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by

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Abstract. We consider smooth endomorphisms $f:M \to M$, M being a compact connected manifold for which $\|Df_x v\| \geq \|v\|$ in some Riemannian norm, where equality is possible only in a finite invariant subset of M. We show, using Bowen's construction, that such f have dense periodic points and Markov partitions of small diameters.

Let M be a compact conected smooth manifold and let $f:M \to M$ be a C^1-map of M onto itself. We say that f is nonstrictly expanding if for some Riemannian metric on M, there exist a finite invariant set of points $A \subset M$ such that $\|Df_x v\| \geq \|v\|$ for all $x \in M$ and $v \in T_x M$, where equality is possible only for $x \in A$. Let us denote by $\mathcal{N}_s$ the class of such mappings.

Concerning $\mathcal{N}_s$ we shall show that analoguous results with the strictly expanding case are valid. The main tool is the technique of tracing pseudo-orbits, developed by Bowen ([B]).

Let us recall it:

Definition. Let X be a metric space and $f:X \to X$ a function. We say that f has the pseudo-orbit tracing property (p.a.t.p) if for every $\alpha > 0$ sufficiently small, there exist $\beta > 0$ such that for every sequence $(x_n)_{n \geq 0} \subset X$ satisfying $d(fx_n, x_{n+1}) < \beta$ for all $n \geq 0$ we can find $x \in M$ such that:

$$d(f^n_x, x_n) < \alpha \quad \text{for every} \quad n \geq 0.$$

Our first result is:

Proposition 1. Every f in $\mathcal{N}_s$ has the p.o.t.p., the tracing orbit being unique if α is sufficiently small.

The metric on M is the induced one by the Riemannian structure considered in the definition of $\mathcal{N}_s$.

Allthough the proof is simple we shall do it by using the following two lemmas.

Lemma 1. Let X be a compact metric space and $f: X \to X$ a continuous covering map with the following property (*): "*There is $\varepsilon > 0$ such that for every $\alpha < \varepsilon$ there exists $\beta > 0$ for which $\varepsilon > d(x,y) > \alpha$ implies* $d(fx, fy)$ $\alpha + \beta$". Then f has the p.o.t.p.

Proof. Let α be as in the p.o.t.p., and let $(x_n)_{n \geq 0}$ be a pseudo-orbit; $d(fx_n, x_{n+1}) < \beta$ for all $n \geq 0$.

We denote by $\bar{B}(x_n, \alpha)$ the closed balls centered at x_n and of radius α. One can suppose that α is so small that by the covering property f^{-1} is defined in the usual sense on every $\bar{B}(x_n, \alpha)$. We shall show that if β is small then the component of $f^{-1}\bar{B}(x_n, \alpha)$ which intersects $\bar{B}(x_{n-1}, \alpha)$ is in fact contained in it.

For every $x \in \bar{B}(x_n, \alpha)$ we have $d(x, fx_{n-1}) \leq \alpha + \beta$ and if β is as in property (*) it follows that $d(f\bar{x}^{1}, x_{n-1}) \leq \alpha$.

The point that traces our pseudo-orbit is obtained by taking a limit point of the sequence $f\bar{x}^n_n$ in $\bar{B}(x_o, \alpha)$.

Lemma 2. Functions in $\mathcal{N}_s$ have property (*).

Proof. The property imposed on the derivative implies the covering property and that there exists $\varepsilon > 0$ such that if $d(x,y) < \varepsilon$ (Riemannian metric) then $d(fx,fy) \geq d(x,y)$. Let $\gamma > 0$ such that $2\gamma < \alpha$ and $A_\gamma = \{x \in M \mid \exists\, y \in A,\ d(x,y) < \frac{\gamma}{n}\}$, n being the cardinality of A. By compactness there exists $\lambda > 1$ such that $\|Df_x v\| \geq \lambda \|v\|$ for all $x \in M \setminus A_\gamma$, $v \in T_x M$. (1)

Let $x,y \in M$ such that $\varepsilon > d(x,y) > \alpha$ and γ a geodesic from fx to fy. It is the image (if ε is small) of a path γ_1 from x to y. But the lenght of the portions of γ_1 that lie outside A_γ is greater then $\alpha - 2\gamma$ and by (1) we obtain

$$\ell(\gamma) = \int \|Df_{\gamma_1(t)} \dot{\gamma}_1(t)\| dt = \int_{\gamma_1 \in A_\gamma} + \int_{\gamma_1 \notin A_\gamma} \geq$$

$$\geq \int_{\gamma_1 \in A_\gamma} \|\dot{\gamma}_1(t)\| dt + \lambda \int_{\gamma_1 \notin A_\gamma} \|\dot{\gamma}_1(t)\| dt \geq \ell(\gamma_1) + (\lambda - 1)(\alpha - 2\gamma) >$$

$$> \alpha + (\lambda - 1)(\alpha - 2\gamma)$$

We can thus take $= (\ -1)(\ -2)$.

By lemmas 1 and 2 the first part of proposition 1 follows. To prove unicity let us note that if x and x′ are two points that α-trace the same pseudo-orbit we have $d(f^n x, f^n x') < 2\alpha$ for every $n \geq 0$, and from the easy seen fact that $f \in \mathcal{N}_s$ has an expansive constant ε_0 it follows that for $2\alpha < \varepsilon_0$ we have $x = x'$.

In order to state the first consequence, we recall the following simple result from [Sh], obviously true in our case also.

Lemma 3. If $f \in \mathcal{N}_s$ and $U \subset M$ is open then $\bigcup_{n=1}^{N} f^n(U) = M$ for some $N \in \mathbb{N}$.

Proposition 2. For $f \in \mathcal{N}_s$ the periodic points are dense in M.

Proof. Let $\gamma > 0$ small be given and let $U=B(x,\gamma)$ be the open ball of radius γ centered at x. From lemma 3 we find $n \in \mathbb{N}$ such that $U \cap f^n(U) \neq \emptyset$, that is there exists $y=f^n z \in U$, $z \in U$ with: $d(y,x) < \gamma$, $d(z,x) < \gamma$. This implies $d(f^n z, z) < 2\gamma$. Let $\alpha < \frac{\varepsilon_0}{2}$, and choose γ such that $2\gamma < \beta(\alpha)$ and $\gamma < \alpha$. Consider the 2γ-pseudo-orbit $(x_k)_{k \geq 0}$ defined as follows:

$$z,\ fz, \ldots, f^{n-1}z,\ z \ldots, f^{n-1}z,\ z, \ldots$$

which is traced by x'. It follows that:

$$d(f^k x', f^k f^n x') \leq d(f^k x', x_k) + d(x_k, f^{k+n} x') < 2\alpha < \varepsilon_0$$

where $x_k = f^{k(\mathrm{mod}\ n)} z$.

By expansivity we must have $x' = f^n x'$ and it follows also that $d(x',x) < \alpha + \gamma < 2\alpha$, thus finishing the proof.

The main result is:

Proposition 3. Every $f \in \mathcal{N}_s$ has Markov partitions of arbitrary small diameters, that is for $\alpha' > 0$ there is a cover $\{P_1, \ldots, P_s\}$ of M by closed sets that intersect only in their boundaries $\mathrm{diam}(P_i) < \alpha'$ for all i, $M \setminus \bigcup_i \partial P_i$ is open and dense, $f(\bigcup_i \partial P_i) \subset \bigcup_i \partial P_i$ and $\mathrm{int}\, P_i \cap f(\mathrm{int}\, P_j) \neq \emptyset$ implies $P_i \subset f P_j$.

Proof. As the proof consists in nothing but adapting Bowen's proof for Markov partitions in [B] let us only sketch it.

Let $\alpha < \frac{\alpha'}{2}$, $\alpha < \frac{\varepsilon_0}{2}$ and β be as in proposition 1 and choose $\gamma < \frac{\beta}{2}$ such that $d(x,y) < \gamma$ implies $d(fx, fy) < \frac{\beta}{2}$. Let $\mathcal{P} = \{p_1, \ldots, p_k\}$ be a γ-dense set in M and consider the space:

$$\Sigma(\mathcal{P})=\{q=(q_n)_{n\geq 0} \mid d(fq_n,q_{n+1})<\beta \text{ for every } n\geq 0\}$$

By proposition 1, for every $q\in\Sigma(\mathcal{P})$ there is a unique $x\in M$ which α-traces (q_n).

Conversely for $x\in M$, consider $p_{i_o}\in\mathcal{P}$ such that $d(x,p_{i_o})<\gamma$ which implies by our assumptions that $d(fx,fp_{io})<\frac{\beta}{2}$, that is $d(fp_{i_o},p_{i_1})<d(fx,p_{i_1})+d(fx,fp_{io})<\gamma+\frac{\beta}{2}<\beta$ if p_{i_1} is choosen such that $d(fx,p_{i_1})<\gamma$, a.s.o.

In this manner we obtain $(q_n)_{n\geq 0}\in\Sigma(\mathcal{P})$ such that the trajectory tracing $(q_n)_{n\geq 0}$ is $(f^n_x)_{n\geq 0}$.

Let us denote by θ the correspondence constructed above $\theta:\Sigma(\mathcal{P})\to M$. It is easy seen by expansivity of f that θ is continuous.

Consider the 0-cillinders in $\Sigma(\mathcal{P})$:

$$C_i=\{q \mid q_o=p_i\}$$

and let $A_i=\theta(C_i)$ which are closed sets by continuity of θ. It is easy seen that $M\setminus\cup\partial A_i$ is open and den e.

By considering all intersections of sets of the form int $(A_i\cap A_j)$, int $(A_i\cap CA_j)$ and taking their closures, a little analysis involving θ show that the cover obtained in this manner has all the anounced properties.

Remarks

1. It is easy seen that for $f:M\to M$, for which the set of points where $\|Dfv\|\geq\|v\|$ nonstrictly is a set of diameter $<\alpha_o$ the p.o.t.p. property is valid for $\alpha>\alpha_o$ and when the expansive constant ε_o satisfies $\varepsilon_o>2\alpha_o$ it follows also unicity.

Some weaker versions of the results in the paper are then valid.

2. Have $f \in \mathcal{N}_s$ absolutely continuous strictly positive invariant measures ? See for instance [W].

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