal{O}_{x,W}$ then the surjection $Cl(W) \longrightarrow Cl(\mathcal{O}_{x,W})$ shows that there exists a sheaf $F \in C(W)$ such that $F_x \simeq \underline{a}$ and hence $\underline{a}$ has (S_3). The converse is obvious.

2.2.3. A noetherian normal domain A is strongly (S_3) if and only if it has (S_3) and no prime of height $\geqslant 3$ is associated to an ideal generated by two elements. This comes from the fact that any divisorial ideal $\underline{a}$ in A is isomorphic as an A-module to an ideal of the form $fA \cap gA$ with $f,g \in A$, hence one may consider an exact sequence :

$$0 \longrightarrow \underline{a} \longrightarrow fA \oplus gA \longrightarrow A \longrightarrow A/(fA+gA) \longrightarrow 0$$

and our assertion follows from the local cohomology sequence.

2.2.4. Let $u:W \longrightarrow V$ be a faithfully flat morphism of noetherian integral normal schemes. If W is strongly (S_3), the same holds for V. Indeed, by Remark 2.2.2 we may assume that $W = \mathrm{Spec}\, B$ and $V = \mathrm{Spec}\, A$, where A and B are noetherian normal domains. Now if there existed a prime $\underline{p} \in \mathrm{Spec}\, A$ with height $\underline{p} \geqslant 3$ and elements $f,g \in A$ such that $\underline{p} \in \mathrm{Ass}_A(A/(fA+gA))$, then taking a minimal prime $\underline{q} \in \mathrm{Spec}\, B$ containing $\underline{p}B$ we would get by flatness that height $\underline{q} \geqslant 3$ and $\underline{q} \in \mathrm{Ass}_B(B/(fB+gB))$ (see [7]) which is a contradiction.

We will give now a class of strongly (S_3) singularities (i.e. non-regular local rings) which are not generally factorial.

Let K be a field, $q = (q_o,\dots,q_r)$ and $d = (d_1,\dots,d_s)$ systems of natural numbers, $S = K[T_o,\dots,T_r]$ the graded polynomial algebra over K with $\deg T_i = q_i$ for $i=0,\dots,r$, $F_1,\dots,F_s \in S$ homogenous polynomials, $A = S/(F_1,\dots,F_s)$, and let $t_o,\dots,t_r$ be the images of $T_o,\dots,T_r$ in A. The scheme $X = \mathrm{Proj}\, A$ will be called a weighted projective scheme. Following [3] the scheme $P(q_o,\dots,q_r) = \mathrm{Proj}\, S$ will be called the weighted projective space of type (q). If $F_1,\dots,F_s$ form a regular sequence in S, we say after [3] that X is a weighted complete intersection of type (q,d)

If A is a regular ring outside the maximal irrelevant ideal $\underline{m}$ we say after [3] that X is quasi-smooth. Now let $N+1$ be the minimum number of members in $\{t_o,..,t_r\}$ generating the K-algebra A. We say that A is normalized (or X is normalized) if any N members of tha set $\{q_o,..,q_r\}$ have no common prime divisors. For any integer n consider the sheaf $\mathcal{O}_X(n) = A(n)^{\sim}$. These sheaves are coherent on X since $\mathcal{O}_P(n)$ are coherent on $P = P(q_o,..,q_r)$ by [2].

<u>PROPOSITION 2.3</u>. Suppose $X = \text{Proj } A$ is a weighted projective scheme such that A is normalized and factorial. Then :

1) For any integer n, $\mathcal{O}_X(n)$ is reflective, of rank one.

2) For any integers n,m the restriction of the canonical morphism $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \longrightarrow \mathcal{O}_X(n+m)$ to the regular locus of X is an isomorphism.

If in addition X is normal, then :

3) $Cl(X)$ is cyclic generated by $\mathcal{O}_X(1)$.

4) $Pic(X)$ is cyclic generated by $\mathcal{O}_X(k)$ for some k.

<u>REMARKS</u> 1) A is factorial for instance when X is a quasi-smooth weighted complete intersection with $\dim X \geq 3$ (see [3])

2) If P is normalized and each q_j is less than any d_i, then X itself is normalized.

3) In the case $X = P$ the proposition fo-

llows by carefully combining several facts from [2],[3] and [8]. However, our general case requires another approach.

<u>Proof of the Proposition</u>. Fix two integers n,m and denote by $\mathcal{O}_X(n)\mathcal{O}_X(m)$ the image of the morphism $\mathcal{O}_X(n)\otimes\mathcal{O}_X(m)\longrightarrow\mathcal{O}_X(n+m)$. Suppose $t_o,\dots,t_N$ is a system of generators for the K-algebra A, with minimal N, and let f be one of these generators. Put $B=A_f$, $H=(\bigcup_{i\geq 0}A_i)\setminus\{0\}$ and $L=H^{-1}A$. We proceed in several steps :

<u>Step 1</u>. $t_o,\dots,t_N$ are prime elements in A.

Indeed it would be sufficient to prove that they are irreducible. Now if for instance

$$t_o=F(t_o,\dots,t_N)G(t_o,\dots,t_N)$$

F and G being homogenous polynomials in S, then by the minimality of N, we get that T_o must occur in at least one of the polynomials F or G. If non of F or G belongs to K, we get that $\deg(FG)>\deg(T_o)$ and since $T_o-FG\in(F_1,\dots,F_s)$ it follows that $T_o\in$ $\in(F_1,\dots,F_s)$, hence $t_o=0$ which contradicts again the minimality of N. Hence F or G is in K and we are done.

<u>Step 2</u>. We have $B_{-n-m}=\{y\in L_{-n-m}\mid yB_nB_m\subseteq B_o\}$

The inclusion "$\subseteq$" is obvious. To prove "$\supseteq$", suppose for instance $f=t_o$, take $y=a/b\in L_{-n-m}$ with $yB_nB_m\subseteq B_o$ such that $a,b\in A$ and fix an index i between 1 and N. Since A is normalised, there exist integers $e_o, e_1, .., e_{i-1}, e_{i+1}, .., e_N$ such that

$$\sum_{\substack{j=0 \\ j\neq i}}^{N} e_j q_j = 1$$

Now there exist integers $h_o, h_1, .., h_{i-1}, h_{i+1}, .., h_N$ such that

$$\sum_{\substack{j=0 \\ j\neq i}}^{N} h_j q_j = 0$$

and $h_o + e_o < 0$, $h_j + e_j > 0$, $j=1,.,i-1,i+1,.,N$. Put $k_j = h_j + e_j$ for $j=0,..N$, $j\neq i$. It follows that the element

$$z = t_1^{k_1} .. t_{i-1}^{k_{i-1}} t_{i+1}^{k_{i+1}} .. t_N^{k_N} / t_o^{-k_o}$$

belongs to B_1. Hence $(a/b)z^{n+m} \in B_o$ and if we write $b = t_o^{r_o} .. t_N^{r_N} c$ where t_j does not divide c for any $j=0,..,N$, it follows that $t_i^{r_i}$ divides a and c divides a too in the ring A. Since this reasoning works for every $i=1,.,N$, we get that b divides $at_o^{r_o}$ in A, hence $a/b \in L_{-n-m} \cap B = B_{-n-m}$ and we are done.

Step 3. There is a canonical isomorphism

$$\varphi : \{ y \in L_{-n-m} \mid yB_nB_m \subseteq B_o \} \longrightarrow \mathrm{Hom}_{B_o}(B_nB_m, B_o)$$

Clearly, φ is injective. To prove that φ is surjective, take $u \in \mathrm{Hom}_{B_o}(B_nB_m, B_o)$; since L_o is the field of quotients of B_o and $B_nB_m \otimes_{B_o} L_o = L_{n+m}$, we may consider the induced map $\bar{u} \in \mathrm{Hom}_{L_o}(L_{n+m}, L_o)$. But since L_{n+m} is an L_o-vector space of dimension one, u must be the multiplication with an element $y \in L_{-n-m}$ and we are done.

Step 4. There is a canonical isomorphism

$$\mathcal{O}_X(-n-m) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X(n)\,\mathcal{O}_X(m), \mathcal{O}_X)$$

In particular, $\mathcal{O}_X(n)$ are reflective.

This comes from the steps 2 and 3 ; the reflectivity follows from the above formula, putting $m=0$.

Step 5. Proof of assertion 2).

The canonical map $\mathcal{O}_X(n)\otimes\mathcal{O}_X(m)|_{\mathrm{Reg}\,X} \longrightarrow \mathcal{O}_X(n+m)|_{\mathrm{Reg}\,X}$ is injective because it is nonzero and its source and adress are invertible. To prove the surjectivity, it would be sufficient to prove that $\mathcal{O}_X(n)\,\mathcal{O}_X(m)|_{\mathrm{Reg}\,X} = \mathcal{O}_X(n+m)|_{\mathrm{Reg}\,X}$. But the sheaves above are both invertible and have the same dual by step 4. Hence they are equal and we are done.

Step 6. Proof of assertions 3) and 4).

Note first that 4) immediately follows from 3) since $\mathrm{Pic}(X) \subseteq \mathrm{Cl}(X)$. To prove 3) it is sufficient to prove that $\mathrm{Pic}(\mathrm{Reg}\,X)$ is cyclic , and it is generated by $\mathcal{O}_X(1)|_{\mathrm{Reg}\,X}$. Consider the morphisms

$$\mathrm{Spec}\,A \setminus \{\underline{m}\} = \mathrm{Spec}(\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n)) \xrightarrow{\ u\ }$$

$$\xrightarrow{\ u\ } \mathrm{Spec}(\bigoplus_{n\geqslant 0} \mathcal{O}_X(n)) \xrightarrow{\ v\ } X$$

where $\underline{m}$ is the irrelevant ideal of A. Put $V=v^{-1}(\mathrm{Reg}\,X)$ and $W=u^{-1}(V)$. By 1) and 2) it follows that V is an $\mathbb{A}^1_K$-bundle over $\mathrm{Reg}\,X$, hence $\mathrm{Pic}(\mathrm{Reg}\,X) \simeq \mathrm{Pic}(V)$.

On the other hand the morphism $W \longrightarrow V$ is the open immersion which corresponds to the excizion of the support of the zero section $0=s \in H^0(\mathrm{Reg}\, X, \mathcal{O}_X(1))$. Since $Cl(W) = Cl(A) = 0$ we get that $Cl(V)$ is generated by the support of s and the assertion 3) is proven.

PROPOSITION 2.4. If X is a quasi-smooth integral weighted complete intersection, then $\mathcal{O}_X(n)$ are Cohen-Macaulay.

Proof. According to [7] it is sufficient to prove that for any closed point $x \in X$, $\mathrm{prof}\, \mathcal{O}_X(n)_x = p = \dim X$. Now by [2] there exists a multiple k of $q_o,..,q_r$ such that $\mathcal{O}_P(k)$ is a very ample invertible sheaf on $P = P(q_o,..,q_r)$ and such that $\mathcal{O}_P(n) \otimes \mathcal{O}_P(mk) = \mathcal{O}_P(n+mk)$ for any integers n,m. Let $i : X \hookrightarrow \Pi$ be the immersion of X into a smooth projective space Π corresponding to the very ample sheaf $\mathcal{O}_X(k)$. Then for any $j \geqslant 0$ we have :

$$H^j(\Pi, (i_* \mathcal{O}_X(n)) \otimes \mathcal{O}_\Pi(-m)) =$$
$$= H^j(\Pi, i_*(\mathcal{O}_X(n) \otimes \mathcal{O}_X(-mk))) = H^j(X, \mathcal{O}_X(n-mk))$$

The last group vanishes for $j=1,..,p-1$ and $m \gg 0$ by [3]. On the other hand $\mathcal{O}_X(n)$ are torsion free, hence $\mathrm{prof}\, i_* \mathcal{O}_X(n)_y \geqslant 1$ for any closed point $y \in \Pi$. Now by [4, Exp. XIII] we get that $\mathrm{prof}\, i_* \mathcal{O}_X(n)_y \geqslant p$ for any closed point $y \in \Pi$ hence $\mathrm{prof}\, \mathcal{O}_X(n)_x \geqslant p$ for any closed point $x \in X$ and we are done.

COROLLARY 2.5. Let X be a weighted complete intersection of dimension $\geqslant 3$, which is normalized, quasi-smooth and normal. Then all its singularities are strongly (S_3).

As it is shown in [3] there are many interesting singularities which appear on such varieties. On the other hand, even the simplest ones fail to be factorial (look for instance at the vertex of the projective cone over a Veronese embedding of a projective space).

3. Grothendieck-Lefschetz preparation. The main idea in proving the result announced in the introduction (see also §4) is to extend every $\mathcal{O}_{Y_i}(n)$ (where Y_i is supposed to be a weighted projective space) to a reflective rank one $\mathcal{O}_{Y_{i+1}}$-module. To do this, we need the preparation below.

We will make use several times of the following:

LEMMA 3.1. If $\overline{W}$ is a noetherian scheme, W is an open subset in $\overline{W}$ and $Z = \overline{W} \setminus W$, and if $\overline{F} \in \mathrm{Coh}(\overline{W})$ with $\mathrm{prof}_Z \overline{F} \geqslant n+2$ for some positive integer n, then

$$H^i(\overline{W},\overline{F}) = H^i(W,F) \qquad \text{for } i=0,\dots,n$$

where $F = \overline{F}|_W$.

The proof of the lemma is standard, using the Leray spectral sequence and the local cohomology sequence.

Now we state the main ingredient of the proof of the result in § 4 :

PROPOSITION 3.2. Let $\overline{Y}$ be an integral K-projective scheme which is an ample Cartier divisor on a normal integral K-projective scheme $\overline{X}$. Put $Y = \mathrm{Reg}\,\overline{Y}$ and let J be the sheaf of ideals of $\overline{Y}$ on $\overline{X}$. Suppose that :

1) $H^i(\overline{Y}, J^m/J^{m+1}) = 0$ for $i=1,2$ and $m \geqslant 1$.

2) $\overline{Y}$ has (R_3) and (S_4)

Then :

a) For any open neighbourhood U of Y in $\overline{X}$, $\mathrm{codim}(\overline{X} \setminus U, \overline{X}) \geqslant 4$; in particular, $\overline{X}$ has (R_3).

b) There is an injective morphism $\theta : \mathrm{Cl}(\overline{X}) \longrightarrow \mathrm{Cl}(\overline{Y})$ and any class in $\mathrm{Cl}(\overline{Y})$ which has (S_3) belongs to the image of θ.

Proof. We proceed in several steps.

Step 1. Proof of a).

Put $Z = \overline{Y} \setminus Y$ = the singular locus of $\overline{Y}$.

Let x be a generic point of an irreducible component of $T = \overline{X} \setminus U$. If $\overline{\{x\}} \cap Z = \emptyset$ then $\overline{\{x\}} \subseteq \overline{X} \setminus \overline{Y}$ which is affine, hence x must be a closed point in $\overline{X}$ and then $\dim \mathcal{O}_{x,X} \geqslant 4$. If there exists a $y \in \overline{\{x\}} \cap Z$ put $A = \mathcal{O}_{y,\overline{X}}$ and let $\underline{p}$, $\underline{a}$ and tA the ideals in A corresponding to $\overline{\{x\}}$, Z and $\overline{Y}$ respectively. Since $\overline{\{x\}} \cap \overline{Y} \subseteq Z$ we get $\mathrm{height}(\underline{p}+tA) \geqslant \mathrm{height}\ \underline{a}$ and since $\mathrm{prof}\ (A/tA)_{\underline{q}} \geqslant 4$ for any $\underline{q} \in \mathrm{Spec}\ A$ with $\underline{q} \supseteq \underline{a}$ we get height $\underline{a} \geqslant 5$. Now since we have $\mathrm{height}(\underline{p}+tA) \leqslant$

$\leq$ height $\underline{p}$ + 1 it follows that $\dim \mathcal{O}_{x,\overline{X}} =$

= height $\underline{p} \geqslant 4$ and we are done.

Put $X = \overline{X} \setminus Z$, let $j : Y \hookrightarrow \overline{Y}$ be the canonical immersion and let $\hat{X}$ be the formal completion of X along Y.

Step 2. $\mathrm{Pic}(\hat{X}) \longrightarrow \mathrm{Pic}(Y)$ is an isomorphism.

By Lemma 3.1 we have $H^i(Y, J^m_{|Y}/J^{m+1}_{|Y}) = H^i(\overline{Y}, J^m/J^{m+1}) =$ $=0$ for $i=1,2$ and $m \geqslant 1$. Now we conclude by [4, Exp XI].

Step 3. For any $F \in LF(X)$ we have an isomorphism $H^o(X,F) \xrightarrow{\sim} H^o(\hat{X},\hat{F})$.

Since we have an exact sequence of the form

$0 \longrightarrow F \longrightarrow E_{1|X} \longrightarrow E_{2|X}$ with $E_1, E_2 \in LF(\overline{X})$ and since H^o is left exact, we may suppose that F itself is the restriction of a sheaf $\overline{F} \in LF(\overline{X})$. For any integer m define $F(mY) = F \otimes \left[\mathcal{O}_{\overline{X}}(m\overline{Y})|_X\right]$ and $F_m = F/J^m_{|Y}F$.

We get an exact sequence

$$H^o(X,F(-mY)) \longrightarrow H^o(X,F) \longrightarrow H^o(X,F_m) \longrightarrow H^1(X,F(-mY))$$

Now by Lemma 3.1 we get $H^i(X,F(-mY)) = H^i(\overline{X},\overline{F}(-m\overline{Y})) = 0$ for $i=1,2$ and $m \gg 0$, hence $H^o(\hat{X},\hat{F}) = \varprojlim_m H^o(X,F_m) =$ $= H^o(X,F)$.

Step 4. We have $\mathrm{Lef}(X,Y)$. In particular, by [4, Exp. XI]

$Pic(U) \longrightarrow Pic(Y)$ is injective for any neighbourhood U of Y in $\overline{X}$.

The proof is standard if we use the preceding steps (see [6]).

<u>Step 5</u>. If $F \in LF(Y)$ then $j_* F \in Coh(\overline{Y})$; if in addition $R^1 j_* F = 0$ then

$$H^i(Y,F(mY)) = \begin{cases} \text{finitely dimensional over } K \text{ for } i=0,1 \\ 0 \quad \text{for } i=0,1 \text{ and } m \ll 0 \\ 0 \quad \text{for } i=1 \text{ and } m \gg 0 \end{cases}$$

The fact that $\overline{F} = j_* F \in Coh(Y)$ follows from [4, Exp.VIII] . Now if $R^1 j_* F = 0$ we get by lemma 3.1 that $H^i(Y,F(mY)) = H^i(\overline{Y},\overline{F}(m\overline{Y}))$ for $i=1,2$ and we are done since prof $\overline{F}_x \geqslant 2$ for any closed point $x \in \overline{Y}$.

<u>Step 6</u>. If $\mathcal{F} \in LF(\hat{X})$, $F = \mathcal{F} \otimes \mathcal{O}_Y$ and $R^1 j_* F=0$ then $H^i(\hat{X}, \mathcal{F})$ has finite dimension over K for $i=0,1$.

Put $F_m = \mathcal{F} / J^m_{|Y} \mathcal{F}$ for $m \geqslant 1$ and consider the exact sequence $0 \to F(-mY) \to F_{m+1} \to F_m \to 0$ which gives an exact sequence

$$H^i(Y,F(-mY)) \longrightarrow H^i(X,F_{m+1}) \xrightarrow{u^i_m} H^i(X,F_m) \longrightarrow$$
$$\longrightarrow H^{i+1}(Y,F(-mY))$$

By Step 5, u^0_m is bijective for $m \gg 0$ and u^1_m is injective for $m \gg 0$. Hence $H^0(\hat{X}, \mathcal{F}) = \varprojlim_m H^0(X,F_m) = H^0(X,F_{m_0})$ for some m_0. Since $\{H^0(X,F_m)\}_m$ sasfies the Mittag-Leffler condition, it follows by 5, 13.3.1] that

$$H^1(\hat{X}, \mathcal{F}) = \varprojlim_m H^1(X, F_m) = \bigcap_{m \geqslant m_1} H^1(X, F_m) \subseteq$$

$$\subseteq H^1(X, F_{m_1}) \quad , \text{ for some } m_1 .$$

would be sufficient to prove that $H^i(X,F_m)$ have inite dimension over K for $i=0,1$ and $m \geqslant 1$. We roceed by induction on m. If $m=1$ this follows rom Step 5. The induction step follows from the xact sequence above.

Step 7. If $\mathcal{F} \in LF(\hat{X})$, $F = \mathcal{F} \otimes \mathcal{O}_Y$ and $R^1 j_* F = 0$ hen $\mathcal{F}$ is algebraisable (i.e. there exists $G \in$ $\in Coh(X)$ such that $\hat{G} \cong \mathcal{F}$). Consequently (by a tandard trick, see [6]) there exists a neighbour-ood U of Y in $\overline{X}$ and $E \in LF(U)$ such that $\hat{E} \cong \mathcal{F}$.

To prove this, it is sufficient to prove that there xists an exact sequence in $LF(X)$

$$(*) \qquad 0 \longrightarrow \mathcal{M} \longrightarrow \hat{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

ith $E \in LF(X)$ and $R^1 j_* M = 0$ where $M = \mathcal{M} \otimes \mathcal{O}_Y$. In-deed, since $\mathcal{M}$ satisfies again the hypothesis of Step 7, one finds an exact sequence $\hat{L} \to \hat{E} \to \mathcal{F} \to 0$ with $L \in LF(X)$ and we may conclude by a standard rick (see [6]).

To prove the existence of the exact sequence ($*$) consider the exact sequence

$$0 \longrightarrow \mathcal{F}((m-1)Y) \longrightarrow \mathcal{F}(mY) \longrightarrow F(mY) \longrightarrow 0$$

where $\mathcal{F}(pY) = \mathcal{F} \otimes \left[\mathcal{O}_X(pY)^\wedge\right]$ for any integer p, and a piece of the long exact sequence :

$$H^o(\hat{X}, \mathcal{F}(mY)) \xrightarrow{u_m} H^o(Y, F(mY)) \longrightarrow H^1(\hat{X}, \mathcal{F}((m-1)Y)) \longrightarrow$$

$$\xrightarrow{v_m} H^1(\hat{X}, \mathcal{F}(mY)) \longrightarrow H^1(Y, F(mY))$$

By Step 5, $H^1(Y,F(mY)) = 0$ for $m \gg 0$, hence v_m is surjective for $m \gg 0$. By the adjunction formula, $R^1 j_*(F(mY)) = (R^1 j_* F)(mY) = 0$, hence we may apply Step 6 to $\mathcal{F}(mY)$ and we get that for $m \gg 0$, the sequence $\{\dim_K H^1(\hat{X}, \mathcal{F}(mY))\}_m$ is a descending sequence of natural numbers, so it must become constant for large m. Consequently, u_m becomes surjective for $m \geqslant m_o$ for a certain m_o. Since by Step 5, $\bar{F} = j_* F$ is coherent, there exists a $m \geqslant m_o$ such that $\bar{F}(m\bar{Y})$ is generated by a finite number of global sections $s_1, \ldots, s_k \in H^o(\bar{Y}, \bar{F}(m\bar{Y}))$ giving an exact sequence

$$0 \longrightarrow \bar{N} \longrightarrow \mathcal{O}^k_{\bar{Y}} \xrightarrow{f} \bar{F}(m\bar{Y}) \longrightarrow 0$$

Since $\mathrm{prof}_Z\, \bar{F}(m\bar{Y}) \geqslant 2$ and $\mathrm{prof}_Z\, \mathcal{O}_{\bar{Y}} \geqslant 3$ we get by the local cohomology sequence that $R^1 j_* N = 0$, where $N = \bar{N}|_Y$. Selecting $t_1, \ldots, t_k \in H^o(\hat{X}, \mathcal{F}(mY))$ such that $u_m(t_i) = s_i|Y$ for $i=1,\ldots,k$ we obtain a morphism

$$\mathcal{O}^k_{\hat{X}} \xrightarrow{\varphi} \mathcal{F}(mY)$$

such that $\varphi \otimes \mathcal{O}_Y = f|_Y$. By Nakayama's lemma, φ must be surjective and if we set $\mathcal{N} = \ker \varphi$, we obviously get $\mathcal{N} \otimes \mathcal{O}_Y = N$. Now we are done by putting $\mathcal{M} = \mathcal{N}(-mY)$ and $E = \mathcal{O}_X(-mY)$.

Finally, if we look at the steps 2,4 and 7 we get an injective morphism

$$\mathrm{Cl}(\overline{X}) = \varinjlim_U \mathrm{Pic}(U) \longrightarrow \mathrm{Pic}(Y) = \mathrm{Cl}(\overline{Y})$$

(where U runs through the set of all open neighbourhoods of Y in $\mathrm{Reg}\, X \setminus Z$) whose image contains every class in $\mathrm{Cl}(\overline{Y})$ which has (S_3). The proposition is proven.

COROLLARY 3.3. Under the hypothesis of Proposition 3.2, if we assume that $\overline{Y}$ has only strrongly (S_3) singularities, then there is an isomorphism

$$\theta : \mathrm{Cl}(\overline{X}) \xrightarrow{\sim} \mathrm{Cl}(\overline{Y})$$

REMARK 3.4. The morphism θ from (3.2) and (3.3) is a "natural" one since it comes from the restriction morphism $\mathrm{Pic}(\mathrm{Reg}\, X) \longrightarrow \mathrm{Pic}(\mathrm{Reg}\, Y)$.

4. Main result. We will prove now the following :

THEOREM 4.1. Suppose $\overline{Y}$ is an ample Cartier divisor on a normal projective variety $\overline{X}$ over a field K. If $\overline{Y} = P(q_o,..,q_r) = \mathrm{Proj}\, K[T_o,..,T_r]$ with $\deg T_i = q_i$ for $i=0,..,r$ and if $\overline{Y}$ is normalised with $\mathrm{codim}(\mathrm{Sing}\, \overline{Y}, \overline{Y}) \geqslant 4$, then there exists a natural q_{r+1} such that $\overline{X}$ is isomorphic to $P(q_o,..,q_r,q_{r+1}) = \mathrm{Proj}\, K[T_o,..,T_r,T_{r+1}]$ with $\deg T_i = q_i$ for $i=0,..,r+1$, $\overline{X}$ is normalised and $\mathrm{codim}(\mathrm{Sing}\, \overline{X}, \overline{X}) \geqslant 4$. Furthermore the inclusion $\overline{Y} \subset \overline{X}$

corresponds to the natural surjection $K[T_o,..,T_{r+1}] \to$ $\to K[T_o,..,T_r]$ which sends T_{r+1} into zero and leaves T_i $(i \leq r)$ fixed.

COROLLARY 4.2. Suppose $Y_o \subset Y_1 \subset Y_2 \subset \cdots$ is a sequence of normal projective varieties over a field K such that Y_o is a smooth projective space of dimension $d \geqslant 3$ and each Y_{i-1} is an ample Cartier divisor on Y_i for $i \geqslant 1$. Then there exists a sequence of natural numbers $q_1, q_2, \ldots$ such that each Y_i with $i \geqslant 1$ is isomorphic to

$$P(1,\ldots,1,q_1,..,q_i) = \mathrm{Proj}\ K[T_o,..,T_d,T_{d+1},..,T_{d+i}]$$

where $\deg T_o = .. = \deg T_d = 1$ and $\deg T_{d+j} = q_j$ for $j \geqslant 1$. Furthermore, for $i \geqslant 1$ the inclusion $Y_{i-1} \subset$ $\subset Y_i$ corresponds to the natural surjection $K[T_o,..,T_{d+i}] \to K[T_o,..,T_{d+i-1}]$ which sends T_{d+i} into zero and leaves T_k $(k \leq d+i-1)$ fixed.

Proof of the theorem. Let $f : \overline{Y} \hookrightarrow \overline{X}$ be the canonical immersion, $Y = \mathrm{Reg}\,\overline{Y}$, $Z = \mathrm{Sing}\,\overline{Y}$, $X = \overline{X} \setminus Z$ and j be the open immersion $Y \hookrightarrow P = \overline{Y}$. By [3] we get that $\mathrm{Pic}(Y) \simeq \mathbb{Z}$ and is generated by some $\mathcal{O}_P(s)$. Hence $f^*\mathcal{O}_{\overline{X}}(\overline{Y}) = \mathcal{O}_P(t)$ for some $t \geqslant 1$. We get by [3] that $H^i(\overline{Y}, \mathcal{O}_{\overline{Y}}(-m\overline{Y})) = H^i(P, \mathcal{O}_P(-mt)) = 0$ for $i=1,2$ and $m \in \mathbb{Z}$, hence the condition 1) in Proposition 3.2 is fulfilled. The condition 2) in Proposition 3.2 follows from the fact that P is Cohen-Macaulay. Since $\mathcal{O}_P(1)$ is reflective of rank one (by Proposition 2.3) and since it has (S_3) (by Proposition 2.4) we get by

Proposition 3.2 that there exists an open neighbouhood U of Y in $\mathrm{Reg}\, X \setminus Z$ and there exists a sheaf $L \in P(U)$ such that $L|_Y = \mathcal{O}_P(1)|_Y$. By Proposition 3.2 we also have $\mathrm{codim}(\overline{X} \setminus U, \overline{X}) \geqslant 4$ and $\mathrm{Pic}(U) \longrightarrow \mathrm{Pic}(Y)$ is injective. If $h: U \hookrightarrow \overline{X}$ is the canonical immersion, put for any integer m, $F^{(m)} = h_*(L^{\otimes m})$. By [4, Exp.VIII] we get that $F^{(m)} \in \mathrm{Coh}(X)$ and $\mathrm{prof}_{\overline{X} \setminus U} F^{(m)} \geqslant 2$. The following statements hold:

1) $F^{(m)}|_{\overline{Y}} = \mathcal{O}_P(m)$ for any integer m.

Indeed, since $Z \subset \overline{X} \setminus U$, it follows that $F^{(m)}|_Y$ has $\mathrm{prof} \geqslant 2$ in Z. On the other hand this holds also for $\mathcal{O}_P(m)$. Since $F^{(m)}|_{\overline{Y}}$ and $\mathcal{O}_P(m)$ are coherent, in order to check 1) it is sufficient to prove that $F^{(m)}/_Y = \mathcal{O}_P(m)/_Y$. But $F^{(m)}|_Y = L|_Y^{\otimes m} = \mathcal{O}_P(1)|_Y^{\otimes m} = \mathcal{O}_P(m)|_Y$, by Proposition 2.3.

2) $F^{(mt)} = \mathcal{O}_{\overline{X}}(m\overline{Y})$ for any integer m.

Since these sheaves are coherent and have $\mathrm{prof} \geqslant 2$ in $\overline{X} \setminus U$, it is sufficient to prove that their restrictions at U are equal. But since their restrictions at U are invertible and $\mathrm{Pic}(U) \longrightarrow \mathrm{Pic}(Y)$ is injective, it is sufficient to prove that $L|_Y^{\otimes mt} = \mathcal{O}_{\overline{X}}(m\overline{Y})|_Y$. But this follows immediately by Proposition 2.3.

3) If m, q, p are integers such that $m = qt + p$, then $F^{(m)} = F^{(p)} \otimes \mathcal{O}_{\overline{X}}(q\overline{Y})$.

This follows from 2) and from the equality $F^{(m)} = F^{(p)} \otimes F^{(qt)}$, which is valid since it is valid on U.

In particular, exactly as in [1] we get that

$H^1(F^{(m)}) = 0$ for $m \ll 0$. Now the exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{X}}(-\bar{Y}) \xrightarrow{\sigma} \mathcal{O}_{\bar{X}} \longrightarrow \mathcal{O}_{\bar{Y}} \longrightarrow 0$$

$\sigma \in H^0(\bar{X}, \mathcal{O}_{\bar{X}}(\bar{Y}))$, yelds an exact sequence

$$F^{(m)} \otimes \mathcal{O}_{\bar{X}}(-\bar{Y}) = F^{(m-t)} \xrightarrow{\sigma} F^{(m)} \longrightarrow \mathcal{O}_P(m) \longrightarrow 0$$

Since in any point $x \in X$, the morphism $F_x^{(m-t)} \longrightarrow F_x^{(m)}$ may be identified with the multiplication with $\sigma_x \neq 0$ in $F_x^{(m)}$ and since $F^{(m)}$ are torsion free, we get that $F^{(m-t)} \longrightarrow F^{(m)}$ is injective. Hence we get an exact sequence $0 \longrightarrow F^{(m-t)} \longrightarrow F^{(m)} \longrightarrow \mathcal{O}_P(m) \longrightarrow 0$. Taking the long cohomology sequence and taking into account that $H^1(\mathcal{O}_P(m)) = 0$ for $m \in \mathbb{Z}$ (see [3]) we get that $H^1(F^{(m-t)}) \longrightarrow H^1(F^{(m)})$ is surjective for any $m \in \mathbb{Z}$. We obtain an exact sequence of graded algebras

$$0 \longrightarrow B \xrightarrow{\sigma} B \longrightarrow \bigoplus_{m \geq 0} H^0(P, \mathcal{O}_P(m)) \longrightarrow 0$$

where

$$B = \bigoplus_{m \geq 0} H^0(X, F^{(m)}) = \bigoplus_{m \geq 0} H^0(U, L^{\otimes m}).$$

By [3] we have

$$\bigoplus_{m \geq 0} H^0(P, \mathcal{O}_P(m)) = K[T_0,..,T_r] = S$$

with $\deg T_i = q_i$ for $i=0,..,r$. Now it is a standard fact that this situation implies that B is isomorphic as a graded algebra with $S[T]$, $\deg T = t$. Since $X \cong \operatorname{Proj} B$, the theorem is proven.

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