INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250-3638

APPROXIMATION PROPERTIES AND EXISTENTIAL COMPLETENESS FOR RING MORPHISMS

by

Serban BASARAB, Vasile NICA and Dorin POPESCU

PREPRINT SERIES IN MATHEMATICS
No. 9/1980

BUCURESTI

APPROXIMATION PROPERTIES AND EXISTENTIAL COMPLETENESS FOR RING MORPHISMS

by

Şerban BASARAB*), Vasile NICA*) and Dorin POPESCU**)

March 1980

*) Institute of Mathematics, Academiei 14, Bucharest, Romania

**) Department of Mathematics, National Institute for Scientific and Technical Creation, Bd.Pacii 220, 77538 Bucharest, Romania

APPROXIMATION PROPERTIES AND EXISTENTIAL COMPLETENESS FOR RING MORPHISMS

by

Serban Basarab, Vasile Nica and Dorin Popescu

Contents:

§1. Introduction

Let L be a system of linear equations with coefficients in the field $\mathbb{Q}$ of rational numbers. Then L has solutions in some field K of characteristic zero iff it has rational solutions. This is a property of linear saturation for $\mathbb{Q}$. In the algebraic case, a system of polynomial equations with complex coefficients has a solution in some extension K of $\mathbb{C}$ iff it has a solution in the field $\mathbb{C}$ of complex numbers. This means a property of algebraic saturation for $\mathbb{C}$. These types of saturation properties are very stronc because they are reffering to each extension of the base field. It is also interesting to investigate weaker saturation properties relatively to a given extension. For instance the pure extensions in the linear case [16], and algebraically pure extensions in the algebraic case [21].

In the algebraic case, there are known noetherian local rings A which have algebraic saturation property with respect to their completions $\hat{A}$ (see [1], [20], [15]). These rings are so called AP-rings, i.e. rings with the approximation property. More precisely, a noetherian complete local ring (A,m) is called an AP-ring, if every " formal " solution (in $\hat{A}$) of an arbitrary finite system of polynomial equations over A can be well approximated by " algebraic " solutions (in A) with respect to the $m\hat{A}$-adic topology of $\hat{A}$. In fact, A is an AP-ring iff the morphism $A \longrightarrow \hat{A}$ is algebraically pure, using the terminology from [21].

In [15], Chap.V and [20] there are investigated some properties which are preserved from A to $\hat{A}$ if A is an AP-ring. These properties can be expressed in the considered context by the compatibility of some systems of polynomial equations. For instance, A is reduced, respectively integral domain iff $\hat{A}$ is so. However, this cannot be done for arbitrary algebraically pure morphisms. As an example, the morphism $\mathbb{R} \longrightarrow \mathbb{R}[[X]]/(X^2)$ is algebraically pure but does not preserve the property of being reduced. The reason is that the proofs from [15] and [20] use the possibility to approximate well the solutions from $\hat{A}$ of an arbitrary system of polynomial equations over A by solutions from A. This is not possible for arbitrary algebraically pure morphisms. However, the corollary 2.6 in [21] shows that if A is a noetherian complete local ring and B is a Cohen A-algebra such that the residue field of B is an ultrapower of the residue field of A, then the solutions from B of an arbitrary system of polynomial equations over A can be " well approximated " by solutions from A. In particular, A is reduced, respectively integral domain iff B is so [21] Proposition 2.10.

Trying to define more precisely the afore mentioned " good approximation " we remark that we must consider not only finite systems of polynomial equations, but more complex fomulas containing quantifiers. Thus it seems convenient to use for this purpose some model-theoretic concepts and methods. It is well-known that the model-theoretic methods are very useful in analysing algebraic questions about certain classes of rings, for instance henselian valued rings [2], [11] , [14] , [28] , [30], [3] .

In the present work, which is an extended version of [19] , we define some types of "good approximations " using the model-theoretic concept of existential completeness in adequate formal languages extending the first order language of rings (Section 4). Then we apply the general theory given in Section 4 to the case of AP-rings (Section 5). As a consequence, we extend the remark (2.18)iv) from [21] showing that for a formally smooth morphism $u:A \to B$ between noetherian complete local rings, u is algebraically pure iff the residue field extension is algebraically pure (5.7). The reader interesting in pure algebraic proofs of some results of this work, can see 19 .

§ 2. Algebraically and analytically pure morphisms

Algebraically and analytically pure morphisms were introduced in [21] in connection with the study of rings which have certain approximation properties. These concepts generalize the linear case of pure module morphisms [16] .

(2.1) Let $(A,\underline{m})$ be a noetherian local ring and $\hat{A}$ the completion of A with respect to the $\underline{m}$-adic topology. Let us consider the following properties:

(AP) For every system of polynomials $F=(F_1,\dots,F_m)\in A[Y]^m$,

$Y = (Y_1,\dots,Y_n)$, for every " formal " solution $\hat{y}\in\hat{A}^n$ of F and for every natural number c, there exists a solution $y\in A^n$ of F such that $y\equiv\hat{y} \mod \underline{m}^c\hat{A}$.

(SAP) For every system of polynomials $F=(F_1,\dots,F_m)\in A[Y]^m$, $Y=(Y_1,\dots,Y_n)$, there exists a function $\nu : \mathbb{N}\longrightarrow\mathbb{N}$ with the following property:

" For every natural number c, for every $\bar{y}\in A^n$, if $F(y)\equiv 0 \mod \underline{m}^{\nu(c)}$ then there exists $y\in A^n$ such that $F(y)=0$ and $y\equiv\bar{y} \mod \underline{m}^c$ ".

It was established in [20] (see also [15], ch.II for an improved proof) that (AP) and (SAP) are equivalent for a noetherian local ring A. A noetherian local ring A is called an AP-ring (i.e. a ring with approximation property) if A satisfies the equivalent conditions (AP) and (SAP).

(2.2) Definition. Let A and B be commutative rings with identity. A ring morphism is called algebraically pure, or A is algebraically pure in B, if every finite system of polynomials $F=(F_1,\dots,F_m)$ with coefficients in A, in an arbitrary number of variables $Y=(Y_1,\dots,Y_n)$ has a solution in A iff it has a solution in B.

(2.3) Properties and examples. i) If A is a local noetherian ring, then the completion morphism $A\longrightarrow\hat{A}$ is algebraically pure iff A is an AP-ring. Moreover, if A is an integral domain then $\hat{A}$ is too and the fraction field extension $Q(A)\hookrightarrow Q(\hat{A})$ is algebraically pure.

ii) The class of algebraically pure morphisms is stable under composition and base change. In addition, if $v\circ u$ is algebraically pure, then u is so.

iii) If $u: A\longrightarrow B$ is a finite presentation morphism, then u is algebraically pure iff u has an A-algebra retract. In particular, if K is a field, and B a finite type K-algebra, the structure morphism $K \longrightarrow B$ is algebraically pure iff Spec (B) has a closed K-rational point. If B is an integral domain and $K \hookrightarrow B$ is algebraically pure, then K is algebraically closed in B.

iv) More generally, an arbitrary ring morphism $u: A \to B$ is algebraically pure iff B is a filtered inductive limit of algebraically pure A-Algebras or a filtered inductive limit of A-algebras such that their structure morphism have retracts.

The afore result furnishes some examples of algebraically pure morphisms.

v) If K is an algebraically closed field, and B an arbitrary K-algebra then the structure morphism $K \hookrightarrow B$ is algebraically pure.

vi) If K is an infinite field and X a variable, then the morphism $K \hookrightarrow K(X)$ is algebraically pure. More generally, any pure transcendental extension of an infinite field is algebraically pure.

vii) Any algebraically pure field extension of a finite field is trivial.

viii) The algebraically pure morphisms are not in general flat. For instance, $A \longrightarrow A[\![X]\!]$ can be not flat (nonnoetherian case), but it is algebraically pure (having a retract).

All these properties and examples are given in [21].

In the case of complete rings, the concept of algebraically pure morphism is extended as follows:

(2.4) Definition. A local morphism of noetherian complete local rings $u: A \longrightarrow B$ is called analytically pure if every system $F=(F_1,\ldots,F_m)$ of formal power series in $A[\![Z]\!][Y]$, where $Z=(Z_1,\ldots,Z_e)$, $Y=(Y_1,\ldots,Y_n)$ are variables, has a solution (z,y) in A iff it has a solution $(\bar{z},\bar{y})$

in B. (Obviously, the components of $z, \bar{z}$ belong to the correspon-ding maximal ideals of A and B).

(2.4.1) Note that for $e=0$, we recover the algebraic case of (2.1). Also in the case of artinian local rings, both definitions coincide. Moreover, both defintions coincide if the maximal ideal of A generates the maximal ideal of B. To show this we need the following result.

(2.5) <u>Theorem</u> (Pfister-Popescu theorem). Let $(A,\underline{m})$ be a noetherian complete local ring $F=(F_1,\dots,F_m)$ a system of formal power series in $A[[Z]][Y]$ where $Z=(Z_1,\dots,Z_e)$, $Y=(Y_1,\dots,Y_n)$ are variables. Then there exists a function $\nu : \mathbb{N}\longrightarrow\mathbb{N}$ with the following property:

" For every natural number c, for every $\bar{z}\in\underline{m}A^e$ and for every $\bar{y}\in A^n$, if $F(\bar{z},\bar{y})\equiv 0 \bmod \underline{m}^{\nu(c)}$ then there exist $z\in\underline{m}A^e$ and $y\in A^n$ such that $F(z,y)=0$ and $z\equiv\bar{z}$, $y\equiv\bar{y} \bmod \underline{m}^c$".

For the proof of this theorem see [20] Theorem 2.5 or [15], ch.II, Theorem 1.4. Another proof, involving ultrapower, was given recently by Denef and Lipshitz in [9]. A nonstandard form of (2.5) was proved by Popescu in [21], Theorem 2.5 and Corollary 2.6. In Section 3 we shall give a nonstandard equivalent of (2.5), which may be seen as a more general form of the non-standard equivalent considered in [21].

(2.6) <u>Propositon</u>. Let $u: A\longrightarrow B$ be an algebraically pure morphism of noetherian local rings. Assume that the maximal ideal $\underline{m}$ of A generates the maximal ideal $\underline{n}$ of B (it is enough to ask for $\underline{m}B$ to be a $\underline{n}$-primary ideal). Then the induced morphism $\hat{u}: \hat{A}\longrightarrow\hat{B}$ is analytically pure. In particular, if A and B are complete then u is algebraically pure iff u is analytically pure.

<u>Proof.</u> Let F be a system of formal power series in $\hat{A}[[Z]][Y]$ which has a solution in $\hat{B}$. By (2.5), there exists a natural number c such that F has a solution in $\hat{A}$ iff it has a solution in $\hat{A}/\underline{m}^c\hat{A}$. However the system F has a solution in $B/\underline{m}^c B \cong \hat{B}/\underline{m}^c\hat{B}$ and thus it has one in $\hat{A}/\underline{m}^c\hat{A} \cong A/\underline{m}^c$ because the induced morphism of artinian local rings $A/\underline{m}^c \longrightarrow B/\underline{m}^c B$ is analytically pure by (2.3) ii) and (2.4.1).

Q.E.D.

(2.7) Let us remark that no relation between the solution $(\bar{z},\bar{y})$ of F in B and the solution (z,y) of F in A is considered in Definition (2.4). Thus, the following question arises: if F has a solution $(\bar{z},\bar{y})$ in B and it is known that F has solutions in A, one can find in A a solution (z,y) which is "near" to $(\bar{z},\bar{y})$ in B? For this, it is necessary to consider a suitable concept of "nearness" between the solutions $(\bar{z},\bar{y})$ and (z,y). One of the possible ways to introduce such a concept is the following one:

Suppose that the solution $(\bar{z},\bar{y})$ in B satisfies the condition: ord $G_j(\bar{z},\bar{y})=c_j$, $j=1,\dots,p$, briefly ord $G(\bar{z},\bar{y})=c$, where $G=(G_1,\dots,G_p)$ are in $A[[Z]][Y]$, $c=(c_1,\dots,c_p)$ are natural numbers and ord $G_j(\bar{z},\bar{y})=c_j$ means $G_j(\bar{z},\bar{y}) \in \underline{n}^{c_j} \setminus \underline{n}^{c_j+1}$, where $\underline{n}$ denotes the maximal ideal of B.

In the above notation, we shall say that the solution (z,y) in A is G-near to the solution $(\bar{z},\bar{y})$ in B if ord $G(z,y)=$ $=$ord $G(\bar{z},\bar{y})=c$.

So, we can specialize the definition (2.4) asking for the lifting of any solution $(\bar{z},\bar{y})$ of F in B to a solution (z,y) in A, which is G-near in the above sense. We shall see in Section 4 that this type of lifting is strongly related to an adequate model-theoretic concept of existentially - completeness.

(2.7.1) Definition. Let $u: A \longrightarrow B$ be a local ring morphism. We say that u lifts well algebraically if for every finite system of polynomials $F=(F_1,\dots,F_m)\in A[Y]^m$, $G=(G_1,\dots,G_s)\in A[Y]^s$, $Y=(Y_1,\dots,Y_n)$, and for every s-tuple $c=(c_1,\dots,c_s)$ of natural numbers, the existence of a solution $b\in B^n$ of F such that $\mathrm{ord}_B\, G(b)=c$ implies the existence of a solution $a\in A^n$ of F such that $\mathrm{ord}_A\, G(a)=c$.

(2.7.2) Definition. Let $u: A \longrightarrow B$ be a local ring morphism between noetherian complete local rings. We say that u lifts well analytically if u satisfies the condition derived from (2.7.1) by replacing polynomials with formal power series in $A[[Z]][Y]$.

§ 3. Background from Logic. Nonstandard equivalents of some approximation properties

The aim of this section is to prepare the reader for the last two sections of the paper, giving some model-theoretic preliminaries and some nonstandard forms of certain approximation properties. These nonstandard equivalents seem to be interesting in themself and also are useful in the proof of some results from Section 5.

(3.1) The first order language L in this paper will be the customary language for ring theory with constant symbols 0 and 1, binary function symbols + and ., and a unary symbol - (for additive inverse). This language may be extended through the introduction of a distinguished set of constant symbols and relation symbols at various times.

The terms of the language L are built up from constants 0 and 1 and variables x,y,...(which range over ring elements) using the function symbols +, -, . . Atomic formulas are of the form "$t_1=t_2$" where t_1 and t_2 are terms Formulas are built up from atomic formulas using logical connectives $\wedge$(and), $\vee$(or), $\neg$(not), $\exists$(there is), $\forall$(for all). The formulas $\varphi \to \psi$ and $\varphi \leftrightarrow \psi$ are used as abbreviations (e.g. $\varphi \to \psi$ stands for $\psi \vee \neg\varphi$). It should be clear what is meant by saying "φ holds in A" or "A satisfies φ", abbreviated $A \models \varphi$, where φ is a sentence in L (i.e. a formula without free variables) and A is a ring.

Usually we denote by T the theory of commutative rings with identity, i.e. the set of those sentences in L which are satisfied by each commutative ring with identity. Observe that every atomic formula is equivalent modulo T with a polynomial equation with integer coefficients.

(3.2) Now let $(A,\underline{m})$ be a noetherian local ring. Consider an arbitrary elementary extension $^*A$ of A, i.e. a ring $^*A$ with the property that for each formula $\varphi(x_1,\dots,x_n)$ in L, where $x_1,\dots,x_n$ are the variables occuring freely in φ, and for each n-tuple $a=(a_1,\dots,a_n)\in A^n$, A satisfies $\varphi(a)$ iff $^*A$ satisfies $\varphi(a)$. In particular A is identified with a subring of $^*A$. For instance, we can take $^*A$ to be an ultrapower $A^I/_D$, where I is an infinite set and D is an ultrafilter on I. It follows easily that $^*A$ is a local ring too with its maximal ideal say $^*\underline{m}$, and $^*\underline{m}^i \cap A=\underline{m}^i$ for $i\in\mathbb{N}$. Since A is noetherian, we have $^*\underline{m}^i=\underline{m}^i\,{}^*A$ for $i\in\mathbb{N}$, and the residue ring extensions $A/_{\underline{m}^i} \to {}^*A/_{^*\underline{m}^i}$ for $i\in\mathbb{N}$ are elementary embeddings. In particular, the residue field extension $k=A/_{\underline{m}} \hookrightarrow {}^*k = {}^*A/_{^*\underline{m}}$ is an elementary embedding.

Denote by $^*\underline{m}^\infty$ the ideal $\bigcap_{i\in\mathbb{N}} {}^*\underline{m}^i$ in $^*A$, by $\tilde{A}$ the factor ring $^*A/_{*m^\infty}$, and by $p: {}^*A \longrightarrow \tilde{A}$ the canonical surjection. Since A is noetherian, $\bigcap_{i\in\mathbb{N}} m^i=(0)$ and hence the composite morphism $A \longrightarrow {}^*A \xrightarrow{p} \tilde{A}$ is injective. Thus we may identify A with a subring of $\tilde{A}$. $\tilde{A}$ is a local ring with its maximal ideal $\tilde{m} = {}^*m/_{*m^\infty}$, $\tilde{m}\cap A=m$, $\tilde{A}$ is separate with respect to the m-adic topology, and $^*A/_{*m^i} \cong \tilde{A}/_{\tilde{m}^i}$ for $i\in\mathbb{N}$. In particular, the residue field of $\tilde{A}$ is $^*k$.

Now, let us assume that $^*A$ satisfies the $\aleph_1$-saturation property, i.e. for each countable system of formulas $(\varphi_i(x))_{i\in\mathbb{N}}$ in the language L extended with individual constants naming the elements of $^*A$, where the variable x is free, the whole system has a solution in $^*A$ whenever every finite subsystem admits a solution in $^*A$. For instance we can take $^*A= A^I/_D$, where I is an infinite set and D is a δ- incomplete ultrafilter on I (i.e. there exists a sequence $F_i\in D$, $i\in\mathbb{N}$,such that $\bigcap F_i\notin D$) [17] Theorem 1.6.4 In particular, we can take $I = \mathbb{N}$ and D a nonprincipal ultrafilter on $\mathbb{N}$. Then it follows easily that $\tilde{A} \cong \varprojlim_{i\in\mathbb{N}} {}^*A/_{*m^i}$ and thus $\tilde{A}$ is a complete local ring.

With these preparations we can give a non-standard characterization of the AP-rings.

(3.2.1) Proposition. Let $(A,\underline{m})$ be a noetherian local ring. Denote by $^*A$ an elementary extension of A and assume that $^*A$ is $\aleph_1$- saturated. Then A is an AP-ring iff the following condition (+) is satisfied: " For every system of polynomials $F=(F_1,\dots,F_m)\in A[Y]^m$, where $Y=(Y_1,\dots,Y_n)$, for every natural number c, and for every $\bar{y}\in {}^*A^n$, if $F(p(\bar{y}))=0$ then there exists $y\in {}^*A^n$ such that $F(y)=0$ and $y\equiv\bar{y} \bmod {}^*m^c$".

Proof. Assume that (A,m) is an AP-ring and the condition (+) is not fulfilled, i.e., there exist a system of polynomials

$F=(F_1,\dots,F_m)\in A[Y]^m$, $Y=(Y_1,\dots,Y_n)$, a natural number c, and some $\bar{y}\in{}^*A^n$ such that $F(p(\bar{y}))=0$ and for every $y\in{}^*A^n$, $y\neq\bar{y}$ mod ${}^*m^c$ if $F(y)=0$. Since A is an AP-ring, A satisfies (SAP) and hence there is a function $\vartheta: \mathbb{N}\longrightarrow\mathbb{N}$ attached to F with the corresponding property (see (2.1)). Let $d=\vartheta(c)$ and denote by φ the sentence in the language L extended with individual constants naming the elements of A, abbreviated as follows:

$$(\forall\bar{Y})F(\bar{Y})\equiv 0 \text{ mod } m^d \longrightarrow (\exists Y)F(Y)=0\wedge Y\equiv\bar{Y} \text{ mod } m^c$$

Since ${}^*A$ is an elementary extension of A, it follows that φ is true on ${}^*A$. Thus there exists $y\in{}^*A^n$ such that $F(y)=0$ and $y\equiv\bar{y}$ mod m^c. Contradiction!

Let us remark that the $\aleph_1$ - saturation property of ${}^*A$ is not necessary for the proof of the previous implication.

Conversely, suppose that the condition (+) is fulfilled and A does not satisfy (SAP), i.e. there exist $F=(F_1,\dots,F_m)\in A[Y]^m$, $Y=(Y_1,\dots,Y_n)$, and a natural number c, such that for every natural number i, there exists $\bar{y}\in A^n$ such that $F(\bar{y})\equiv 0$ mod m^i and for every $y\in A^n$, $y\neq\bar{y}$ mod m^c if $F(y)=0$. Let us consider the countable system of formulas in the language L extended with individual constants naming the elements of A, abbreviated as follows:

$$\varphi_i(X) := \text{"}F(X)\equiv 0 \text{ mod } m^i\wedge(\forall Y)F(Y)=0\rightarrow Y\neq X \text{ mod } m^c\text{"}$$

Since $A\models(\exists X)\varphi_i(X)$ for every natural number i, the considered countable system is finitely satisfiable on A, and hence on ${}^*A$, because ${}^*A$ is an elementary extension of A. Since ${}^*A$ is $\aleph_1$- saturated it follows that there exists $x\in{}^*A^n$ such that

$F(x)\in{}^*\underline{m}^{\infty}$, i.e. $F(p(x))=0$ and $x\neq y \bmod {}^*\underline{m}^c$ for each solution $y\in{}^*A^n$ of F. Contradiction!

Q.E.D.

(3.3) Since the first order predicate calculus seems to be not adequate enough for the work with formal power series, we shall use for this purpose the methods of nonstandard analysis or, as we should rather say in our case, of nonstandard arithmetic, developed by A.Robinson [24] , [25] . The principal fact we shall use is the existence of nonstandard models with good saturation properties of an arbitrary mathematical structure, for a higher order language. The reader who is not yet acquainted with nonstandard arguments may consult [24] , or [17] , or [26] . Here we shall explain only some basic facts directly related to the concrete context of the present paper. Let us remark that another framework which seems to be adequate too for our purpose is the so called topological model theory (see [22]).

(3.4) First let us give some preliminaries concerning the higher order nonstandard models. Following A.Robinson, the higher-order structures and higher-order languages may be founded on the notion of a <u>type.</u>

The class T of types is defined inductively as follows:

i) 0 is a type.

ii) If $t_1,\dots,t_k$, $k\geq 1$, are types, then $(t_1,\dots,t_k)$ is a type.

iii) T is the smallest class satisfying i) and ii).

Let X be a non-empty set. To the elements of X we assign the type 0, and then we introduce the <u>admissible relations</u> in X and at the same time assign to them types by the following rule. If $t=(t_1,\dots,t_k)$ is a type, $k\geq 1$, and M is a set of k-tuples

$(M_1,\dots,M_k)$, where M_i is of the type $t_i (i=1,\dots,k)$, then M is a relation of type t. For example, a subset of X is of type (0), a subset of $X \times X$ is of type $(0,0)$, and the set $P(X \times X)$ of subsets of $X \times X$ is of type $((0,0))$. Observe that the empty set has all types $\neq 0$. For every $t \in T$ we denote by X_t the set of all relations of type t built up from X. Thus $X_0 = X$, $X_{(0)} = P(X)$, and X_t is itself a relation of type (t).

A higher-order structure or simply a structure $\underline{M} = \underline{M}(M_t : t \in T)$ is a family of sets indexed by T such that for some nonempty set X, every M_t is a subset of X_t subject to: $M_0 = X_0 = X$, and for every $t \neq 0$, $t = (t_1,\dots,t_k)$, $k \geqslant 1$, if $A \in M_t$ and $(A_1,\dots,A_k) \in A$, then $A_i \in M_{t_i}$ for $i=1,\dots,k$. The elements of $\underline{M}$ are called the entities of $\underline{M}$. The entities of $\underline{M}$ of type 0 are called the individuals of $\underline{M}$. A structure $\underline{M}$ is said to be full whenever $M_t = X_t$ for all $t \in T$, where $X = M_0$ is the set of the individuals of $\underline{M}$ and X_t is the set of all relations of type t based in X.

Denote by $L = L(\underline{M})$ the formal language of higher order predicate calculus over the structure $\underline{M}$. This language contains names for the individuals of $\underline{M}$, as well for all entities of higher type in $\underline{M}$. Starting from these constants of the language and from a large enough supply of variables, the formulas of the language L are built up in finitely many steps, according to the rules of predicate calculus, with the help of the logical connectives and quantifiers. Quantification is permitted not only with respect to individuals but also with respect to entities of any given type.

A higher-order nonstandard model of the structure $\underline{M} = \underline{M}(M_t : t \in T)$ is a structure $^*\underline{M} = {}^*\underline{M}(^*M_t : t \in T)$ satisfying the following principle of permanence: Every mathematical statement φ about $\underline{M}$, expressed in the language L, has an interpretation $^*\varphi$ in $^*\underline{M}$, and

$^*\varphi$ is true in $^*\underline{M}$ iff φ is true in $\underline{M}$.

This interpretation is made according to the following rules:

i) The interpretation of the logical connectives is the usual one.

ii) To every entity a in $\underline{M}$ of type t there corresponds an entity $^*a$ in $^*\underline{M}$ of the same type t.

iii) " there exists x of type t" is interpreted in $^*\underline{M}$ as " there exist $x \in {}^*M_t$ ", and similarly for " for all x of type t"

The mapping $a \longmapsto {}^*a$ of the entities of $\underline{M}$ into the entities of $^*\underline{M}$ is easily seen to be one to one and defines an embedding of $\underline{M}$ into $^*\underline{M}$. The entities a of $\underline{M}$ are frequently identified with the corresponding entities $^*a$ of $^*\underline{M}$, and we write $M_t \subset {}^*M_t$ $(t \in T)$, if no confusion is possible.

A nonstandard model $^*\underline{M}$ of $\underline{M}$ need not be full even if $\underline{M}$ is full. The entities of $^*\underline{M}$ are called <u>internal</u> and the relations of $({}^*X)_t \setminus {}^*M_t$ where $({}^*X)_t$ denotes the set of all relations of type t based on $^*X = {}^*M_o$, are called <u>external</u>. An entity a of $^*\underline{M}$ which belongs to M, i.e. $a = {}^*b$ for some entity b in $\underline{M}$, is called a <u>standard entity</u> of $^*\underline{M}$.

For the concrete purposes of the present work the higher-order nonstandard models afore defined are not comprehensive enough. We need an additional saturation property. This property is concerned with internal binary relations in a nonstandard model $^*\underline{M}$ of $\underline{M}$.

Let R be a binary relation in a higher-order structure $\underline{N}$. Let a be an entity in $\underline{N}$. If there exists b in $\underline{N}$ such that R(a,b) holds in $\underline{N}$, then a is said to be in the <u>domain</u> of R. The relation R is said to be <u>concurrent or finitely satisfiable</u> on a subset A of the domain of R if for arbitrary finitely many elements $a_1,\dots,a_k$ in A there exists b in $\underline{N}$ such that the relations $R(a_i,b)$ hold simultaneously in $\underline{N}$ for $1\leq i\leq k$.

The higher-order structure $\underline{N}$ is called $\aleph_1$-saturated if for every admissible binary relation R in $\underline{N}$, which is concurrent on a countable subset A of the domain of R, there exists an entity b in $\underline{N}$ such that R(x,b) holds in $\underline{N}$ for all $x\in A$. Observe that the relation R is required to be an entity of $\underline{N}$. However the subset A on which R is required to be concurrent is not necessarily an entity of $\underline{N}$; the only restriction is concerned with the cardinality of A.

A natural way to obtain higher-order nonstandard models of a structure $\underline{M}$, which are $\aleph_1$ - saturated is by means of ultrapowers.

Thus, if I is an infinite set and D is an ultrafilter on I, the structure $^*M={}^*M({}^*M_t:\ t\in T)$, where $^*M_t=M_t^I/_D$, is, by Los'theorem [17], I.3., a higher-order nonstandard model of $\underline{M}$. If, in addition, the set M_o of the individuals of M is infinite and the ultrafilter D is δ - incomplete, then $^*M$ is a proper $\aleph_1$-saturated extension of M [17] Theorem 1.6.4. On the other hand, for every infinite subset A of M of a certain type, the standard set $^*A$ of $^*\underline{M}$ satisfies $^*A\setminus A\neq\emptyset$ and $^*A\setminus A$ is external [17] Theorem 1.4.1.

(3.5) Now let us apply the general theory afore sketched to a more concrete problem. Let $(A,\underline{m})$ be a noetherian complete local ring and $k=A/_m$, its residue field. By structure theorem of noetherian complete local rings we have $A\cong R[[X]]/_a$, where R is either the field k or a complete discrete valuation ring of characteristic 0, with the residue field k, and $X=(X_1,\dots,X_s)$. Let us denote by $\underline{M}=M(M_t:t\in T)$ the full higher-order structure having as set of individuals M_o, the disjoint union $R \cup \mathbb{N}$.

Thus the set $\mathbb{N}$ of natural numbers, the ring $\mathbb{Z}$ of integers, the field $\mathbb{Q}$ of rationals, the ring R and its ideals, the ring $R[\![X]\!]$ and its ideals, the elements of $R[\![X]\!]$, etc. are entities of certain types of the structure $\underline{M}$. Since the factor ring $A = R[\![X]\!]/_{\underline{a}}$ is completely determined by the ideal $\underline{a}$, we may also consider the ring A and its ideals, as well as the rings of formal power-series $A[\![Y]\!][Z]$, where $Y=(Y_1,\ldots,Y_n)$, $Z=(Z_1,\ldots,Z_e)$, etc as entities of $\underline{M}$. The evaluation function

$$A[\![Y]\!][Z] \times \underline{m}A^n \times A^e \longrightarrow A: \ (f,y,z) \longmapsto f(y,z)$$

may be also considered an entity of the structure M.

Denote by ${}^*M = {}^*M({}^*M_t: t \in T)$ a proper $\aleph_1$- saturated nonstandard model of $\underline{M}$.

Let ${}^*\mathbb{N}$, ${}^*\mathbb{Z}$, ${}^*R$, ${}^*k$, ${}^*A$, and ${}^*\underline{m}$ be the standard entities of ${}^*\underline{M}$ attached to $\mathbb{N}$, $\mathbb{Z}$, R, k, A and $\underline{m}$. It follows easily that ${}^*R$ is either a valuation ring with ${}^*\mathbb{Z}$ as value group or ${}^*k$, ${}^*A$ is a local ring with ${}^*m$ its maximal ideal and with the residue field ${}^*k$. ${}^*R$ may be identified with an unramified extension of R in the sense that the maximal ideal of ${}^*R$ is generated by the local parameter, say π, of R. Denote by $(\pi)^\infty$ the ideal $\bigcap_{i \in \mathbb{N}} \pi^i\, {}^*R$ (if $R=k$ then $\pi=0$ and hence $(\pi)^\infty = (0)$), by $\tilde{R}$ the factor ring ${}^*R/_{(\pi)^\infty}$ and by $p_o: {}^*R \longrightarrow \tilde{R}$ the canonical surjection (if $R=k$, then $\tilde{R} = {}^*R = {}^*k$, and $p_o = 1_{{}^*k}$). If $R \neq k$, then $\tilde{R}$ is a discrete valuation ring with the local parameter $\pi \bmod (\pi)^\infty$ and the residue field ${}^*k$. The composite morphism $R \longrightarrow {}^*R \xrightarrow{p_o} \tilde{R}$ is injective and hence $\tilde{R}$ may be identified with an unramified extension of R. Moreover, since ${}^*R$ is $\aleph_1$- saturated, it follows easily that $\tilde{R} \cong \varprojlim_{i \in \mathbb{N}} \tilde{R}/_{\pi^i \tilde{R}}$.

Thus, if $R \neq k$, we conclude that $\tilde{R}$ is a complete discrete valuation ring and hence $\tilde{R}$ is the Cohen R-algebra with the residue field $^*k$; the separability of the residue field extension $k \subset {}^*k$ follows easily from the fact that the embedding $k \hookrightarrow {}^*k$ is elementary.

Similarly, A may be identified with a proper subring of $^*A$, and $^*m \cap A = m$. Since A is noetherian we conclude that $^*m^i = m^i\, {}^*A$ for all $i \in \mathbb{N}$. Denote by $^*m^\infty$ the ideal $\bigcap_{i\in\mathbb{N}} m^i$ of $^*A$, by $\tilde{A}$ the factor ring $^*A/_{^*m^\infty}$, and by p: $^*A \longrightarrow \tilde{A}$ the canonical surjection. $\tilde{A}$ is a local ring with its maximal ideal $\tilde{m} = {}^*m/_{^*m^\infty}$ and the residue field $^*k$. Since A is noetherian, $\bigcap_{i\in\mathbb{N}} m^i = (0)$ and hence the composite morphism $A \longrightarrow {}^*A \longrightarrow \tilde{A}$ is injective. Thus we may identify A with a subring of $\tilde{A}$; p is a local morphism of A-algebras.

(3.6) <u>Proposition</u>. $\tilde{A}$ is a Cohen A-algebra, i.e. $\tilde{A}$ is a flat, noetherian complete local A-algebra and $\tilde{A}/_{m\tilde{A}}$ is a separable field extension of $k = A/m$.

<u>Proof</u>. Since $m\tilde{A} = \tilde{m}$, we have $\tilde{A}/m\tilde{A} = {}^*k$ and hence the separability condition is satisfied. On the other hand, since $A = R[\![X]\!]/a$, where $X = (X_1, \ldots, X_s)$, and $\tilde{R}$ is the Cohen R-algebra with the residue field $^*k$, we conclude that the Cohen A-algebra with the residue field $^*k$ is isomorphic to the factor ring $\tilde{R}[\![X]\!]/_{a\tilde{R}[\![X]\!]}$. Since $\tilde{A} = {}^*(R[\![X]\!])/_{a\,{}^*(R[\![X]\!]) + (\pi,X)^\infty} \cong \widetilde{R[\![X]\!]}/_{a\widetilde{R[\![X]\!]}}$ where $(\pi,X)^\infty = \bigcap_{i\in\mathbb{N}} (\pi,X)^i$ and $\widetilde{R[\![X]\!]} \cong {}^*(R[\![X]\!])/_{(\pi,X)^\infty}$, it remains to show that $\widetilde{R[\![X]\!]} = \tilde{R}[\![X]\!]$. Since $^*(R[\![X]\!])$ is $\aleph_1$-saturated, we have $\widetilde{R[\![X]\!]} \cong \varprojlim_{i\in\mathbb{N}} {}^*(R[\![X]\!])/_{(\pi,X)^i}$. However the last ring is isomorphic to $\varprojlim_{i\in\mathbb{N}} {}^*R[X]/_{(\pi,X)^i\, {}^*R[X]} \cong$ $= \varprojlim_{i\in\mathbb{N}} \tilde{R}[X]/_{(\pi,X)^i \tilde{R}[X]} \cong \tilde{R}[\![X]\!]$. Observe that we identified the ring of polynomials $^*R[X]$ with a proper subring of the internal ring of

internal power series $^*(R[[X]])$. Indeed the embedding $R[X] \subset R[[X]]$ is naturally extended to an internal embedding $^*(R[X]) \subset {}^*(R[[X]])$, and, on the other hand, the ring $^*R[X]$ may be naturally embedded in the ring $^*(R[X])$ of internal polynomials in X with coefficients in the internal ring $^*R$.

Q.E.D.

Now let us show that the following statement may be seen as the nonstandard equivalent of Theorem 2.5.

(3.7) <u>Theorem</u>. Let A, m, k, $\underline{M}$ and $^*\underline{M}$ be as above. Let $F=(F_1,\dots,F_m)$ be a system of formal power series in the ring $A[[Z]][Y]$, where $Z=(Z_1,\dots,Z_e)$, $Y=(Y_1,\dots,Y_n)$. Let $^*F=(^*F_1,\dots,{}^*F_m)$ be the system of internal power series in the internal ring $^*(A[[Z]][Y])$ attached to F. Then, for every natural number $c\geq 1$, for every $\bar{z}\in m\,^*A^e$ and for every $\bar{y}\in A^n$, if $F(p(\bar{z}), p(\bar{y}))=0$ then there exist $z\in m\,^*A^e$ and $y\in {}^*A^n$ such that $^*F(z,y)=0$ and $z\equiv\bar{z}$, $y\equiv\bar{y}$ mod $m^c\,^*A$.

Let us show that the statement of Theorem 2.5 implies the statement of Theorem 3.7. Assume the contrary, i.e. there exist a natural number $c\geq 1$, $\bar{z}\in m\,^*A^e$ and $\bar{y}\in A^n$, such that $F(p(\bar{z}), p(\bar{y}))=0$ and for every $z\in m\,^*A^e$ and for every $y\in {}^*A^n$, $(z,y)\not\equiv(\bar{z},\bar{y})$ mod $m^c\,^*A$ if $^*F(z,y)=0$. On the other hand, according to the statement of Theorem 2.5, there exists a natural number $d=\nu(c)$ such that the following sentence φ is true in the structure $\underline{M}$:

$$\varphi := (\forall \bar{Z}\in mA^e)(\forall \bar{Y}\in A^n)F(\bar{Z},\bar{Y})\equiv 0 \text{ mod } m^d \longrightarrow$$

$$(\exists Z\in mA^e)(\exists Y\in A^n)F(Z,Y)=0 \wedge (Z,Y)\equiv(\bar{Z},\bar{Y}) \text{ mod } m^c.$$

Since $F(p(\bar{z}), p(\bar{y}))=0$ implies $^*F(\bar{z},\bar{y})=0$ mod $m^d\,^*A$, it follows by the permanence principle that there exist $z\in m\,^*A^e$, $y\in {}^*A^n$ such that $^*F(z,y)=0$ and $(z,y)\equiv(\bar{z},\bar{y})$ mod $m^c\,^*A$, which gives a contradiction. Thus we showed that the statement of Theorem 2.5 implies the statement of Theorem 3.7.

Conversely, let assume that the statement of Theorem 2.5 is false, i.e. there exists a natural number $c \geq 1$, such that for every natural number i, there exist $\bar{z} \in \underline{m}A^e$ and $\bar{y} \in A^n$ subject to: $F(\bar{z},\bar{y}) \equiv 0 \bmod \underline{m}^i$ and for every $z \in \underline{m}A^e$ and $y \in A^n$, $(z,y) \not\equiv (\bar{z},\bar{y}) \bmod \underline{m}^c$ if $F(z,y)=0$. By $\mathcal{X}_1$-saturation of ${}^*\underline{M}$, we conclude that there exist $\bar{z} \in \underline{m}\ {}^*A^e$ and $\bar{y} \in {}^*A^n$ such that ${}^*F(\bar{z},\bar{y})=0 \bmod \underline{m}^i\ {}^*A$ for every natural number $i \geq 1$ and for every $z \in \underline{m}{}^*A^e$ and $y \in {}^*A^n$, $(z,y) \not\equiv (\bar{z},\bar{y}) \bmod \underline{m}^c\,{}^*A$ if ${}^*F(z,y)=0$. However ${}^*F(\bar{z},\bar{y})=0 \bmod \underline{m}^i\ {}^*A$ for all $i \in \mathbb{N}$ implies $F(p(\bar{z}), p(\bar{y}))=0$, and, by the statement of Theorem 3.7, we conclude that there exist $z \in \underline{m}{}^*A^e$, $y \in {}^*A^n$ subject to ${}^*F(z,y)=0$ and $(z,y) \equiv (\bar{z},\bar{y}) \bmod \underline{m}^c\,{}^*A$, i.e. a contradiction.

(3.7.1) Corollary. Let A, m, k, $\underline{M}$, F, ${}^*\underline{M}$, ${}^*F$, $\tilde{A}$ and p as above. Then for every $\tilde{z} \in \underline{m}\tilde{A}^e$ and for every $\tilde{y} \in \tilde{A}^n$, if $F(\tilde{z},\tilde{y})=0$ then there exist $z \in \underline{m}{}^*A^e$ and $y \in {}^*A^n$ such that ${}^*F(z,y)=0$ and $p(z)=\tilde{z}$, $p(y)=\tilde{y}$.

This corollary is equivalent with Theorem (3.7). Indeed, let $\bar{z},\bar{y}$ be in ${}^*A$ such that $p(\bar{z})=\tilde{z}$, $p(\bar{y})=\tilde{y}$. By (3.7), every finite subsystem of the following countable system of equations and congruences

(S) $F(Y,Z)=0, \quad Z \equiv \tilde{z}, \quad Y \equiv \tilde{y} \bmod \underline{m}^c$ is compatible in ${}^*A$. Thus S is compatible in ${}^*A$ (${}^*\underline{M}$ being $\mathcal{X}_1$-saturated) and a solution of it satisfies our conditions.

The Theorem 3.7 may be seen as the nonstandard equivalent of Pfister-Popescu theorem. A special case, where ${}^*\underline{M}$ is given by an ultrapower indexed by the set $\mathbb{N}$ of natural numbers with respect to a nonprincipal ultrafilter, is considered by Popescu in [21].

§4. Existentially complete morphisms

A basic concept of model theory is the existential completeness. This section is devoted to apply this concept and some general related results to the concrete framework of ring theory.

(4.1) Preliminaries. Let L be an arbitrary first order language. The L-structures are sets equipped with certain interpretations of the constant symbols, function symbols and relation symbols. A morphism $u:A \longrightarrow B$ between two L-structures is an injective map which preserves the corresponding interpretations of constant, function and relation symbols of the language L. A special mention for the preservation of relation symbols: if R is a relation symbol with n places and $a=(a_1,\dots,a_n) \in A^n$, then $A \models R(a)$ iff $B \models R(u(a))$. If T is a theory in the language L, then the category of models of T is a category whose objects are the L-structures which are models of T and whose morphisms are L-structure morphisms.

A morphism $u:A \longrightarrow B$ of L-structures is called existentially complete (we can also say that A is existentially complete in B) if whenever $\varphi(x_1,\dots,x_n)$ is an existential formula in L, $a_1,\dots,a_n$ are elements in A, and B satisfies $\varphi(a_1,\dots,a_n)$, then A satisfies $\varphi(a_1,\dots,a_n)$ too.

A primitive formula is a formula in prenex form whose prefix contains only existential quantifiers and whose matrix is a conjunction of atomic formulas and negated atomic formulas. Every existential formula is logically equivalent to a disjunction of primitive formulas. Consequently, existential formulas could be replaced by primitive formulas in the preceding definition of existential completeness.

A very useful result concerning existential completeness is Scott's lemma (see [6], lemma 8.1.3 and corollary 9.3.11):

(4.1.1) <u>Lemma</u>. A morphism of L-structures, $u:A \longrightarrow B$ is existentially complete if there exists a morphism of L-structures v from B into an ultrapower $^*A=A^I/_D$ such that $i=vu$, where i denotes the canonical embedding of A into $^*A$.

(4.2) Let L be the first order language of rings. In the theory of rings, each primitive formula is equivalent to one of the form

$$(\exists x) \bigwedge_{i=1}^{e} f_i(x,y)=0 \wedge \bigwedge_{j=1}^{s} g_j(x,y)\neq 0$$

where $f_i(x,y)$ for $i=1,\ldots,e$ and $g_j(x,y)$ for $j=1,\ldots,s$, are polynomials in variables $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_m)$ with integer coefficients.

Consequently, a ring morphism $u:A \longrightarrow B$ is existentially complete if for arbitrary finite systems $F=(F_1,\ldots,F_e)$ and $G=(G_1,..,G_s)$ of polynomials in $A[X]$, where $X=(X_1,\ldots,X_n)$ are variables, the system of equalities and inequalities

$$F_1(x)=\ldots=F_e(x)=0, \quad G_1(x)\neq 0,\ldots,G_s(x)\neq 0$$

has a solution in A iff it has one in B.

(4.2.1) <u>Remark</u>. If the ring morphism $u:A \longrightarrow B$ is existentially complete then the following hold:

i) u is algebraically pure.

ii) A is an integral domain iff B is so, and in this case the field of quotients $Q(A)$ of A is existentially complete in the field of quotients $Q(B)$ of B.

iii) A is a reduced ring iff B is so.

iv) A finitely generated ideal a in A is generated by m elements iff aB is generated by m elements too.

(4.2.2) <u>Remark</u>. Let $u:A \to B$ be an existentially complete ring morphism and C an A-algebra which is free as A-modul (in particular C can be a polynomial A-algebra). Then the induced morphism $u':C \to B\otimes_A C$ is existentially complete. Indeed, by (4.1.1) there exist an elementary extension $i:A \longrightarrow {}^*A \cong A^I/_D$ and an injective ring morphism $v:B \longrightarrow {}^*A$ such that $i=vu$. Let $\mathcal{E}=\{e_i\}_{i\in I}$ be a base of C over A and $i':C \longrightarrow {}^*C:=C^I/_D$ the canonical elementary morphism. Clearly, i' maps $\mathcal{E}$ on a system ${}^*A$-linearly independent of ${}^*C$ and so the morphism $v':B\otimes_A C \longrightarrow {}^*C$ given by $\sum b_i \otimes e_i \rightsquigarrow \sum v(b_i)i'(e_i)$ is an injective one and satisfies $i'=v'u'$. By (4.1.1), u' is existentially complete.

(4.3) Let us consider the particular case of field extensions. In the theory of fields, each primitive formula is equivalent to one of the form

$$(\exists x) \bigwedge_{i=1}^{e} f_i(x,y)=0$$

where each $f_i(x,y)$ is a polynomial in variables $x=(x_1,..,x_n)$ and $y=(y_1,...,y_m)$ with integer coefficients. Indeed, the inequality $z\neq 0$ is equivalent in the theory of fields with the formula $(\exists x)xz-1=0$. Consequently, a field K is existentially complete in a field extension L iff K is algebraically pure in L. According to [29] Theorem 2.1., a field K is existentially complete in a field extension L iff the extension L/K is regular (i.e. K is algebraically closed in L and L/K is separable) and each absolutely irreducible variety over K which has an L-rational point has a K-rational point.

(4.4) Now let us consider the special case where L is a finitely generated field extension of K. Denote by $\mathcal{S}=\mathcal{S}(L/K)$ the set of those places of L which are trivial on K. We call $\mathcal{S}$ the <u>Riemann</u>

<u>space</u> of L/K. The set $\mathcal{S}$ is non-empty; for instance, the identity 1_L of L belongs to $\mathcal{S}$. If P is a member of $\mathcal{S}$, then the residue field L.P is a field extension of the base field K. If x is a subset of L, let us denote by $\mathcal{S}^x$ the subset of $\mathcal{S}$ consisting of those places $P \in \mathcal{S}$ for which the elements of x are holomorphic at P, i.e. x is contained in the valuation ring $\mathcal{O}_P$ attached to P. We have $1_L \in \mathcal{S}^x$ for all subsets x of L. The Riemann space S of L/K admits a Zariski topology with a basis of open sets given by the family $(\mathcal{S}^x)$, where x ranges over the set of all finite subsets of L. Denote by $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}(L/K)$ the subset of $\mathcal{S}$ consisting of those places $P \in S$ which are rational over K, i.e. L.P=K. We call $\tilde{S}$ the K-Riemann space of L/K. Let $\tilde{\mathcal{S}}^x = \tilde{\mathcal{S}} \cap \mathcal{S}^x$ where x is a subset of L.

(4.4.1) <u>Proposition</u>. Let L/K be a finitely generated field extension. Then the following assertions are equivalent:

a) K is existentially complete in L.

b) $\tilde{\mathcal{S}}$ is dense in S, i.e. $\tilde{S}^x$ is non-empty for every finite subset x of L.

<u>Proof</u>. a) implies b): Assume that K is existentially complete in L, and let x be a finite subset of L. If $\tilde{\mathcal{S}}^{x'}$ is non-empty for some finite subset x' of L, extending x, then clearly $\mathcal{S}^x$ is non-empty. So we may enlarge x, if convenient, by adding finitely many elements of L. Therefore we may assume that $x=(x_1,\ldots,x_n)$ is a system of generators of L/K, i.e. L=K(x). Denote by V the affine model of L/K, whose generic point over K is x. Now we envisage V as being defined by a finite system of polynomial equations over K: $f_i(X)=0$ for $i=1,\ldots,r$, where $X=(X_1,\ldots,X_n)$ and $f_i \in K[X]$. Since x is a generic point of V over K, x is a simple point on V, and hence there exists a minor $h \in K[X]$ of order n-dim(V) of the Jacobian matrix $\left(\frac{\partial f_i}{\partial X_j}\right)_{\substack{1\leq i\leq r\\ 1\leq j\leq n}}$ such that $h(x)\neq 0$.

Thus the following existential sentence in the language L extended with constants which are names for the elements of K:

$$\varphi := (\exists X) \bigwedge_{i=1}^{r} f_i(X)=0 \wedge h(X)\neq 0$$

is true on L. Since, by hypothesis K is existential complete in L, we conclude that φ is true on K too, i.e. there exists $b \in K^n$, subject to: $f_i(b)=0$ for $i=1,\dots,r$ and $h(b)\neq 0$. It follows that b is a K-rational simple point on V. By a well known result of the algebraic geometry (see, for instance [13] Corollary A3) the specialization $x \longrightarrow b$ can be extended to a K-rational place P of L. So we obtained a place P belonging to the set $\tilde{S}^x$.

b) implies a). Assume that $\tilde{S}^x$ is non-empty for every finite subset x of L. We have to show that K is existentially complete in L, or equivalently, L is algebraically pure over K. Since $L = \bigcup_x K[x]$, where x ranges over the set of all finite subsets of L, it suffices, by (2.2) iv), to show that the canonical morphism $K \longrightarrow K[x]$, where x is an arbitrary finite subset of L, admits a retract. Since $\tilde{S}^x$ is non-empty, there exists a place P of L/K subject to: $K[x] \subset \mathcal{O}_P$ and L.P=K. Then the restriction of P to K[x] is a retract of the structure morphism $K \longrightarrow K[x]$.

Q.E.D.

(4.4.2) <u>Corollary</u>. Let L/K be a proper finitely generated field extension. If K is existentially complet in L then the sets $\tilde{S}^x$ are infinite for all finite subsets x of L.

<u>Proof</u>. Suppose that $\tilde{S}^x$ is finite for some finite subset $x=(x_1,\dots,x_n)$ of L. By (4.4.1), the set $\tilde{S}^x$ is non-empty. Let $P_1,\dots,P_e$ $(e\geq 1)$ be the members of $\tilde{S}^x$. Since L is a proper extension of K, there exist $y_1,\dots,y_e \in L$ such that $y_iP_i=\infty$ for $i=1,\dots,e$. Let x' denote the set $\{x_1,\dots,x_n, y_1,\dots,y_e\}$. We conclude that

the set $\tilde{S}^{x'}$ is empty, contradicting (4.4.1).

Q.E.D.

The following result is of the same type with (4.4.1).

(4.4.3) Proposition. Let V be an affine variety defined over K. Denote by $x=(x_1,\dots,x_n)$ a generic point of V over K, by $K[x]$ its coordinate ring and by $L=K(x)$ the field of rational functions on V over K. Let V(K) be the set of K-rational points of V. Assume that $\{x_1,\dots,x_s\}$, $s=\dim V\leq n$, is a basis of transcendency of L/K. Then the following assertions are equivalent:

i) V(K) is dense in V, i.e. for every $f\in K[x]$, $f\neq 0$, there exists $a\in V(K)$ such that $f(a)\neq 0$.

ii) For every $f\in K[x_1,\dots,x_s]$, $f\neq 0$, there exists $a\in V(K)$ such that $f(a)\neq 0$.

iii) K is existentially complete in L.

Proof. The implication i) $\longrightarrow$ ii) is trivial.

i) and iii) are equivalent. Indeed, since $L=\bigcup_f K[x]_f$, where f ranges over the set $K[x]\setminus\{0\}$, we conclude by (2.2) iv) that K is existentially complete in L iff the finite presentation morphism $K\longrightarrow K[x]_f$ has a retract for all $f\in K[x]\setminus\{0\}$. However the last condition is equivalent with the statement i).

It remains to show that ii) implies i). Let $f\in K[x]\setminus\{0\}$. Since L is algebraic over $K(x_1,\dots,x_s)$, there is a minimal degree equation over $K[x_1,\dots,x_s]$ satisfied by f, say $g_tf^t+\dots+g_o=0$, where $t\geq 1$, and $g_i\in K[x_1,\dots,x_s]$ for $i=0,1,\dots,t$. Observe that $g_o\neq 0$, because otherwise the equation satisfied by f is not of minimal degree. Applying the hypothesis ii) to go, we conclude that there exists $a\in V(K)$ such that $g_o(a)\neq 0$, and hence $f(a)\neq 0$.

Q.E.D.

(4.4.4) Corollary. Let V be a one-dimensional affine variety defined over K and L be its field of functions. Then K is existentially complet in L iff the set V(K) is infinite.

(4.4.5) Examples. a) Let $\mathbb{R}$ be the field of real numbers. Then $\mathbb{R}$ is existentially complete in $Q(\mathbb{R}[X,Y]/_{(Y^2-X^3)})$, but $\mathbb{R}$ is not existentially complete in $Q(\mathbb{R}[X,Y]/_{(X^2+Y^2)})$.

b) The field $\mathbb{Q}$ of rationals is not existentially complete in $Q(\mathbb{Q}[X,Y]/_{(X^4-2Y^2-17)})$. Indeed the elliptic curve $2Y^2=X^4-17$ has not rational points (see Lind's argument, rediscovered by Mordell, reproduced in [8]).

(4.4.6) Remarks. a) A field K is algebraically closed iff K is existentially complete in each (finitely generated) field extension of K.

b) A field K is pseudo-algebraically closed iff K is existentially complete in every totally transcendental field extension ([29] Theorem 2.2).

c) A field K is separable closed iff K is existentially complete in each separable field extension of K. Indeed assume that K is separable closed, and let L be a finitely generated separable field extension of K. Then $L=K(x)(y)$, where $x=(x_1,\dots,x_n)$ is a basis of transcendency of L/K and y is a separable element over $K(x)$. Since K is existentially complete in $K(x)$ there is an embedding of $K(x)/K$ into an elementary field extension $^*K$ of K. Consider the field diagram

$$\begin{array}{ccc}
{}^*K & \hookrightarrow & {}^*KL = {}^*K(y) \\
\uparrow & & \uparrow \\
 & & L=K(x)(y) \\
K(x) & \hookrightarrow & \\
\uparrow & & \\
K & &
\end{array}$$

Since y is separable over K(x) it follows that y is separable over $^*K$ too. Since K is separable closed and $^*K/K$ is elementary we conclude that $^*K$ is separable closed too and hence $y \in K$ and L is a subextension of $^*K/K$. Therefore K is existentially complete in L.

d) A formally real field K is real closed iff K is existentially complete in each formally real field extension of K. Indeed, let K be a real closed field, and L a formally real field extension of K. Let $\leq_L$ be an ordering of L. Since K is uniquely orderable, $(L, \leq_L)$ is an ordered field extension of $(K, \leq_K)$. Denote by $(\tilde{L}, \leq_{\tilde{L}})$ the real closure of $(L, \leq_L)$. By Robinson's theorem [27] Theorem 17.3, the extension $(\tilde{L}, \leq_{\tilde{L}})/(K, \leq_K)$ is elementary and hence K is existentially complete in L.

e) A similar situation occurs when K is a formally p-adic field in Kochen-Roquette'sense [13]. Indeed, let K be a field equipped with a place p and the corresponding valuation v subject to the conditions:

1) The residue field K_v is finite, say with q elements.

2) The value group v(K) admits a smallest positive element, say $v(\pi)$ with $\pi \in K$; thus $\mathbb{Z}$ can be identified with an isolated subgroup of v(K).

A field extension F of K is called <u>formally p-adic</u> if there exists a valuation w of F such that w extends v, $v(\pi)$ remains the smallest positive element in w(F), and $F_w = K_v = \mathbb{F}_q$. F is called <u>p-adically closed</u> if F is formally p-adic and does not admit proper algebraic formally p-adic field extensions. This is equivalent to the fact that F has a valuation w which makes F formally p-adic and satisfies the supplementary conditions: the valuation ring $\mathcal{O}_w$ is henselian and the value group w(F) is a $\mathbb{Z}$-group, i.e. $w(F)/\mathbb{Z}$ is divisible. Moreover the valuation w is uniquely determined [13].

Now assume that the base field K satisfies the additional conditions: the value v(p) of the residue characteristic p is a multiple of $v(\pi)$, say $v(p)=e\ v(\pi)$ with e a positive integer. In other words, the absolute ramification index of (K,v) is finite ≥ 1. In particular K is of characteristic zero. Then K is p-adically closed iff K is existentially complete in each formally p-adic field extension of K. This fact is a consequence of the model-completeness of the theory of p-adically closed fields with finite absolute ramification index [2], [11], [14], [28], [30], [3].

f) A similar situation holds when K is a formally p-adic field in a more general sense as defined in [4].

(4.5) Now we shall introduce another concept of existential completeness for rings which is stronger then the concept considered in (4.2), and seems to be more adequate for the purposes of the present work. For this we shall extend in a convenient way the language L of rings by adding certain new relation symbols. Denote by L′ the extended language. Then we shall extend the theory T of commutative rings with identity to a theory T′ in the language L′ such that each ring has an unique expansion to a model of T′, and hence T′ is a non-essential extension of T, in the sense that no new theorems in the language L of rings are derivable in T′. However, the morphisms in the category of models of T′ are ring morphisms with supplementary properties.

For every positive integer n, we add to the language L a new relation symbol P_n with n(n+1) places. Denote by L′ the extended language and by T′ the theory in L′ obtained by adding to the axioms of commutative rings with identity the following family of new axioms:

$$P_n(x,y) \longleftrightarrow (\exists z) \bigwedge_{i=1}^{n} \sum_{j=1}^{n} y_{ij} z_j = x_i \quad ,$$

where $x=(x_1,\ldots,x_n)$, $y=(y_{ij})_{1\leq i,\ j\leq n}$, $z=(z_1,\ldots,z_n)$.

The models of T′ are the commutative rings with identity, and the new relation symbols P_n have canonical interpretation on any ring A: Let $a=(a_1,\ldots,a_n)\in A^n$ and $b=(b_{ij})_{1\leq i,\ j\leq n}$ a matrix with entries in A. Then A satisfies $P_n(a,b)$ iff a belongs to the submodule of A^n generated by $b_i=(b_{1i},\ldots,b_{ni})$ for $i=1,\ldots,n$.

(4.5.1) A morphism $u:A \longrightarrow B$ in the category of models of T′ is an injective ring morphism satisfying the condition: every system of linear equations $\sum_{j=1}^{n} a_{ij}Y_j = a_i$, $i=1,\ldots,m$ (a_{ij} , $a_i \in A$) with a solution in B^n has a solution in A^n; i.e. B is a pure A-module [16].

We shall call the morphisms of the category of models of T′ pure morphisms. (In particular, a faithfully flat morphism is a pure morphism).

(4.5.2) In this extended theory, each primitive formula is equivalent to one of the form

$$(\exists x) \bigwedge_{i=1}^{e} f_i(x,x')=0 \wedge \bigwedge_{i=1}^{e} \neg P_n(F_i(x,x'),\ G_i(x,x')) \quad ,$$

where $x=(x_1,\ldots,x_m)$, $x'=(x'_1,\ldots,x'_m)$ are variables, $f_i \in \mathbb{Z}[x,y]$, $F_i=(F_{i1},\ldots,F_{in}) \in \mathbb{Z}[x,x']^n$ and $G_i=(G_{i;j,k})_{1\leq j,h\leq n}$ are matrices with entries in $\mathbb{Z}[x,x']$.

Consequently, a ring morphism $u:A \longrightarrow B$ is existentially complete with respect to the extended theory T′ (we say that u is <u>T′-existentially complete</u>) iff for arbitrary $f_i \in A[X]$, $F_i=(F_{i1},\ldots,F_{in}) \in A[X]^n$, $G_i=(G_{i;j,k})_{1\leq j,k\leq n}$ matrices with entries in $A[X]$, where $i=1,\ldots,e$ and $X=(X_1,\ldots,X_m)$ are variables, if there exists $b \in B^m$ such that the following conditions are satisfied:

i) $f_i(b)=0$ for $i=1,\ldots,e$,

ii) $F_i(b)$ does not belong to the submodule of B^n generated by $(G_{i;1,k}(b),\ldots,G_{i;n,k}(b)) \in B^n$ $(k=1,\ldots,n)$, for $i=1,\ldots,e$, then there exists $a \in A^m$ such that the conditions derived from the previous conditions i) and ii) by replacing b, respectively B, by a, respectively A, are satisfied. In particular, u is an algebraically pure morphism.

According to (4.1.1), a ring morphism $u:A \longrightarrow B$ is T'-existentially complete iff there exists a ring morphism v from B into an ultrapower $^*A=A^I/_D$, such that v is a pure morphism and $i=vu$, where i is the canonical embedding of A into $^*A$.

(4.5.3) Remarks. i) For field extensions, existential completeness coincides with T'-existential completeness.

ii) The class of T'-existentially complete ring morphisms is stable under composition.

iii) If $f:A \longrightarrow B$ and $g:B \longrightarrow C$ are ring morphisms, such that g is a pure morphism and gf is T'-existentially complete then f is T'-existentially complete too.

(4.5.4) Proposition. Let L/K be a field extension and A be a K-algebra of finite type. If K is algebraically pure in L then A is T'-existentially complete in $L \otimes_K A$.

Proof. Let $A=K[X]/_{\underline{a}}$ where $X=(X_1,\ldots,X_n)$ are variables and $\underline{a}$ is an ideal in $K[X]$. Suppose that K is algebraically pure in L, or, equivalently, K is existentially complete in L. By (4.1.1) there exists an embedding over K of L into an ultrapower $K^I/_D$. Let M denote the higher order structure based on the disjoint union $K \cup \mathbb{N}$

and $^*M$ be either $M^I/_D$, if the ultrafilter D is δ -incomplete, or $(M^I/_D)^{\mathbb{N}}/_E$, where E is a nonprincipal ultrafilter on $\mathbb{N}$. Thus $^*M$ is a proper $\aleph_1$-saturated nonstandard model of the structure M. Let $^*\mathbb{N}$, $^*K$, $^*(K[X])$, $^*\underline{a}$, $^*A$ be the standard entities in $^*M$ attached respectively to $\mathbb{N}$, K, K[X], $\underline{a}$ and A.

Since K[X] is noetherian, $\underline{a}$ is finitely generated, and hence $^*\underline{a}=\underline{a}\ ^*(K[X])$ and $^*A= {}^*(K[X])/_{\underline{a}\ ^*(K[X])}$. On the other hand the ring of polynomials $^*K[X]$ can be identified in a canonical way with a proper subring of $^*(K[X])$. The field extensions $K\subset L\subset {}^*K$ induce the sequence of ring morphisms:

$$A=K[X]/_{\underline{a}} \longrightarrow L[X]/_{\underline{a}L[X]} \longrightarrow {}^*K[X]/_{\underline{a}\ ^*K[X]} \longrightarrow {}^*(K[X])/_{\underline{a}\ ^*(K[X])} = {}^*A$$

Since the composite morphism $A \longrightarrow {}^*A$ is an elementary embedding, it suffices by (4.1.1) to show that the composite morphism $B=L[X]/_{\underline{a}L[X]} \longrightarrow {}^*A$ is a pure morphism. Moreover, we shall show that the above morphism is faithfully flat. Indeed, the morphism $L[X]/_{\underline{a}L[X]} \longrightarrow {}^*K[X]/_{\underline{a}\ ^*K[X]}$ is faithfully flat by base change, and the morphism $^*K[X]/_{\underline{a}\ ^*K[X]} \longrightarrow {}^*(K[X])/_{\underline{a}\ ^*(K[X])}$ is also faithfully flat according to [10] p.130. We conclude that the morphism $B \longrightarrow {}^*A$ is faithfully flat and hence A is T′-existentially complete in B.

Q.E.D.

(4.6) The T′-existentially complete ring morphisms have good preservation properties. In the following we shall mention some of them.

(4.6.1) Proposition. Let $u:A \longrightarrow B$ be a T′-existentially complete ring morphism. Then the following assertions hold:

i) Let $\underline{a}$ be a finitely generated ideal in A, $F=(F_1,\ldots,F_m)$ and $G=(G_1,\ldots,G_{m'})$ be finite systems of polynomials in A[X], where

$X=(X_1,\ldots,X_n)$. If the system $F_1(x)=\ldots=F_m(x)=0$, $G_1(x)\neq 0,\ldots,G_m,(x)\neq 0$ has a finite number of solutions in aA^n then no other solutions exist in aB^n.

ii) If C is an A-algebra which is finite presentation as A-modul, then C is T′-existentially complete in $B\otimes_A C$.

iii) If p is a finitely generated prime ideal in A then pB is a prime ideal in B.

iv) If p is a finitely generated prime ideal in A, then the canonical field extension $Q(A/p)\longrightarrow Q(B/_{pB})$ is algebraically pure.

Proof. i) is immediate from definitions. Moreover it suffices to assume that u is an existentially complete ring morphism.

ii) Let a be a finitely generated ideal in A. Since A is T′-existentially complete in B, there exists a ring pure morphism $v:B\longrightarrow {}^*A=A^I/_D$ such that $i=vu$, where i denotes the canonical embedding $A\longrightarrow{}^*A$. Since i is an elementary embedding and C is finite presentation, A-modul it follows that the composite morphism $C\longrightarrow B\otimes_A C\longrightarrow {}^*A\otimes_A C\cong {}^*(A\otimes_A C)\cong {}^*C$ is an elementary embedding. As $B\otimes_A C\longrightarrow {}^*A\otimes_A C$ is a pure morphism (base change preserve pure morphisms), it results that C is T′-existentially complete in $B\otimes_A C$.

iii) If p is a finitely generated prime ideal in A, it follows by ii) that A/p is existentially complete in B/pB. Since A/p is an integral domain we conclude that B/pB is an integral domain too and hence pB is a prime ideal in B.

iv) By ii) and iii) the problem is reduced to show that Q(A) is algebraically pure in Q(B), where A and B are integral domains and A is existentially complete in B what it is immediate.

Q.E.D.

(4.6.2) <u>Corollary</u>. Let A be a noetherian ring and $u:A \longrightarrow B$ a T'-existentially complete ring morphism. Then the following hold:

i) u is a faithfully flat morphism.

ii) If q is a p-primary ideal in A then qB is a pB-primary ideal in B.

iii) If $a=q_1\cap\ldots\cap q_s$ is a reduced primary decomposition of the ideal a in A, and p_i are the associated prime ideals of q_i , then $aB=q_1B\cap\ldots\cap q_sB$ is a reduced primary decomposition of the ideal aB and p_iB are the associated prime ideals of q_iB, and $\sqrt{aB}=\sqrt{a}B$.

<u>Proof</u>. i) follows easily from the fact that there exists an A-algebra morphism $v:B \longrightarrow {}^*A$, where ${}^*A$ is an elementary extension of A and v is a pure morphism. (If L is a linear equation with coefficients in A, then the A-module S of the solutions of L in A is finitely generated. We get easily that S generates the ${}^*A$-module of the solutions of L in ${}^*A$. So ${}^*A$ is an A-flat module. The assertions ii) and iii) are immediate consequences of i) and of (4.6.1) iii) according to [18] , Theorem 13,page 60.

Q.E.D.

(4.6.3) <u>Corollary</u>. Let A and B be noetherian rings, and $u:A \rightarrow B$ be a T'-existentially complete ring morphism. Then the following hold:

i) Every saturated prime chain in A remains a saturated prime chain by extension to B.

ii) The height of an ideal a in A is preserved by extension to B, i.e. ht(aB)=ht(a).

<u>Proof</u>. i) By base change, we reduce to the case when A, B are integral domains and $(0)\subset q$ is a saturated prime chain in A (see (4.6.1) iii)). As ht(q)=1 we get that qB is a nonzero ideal with height one according to [18] , Theorem 19 page 79 (B is a flat module

by (4.6.2)i)). Thus the prime chain $(0) \subset qB$ is saturated.

ii) By (4.6.2) iii), we reduce to the case when a is a prime ideal. Using [18] theorem 19, we get immediately $ht(aB)=ht(a)$.

(4.6.4) Corollary. Let $u:A \to B$ be as above.

i) Let a be an ideal in A and p a be a prime ideal in A. Then aA_p can be generated in A_p by m elements iff aB_{pB} can be generated in B_{pB} by m elements.

ii) $p \in Reg(A)$ iff $pB \in Reg(B)$, where $Reg(A)$ denotes the set of prime ideals p of A such that A_p is regular.

Proof. i) The diagram $A \xrightarrow{u} B \xrightarrow{v} {}^*A$, where ${}^*A = A^I/_D$ and v is a pure morphism, induces, by (4.6.1)iii), the diagram of injective local morphisms $A_p \longrightarrow B_{pB} \longrightarrow {}^*A_{p^*A} \cong {}^*(A_p)$. If the ideal $a\,B_{pB}$ is generated in B_{pB} by m elements then clearly $a\,{}^*A_{p^*A} = {}^*(a\,A_p)$ is generated also by m elements. Since the ring morphism $A_p \longrightarrow {}^*(A_p)$ is an elementary embedding, we conclude that the ideal aA_p is generated in A_p by m elements.

ii) is a consequence of i) and of (4.6.3)ii).

Q.E.D.

(4.6.5) Corollary. Let A and B be noetherian integral domains and $u:A \to B$ a T'-existentially complete ring morphism. Then A is a factorial ring if B is so.

Proof. Let p be a prime ideal in A with height one. By (4.6.1) iii) and (4.6.3) ii), pB is a prime ideal with height one and hence pB is principal. Since A is existentially complete in B, we conclude that p is principal too ((4.2)iv).

Q.E.D.

(4.6.6) Remark. If u is a faithfully flat morphism but not T'-existentially complete then the above corollary is false. For instance, if $u:A:=\mathbb{R}[[X,Y]]/_{(X^2+Y^2-1)} \longrightarrow \mathbb{C}[[X,Y]]/_{(X^2+Y^2-1)}=:B$ is the canonical inclusion, then B is a factorial ring but A not.

(4.6.7) Proposition. Let $u:A \longrightarrow B$ be a T'-existentially complete ring morphism. Suppose that A and B are integral domains.

i) A is integrally closed in Q(A) iff B is integrally closed in Q(B).

ii) A is a Prüfer ring iff B is a Prüfer ring.

Proof. i) We have $Q(A) \cap B = A$, because u is in particular a pure morphism. Thus, A is integrally closed in Q(A) if B is integrally closed in Q(B). Conversely, if B is not integrally closed in Q(B), there exists $b \in Q(B) \setminus B$ such that $b^n+a_1b^{n-1}+\ldots+a_n=0$ with $n \geq 1$, $a_i \in B$. Thus there exist $a_1,\ldots,a_n,c,d \in B$ subject to: $d \neq 0$, c does not belong to the ideal in B generated by d, and $c^n+a_1c^{n-1}d+\ldots+a_nd^n=0$. Since A is T'-existentially complete in B we conclude that there exist $a_1,\ldots,a_n,c,d$ in A subject to the same conditions in A. It follows that A is not integrally closed in Q(A).

ii) Acording to [7],VII, § 2, Ex.12, an integral domain A is a Prüfer ring iff A is integrally closed in Q(A), and for every $z \neq 0$ in Q(A) there exist $x,y \in A$, such that $z=x+yz^2$.

Assume that A is a Prüfer ring. Then, by i), B is integrally closed in Q(B). If B is not a Prüfer ring, B satisfies the following existential sentence in the language L': "there exist $z \neq 0$, $z' \neq 0$ such that zz' does not belong to the ideal generated by z^2 and z'^2 ". Since A is T'-existentially complete in B, the same sentence is true in A and hence A is not a Prüfer ring, which gives a contradiction. The inverse implication follows in a similar way.

Q.E.D.

(4.6.8) Proposition. Let $u:A \rightarrow B$ be a T′-existentially complete ring morphism. Then the following assertions hold:

i) A is a field iff B is so.

ii) A is a local ring iff B is so.

iii) If A is a local ring and its maximal ideal $\underline{m}$ is finitely generated then $\underline{m}B$ is the maximal ideal of the local ring B.

iv) A is a valuation ring iff B is a valuation ring.

Proof. i) The ring A is not a field iff the following existential sentence in L′ is true on A: "there exists $x \neq 0$ such that 1 does not belong to the ideal generated by x".

ii) The ring A is not a local ring iff the following existential sentence in L′ is valid on A: "there exist x and y such that 1 does not belong to the ideals generated by x and by y and belongs to the ideal generated by x+y".

iii) Assume that A is a local ring and its maximal ideal $\underline{m}$ is finitely generated. Then, by (4.6.1) ii), the residue field A/m is T′-existentially complete in B/mB. By i), B/mB is a field and hence $\underline{m}B$ is the maximal ideal of B.

iv) An integral domain A is not a valuation ring iff the following existential sentence in L′ is satisfied by A: "there exist two elements $x \neq 0$ and $y \neq 0$ such that x does not belong to the ideal generated by y and y does not belong to the ideal generated by x".

Q.E.D.

(4.7) Now we shall define another type of existential completeness for rings which is stronger than the concept considered in (4.5). We proceed in the same way as in (4.5) by adding to the language L of rings certain new relation symbols.

For every finite system $G=(g_1,\ldots,g_e)$ of polynomials in

$\mathbb{Z}[X,Y]$, where $X=(X_1,\dots,X_n)$ and $Y=(Y_1,\dots,Y_m)$ are variables, we add to the language L a new relation symbol Q_G with n places. Denote by L" the extended language and by T" the theory in L" obtained by adding to the axioms of commutative rings with identity the family of new axioms:

$$Q_G(x) \longleftrightarrow (\exists y) \bigwedge_{i=1}^{e} g_i(x,y)=0$$

where G ranges over the set of all finite systems of polynomials in $\mathbb{Z}[X,Y]$, $X=(X_1,\dots,X_n)$, $Y=(Y_1,\dots,Y_m)$, e,n and m arbitrary natural numbers.

The models of T" are the commutative rings with identity and the new relation symbols Q_G have canonical interpretation on any ring A: if $a=(a_1,\dots,a_n)\in A^n$, then $A \models Q_G(a)$ iff the system of polynomial equations with coefficients in A,

$$g_1(a,Y)=\dots=g_e(a,Y)=0$$

has a solution in A^m.

Thus the morphisms in the category of models of T" are exactly the algebraically pure ring morphisms. In this extended theory T", each primitive formula is equivalent to one of the form

$$(+) \qquad (\exists\ x) \bigwedge_{i=1}^{e} f_i(x,x')=0 \quad \bigwedge_{i=1}^{e} \neg Q_{G_i}(x,x') \ ,$$

where $x=(x_1,\dots,x_n)$, $x'=(x'_1,\dots,x'_{n'})$ are variables, $f_i\in \mathbb{Z}[x,x']$, $G_i=(g_{i1},\dots,g_{ie})\in \mathbb{Z}[x,x';y]^e$, where $y=(y_1,\dots,y_m)$ are variables. The formula (+) is equivalent modulo T" with the following formula in the language L:

$$(\exists x)(\forall y) \bigwedge_{i=1}^{e} f_i(x,x')=0 \wedge \bigwedge_{i=1}^{e} \bigvee_{j=1}^{e} g_{ij}(x,x';y)\neq 0$$

Consequently, a ring morphism $u:A \longrightarrow B$ is existentially complete with respect to the extended theory T" (we say that u is <u>T"-existentially complete</u>) iff for arbitrary $f=(f_1,\dots,f_e) \in A[X]^e$, $G=(g_{ij})_{1\leq i,j\leq e}$ a matrix with entries in $A[X,Y]$, where $X=(X_1,\dots,X_n)$ and $Y=(Y_1,\dots,Y_m)$ are variables, if there exists $b \in B^n$ such that the following conditions are satisfied:

i) $f_1(b)=\dots=f_e(b)=0$,

ii) Every system $g_{i1}(b,Y)=\dots=g_{ie}(b,Y)=0$, $(i=1,\dots,e)$ has no solution in B^m, then there exists $a \in A^n$ such that the conditions derived from the previous ones by replacing b, respectively B, by a, respectively A, are satisfied too. In particular u is an algebraically pure morphism.

According to (4.1.1), a ring morphism $u:A \longrightarrow B$ is T"-existentially complete iff there exists an algebraically pure morphism v from B into an ultrapower $^*A=A^I/_D$, such that $i=v\circ u$, where i is the canonical embedding of A into $^*A$.

(4.7.1) <u>Remarks</u>. i) For field extensions, T"-existential completeness coincides with $\exists\forall$-completeness. More precisely, a field extension L/K is T"-existentially complete iff for each sentence in the language L extended with constants which are names for the elements of K, having the form $\varphi=(\exists x)(\forall y)\psi(x,y)$, where $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_m)$ are variables, and $\psi(x,y)$ is quantifier-free, φ is satisfied by K iff φ is satisfied by L. This is a consequence of the fact that a field extension is existentially complete iff it is algebraically pure.

ii) The class of T"-existentially complete ring morphisms is stable under composition.

iii) If $g \circ f$ is T"-existentially complete and g is algebraically pure then f is T"-existentially complete.

Since T"-existential completeness implies T′-existential completeness, all preservation properties of T′-existentially complete ring morphisms given in (4.6) are also satisfied by T"-existentially complete ring morphism. We add to these properties the following one.

(4.7.2). Proposition. Let A and B be noetherian local integral domains, and $u: A \to B$ a T"-existentially complete ring morphism. Then A is factorial iff B is so.

Proof. If B is factorial then A is factorial by (4.6.5). Conversely, assume that B is not factorial, i.e. there exists an ireducible element $x \in B$ which is not prime. In other words x cannot be write as a product $x'x"$ with $x', x"$ in the maximal ideal of B, and there are two elements $y, z \in B$ such that $yz \in xB$ and $y \notin xB$, $z \notin xB$.

Denote by $\underline{m}$ the maximal ideal in A, and let $a_1, \ldots, a_n$ be some generators of $\underline{m}$. By (4.6.8)iii), $\underline{m}B$ is the maximal ideal of B. Thus there exists a solution $(x,y,z,z') \in B^4$ of the equation $XZ'-YZ=0$ such that each of the following three equations:

$$xU-y=0, \quad xU'-z=0, \quad \sum_{i,j=1}^{n} a_i a_j T_i T_j' - x = 0$$

where $U, U', T_1, \ldots, T_n, T_1', \ldots, T_n'$ are variables, have no solution in B. Since A is T"-existentially complete in B, we conclude that the same situation occurs on A, i.e. there exists an irreducible element in A which is not prime. In other words, A is not a factorial ring.

Q.E.D.

(4.8) In the case of complete rings, the concepts of existential completeness considered in this section are extended as follows.

(4.8.1) <u>Definition</u>. A local morphism of noetherian complete local rings $u:A \longrightarrow B$ is called <u>analytically existentially complete</u> if for arbitrary finite systems $F=(F_1,\dots,F_e)$ and $G=(G_1,\dots,G_s)$ of formal power series in $A[[Z]][Y]$, where $Z=(Z_1,\dots,Z_n)$, $Y=(Y_1,\dots,Y_m)$ are variables, the system

$$F_1(z,y)=\dots=F_e(z,y)=0, \quad G_1(z,y)\neq 0,\dots,G_s(z,y)\neq 0$$

has a solution in A iff it has one in B (The components of z belong to the corresponding maximal ideals of A and B).

Note that for n=0, we recover the algebraic case of (4.2). Also in the case of artinian local rings, both definitions coincide.

(4.8.2) A local morphism of noetherian complete local rings $u:A\rightarrow B$ is called <u>analytically</u> T'-<u>existentially</u> complete if for arbitrary $f_i \in A[[Z]][Y]$, $F_i=(F_{i1},\dots,F_{is}) \in A[[Z]][Y]^s$, $G_i=(G_{i;j,k})_{1\leq j,k\leq s}$ matrices with entries in $A[[Z]][Y]$, where $i=1,\dots,e$, and $Z=(Z_1,\dots,Z_n)$, $Y=(Y_1,\dots,Y_m)$ are variables, the existence of some $(b,b') \in m_B B^n \times B^m$ (m_B denotes the maximal ideal of B) subject to conditions:

i) $f_i(b,b')=0$ for $i=1,\dots,e$,

ii) $F_i(b,b')$ does not belong to the submodule of B^n generated by $(G_{i;1,k}(b,b'),\dots,G_{i;s,k}(b,b')) \in B^s$ $(k=1,\dots,s)$, for $i=1,\dots,e$,

implies the existence of some $(a,a') \in mA^n \times A^m$ (m denotes the maximal ideal in A) satisfying the conditions obtained from the previous ones by replacing b, b', B respectively by a, a', A.

Note that for n=0, we recover the algebraic case of (4.5.2). Also in the case of artinian local rings, both definitions coincide.

(4.8.3) A local morphism of neotherian complete local rings $u:A \longrightarrow B$ is called T"-<u>existentially complete</u> if for arbitrary

$f=(f_1,\ldots,f_e)\in A[[Z]][Y]^e$, $G=(g_{ij})_{1\leq i,j\leq e}$ a matrix with entries in $A[[Z]][Y,U]$, where $Z=(Z_1,\ldots,Z_n)$, $Y=(Y_1,\ldots,Y_m)$, $U=(U_1,\ldots,U_s)$ are variables, if there exists $(b,b')\in m_B B^n \times B^m$ such that the following conditions are satisfied:

i) $f_1(b,b')=\ldots=f_e(b,b')=0$,

ii) Every system $g_{i1}(b,b',U)=\ldots=g_{ie}(b,b',U)=0$, $(i=1,\ldots,e)$ has no solution in B^m, then there exists (a,a') $mA^n \times A^m$ subject to the conditions derived from the preceding ones by replacing b,b',B respectively by a,a', A.

§5. Existential completeness for AP-rings

The main goal of this section is to apply the general theory of Section 4 to the special case of ring morphisms between AP-rings.

The first result gives a characterization of AP-rings in terms of existential completeness.

(5.1) Proposition. Let A be a noetherian local ring, m its maximal ideal, and $\hat{A}$ its completion in the m-adic topology. Then the following assertions are equivalent:

i) A is an AP-ring.

ii) A is existentially complete in $\hat{A}$.

iii) A is T'-existentially complete in $\hat{A}$.

iv) A is T''-existentially complete in $\hat{A}$.

Proof. We have to show that i) implies iv). Let $f=(f_1,\ldots,f_e)\in A[X]^e$ and $G=(G_1,\ldots,G_e)$, where $G_i=(g_{i1},\ldots,g_{ie})\in A[X,Y]^e$ for $i=1,\ldots,e$, $X=(X_1,\ldots,X_n)$, $Y=(Y_1,\ldots,Y_m)$. Denote by $\varphi(X)$ the formula:

$$(\forall Y) \bigwedge_{i=1}^{e} f_i(X)=0 \wedge \bigwedge_{i=1}^{e} \bigvee_{j=1}^{e} g_{ij}(X,Y)\neq 0$$

Assume that the sentence $(\exists X)\varphi(X)$ is true on $\hat{A}$. We have to show that the same sentence is true on A. Let $b\in \hat{A}^n$ be such that $\hat{A}$ satisfies $\varphi(b)$. Since $\hat{A}$ is an AP-ring it follows that there exists a natural number c such that for every natural number $s\geq c$, φ (b mod $\underline{m}^s A$) is true on $\hat{A}/_{\underline{m}^s\hat{A}}\cong A/_{\underline{m}^s}$. As A is an AP-ring there exists $a\in A^n$ such that $f_1(a)=\ldots=f_e(a)=0$ and $a\equiv b$ mod $\underline{m}^c\hat{A}$. We conclude that $\varphi(a)$ is true on A.

Q.E.D.

(5.1.1) Remark. Using (5.1), (4.6) and (4.7.2) we get some properties which are preserved from A to $\hat{A}$, if A is an AP-ring (see [15], chap.V).

The next step is to characterize the T'-existentially complete ring morphisms $u:A\longrightarrow B$, where A is an AP-ring and B a noetherian ring. If u is T'-existentially complete it follows by (4.6.2)i), (4.6.8)ii), iii), (4.6.1)ii), that B is a local ring with the maximal ideal generated by the maximal ideal $\underline{m}$ of A, u is faithfully flat, and the residue field extension $A/\underline{m}\hookrightarrow B/_{\underline{m}B}$ is algebraically pure. Moreover, the converse is also true:

(5.2) Theorem. Let A be an AP-ring, $\underline{m}$ its maximal ideal, B a noetherian local ring, and $u:A\longrightarrow B$ a local morphism such that $\underline{m}B$ is the maximal ideal of B, and u is flat. Then the following assertions are equivalent:

i) u is T'-existentially complete.

ii) u lifts well algebraically.

iii) u is existentially complete.

iv) u is algebraically pure.

v) The residue field extension $A/_{m} \longrightarrow B/_{mB}$ is algebraically pure.

Proof. The implications i) $\longrightarrow$ ii) $\longrightarrow$ iii) $\longrightarrow$ iv) $\longrightarrow$ v) are trivial. To prove the implication v) $\longrightarrow$ i), let us consider the commutative diagram in the category of models of T′ (observe that all morphisms are faithfully flat):

$$\begin{array}{ccc} A & \xrightarrow{\ u\ } & B \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_B} \\ \hat{A} & \xrightarrow{\ \hat{u}\ } & \hat{B} \end{array}$$

To conclude that u is T′-existentially complete it suffices to show that the composite morphism $\hat{u} \circ i_A = i_B \circ u$ is T′-existentially complete. Since A is an AP-ring, i_A is T′-existentially complete by (5.1). Thus, it remains to show that $\hat{u}$ is T′-existentially complete if the hypothesis v) is fulfilled. This implication is the object of the following theorem.

(5.3) *Theorem*. Let A be noetherian complete local ring and B a Cohen A-algebra (i.e. a flat, noetherian, complete local A-algebra such that $K=B/_{mB}$ is a separable field extension of the residue field $k=A/_{m}$ of A). Then the following assertions are equivalent:

i) The structure morphism $u:A \longrightarrow B$ is analytically T′-existentially complete.

i′) u is T′-existentially complete.

ii) u lifts well analytically.

ii′) u lifts well algebraically.

iii) u is analytically existentially complete.

iii′) u is existentially complete.

iv) u is analytically pure.

iv′) u is algebraically pure.

v) The residue extension K/k is algebraically pure.

<u>Proof</u>. The implications

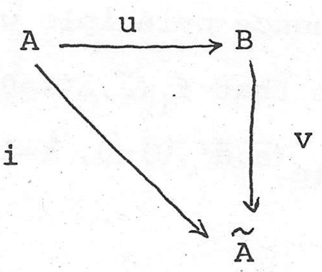
$$\begin{array}{ccccccccc} i) & \longrightarrow & ii) & \longrightarrow & iii) & \longrightarrow & iv) & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ i') & \longrightarrow & ii') & \longrightarrow & iii') & \longrightarrow & iv') & \longrightarrow & v) \end{array}$$

are trivial. It remains to show that v) implies i).

Suppose that k is algebraically pure in K, and hence, by (4.3), k is existentially complete in K. According to (4.1.1), K can be embedded over k into an ultrapower $k^I/_D$.

On the other hand, by structure theorem of noetherian complete local rings, we have $A \cong R[[X]]/a$, where R is either the field k or a complete discrete valuation ring of characteristic zero, with residue field k, and X are variables. Denote by M the higher order structure based on $R \cup \mathbb{N}$ and take $^*M$ to be either the structure $M^I/_D$ if the ultrafilter D is δ-incomplete, or the structure $(M^I/_D)^{\mathbb{N}}/_E$, where E is a nonprincipal ultrafilter on $\mathbb{N}$, on the contrary. Thus $^*M$ is an $\aleph_1$-saturated nonstandard model of the structure M. Denote by $^*A$, $^*m$, $^*k$, $^*\mathbb{N}$, etc., the standard entities in $^*M$ attached to A, m, k, $\mathbb{N}$, etc.

Thus we have the field extensions $k \hookrightarrow K \hookrightarrow {}^*k$. By [12], III, 0, proposition 10, 3, 1 there exist a noetherian complete local ring C, with $\underline{n}$ its maximal ideal, and a local morphism $w:B \longrightarrow C$ such that w is flat, $\underline{n}=\underline{m}C$ and $C/\underline{n} \cong {}^*k$. It follows that $w \circ u : A \longrightarrow C$ is the structure morphism of a Cohen A-algebra with residue field ${}^*k$. According to (3.6), the Cohen A-algebra C is isomorphic to $\tilde{A}={}^*A/_{{}^*\underline{m}^\infty}$. Thus we get the commutative diagram of faithfully flat morphisms



So, it remains to show that the canonical embedding $i:A \longrightarrow \tilde{A}$ is analytically T′-existentially complete to conclude that u is analytically T′-existentially complete. However, the fact that i is analytically T′-existentially complete is a consequence of the following stronger result.

(5.4) Proposition. Let $i:A \longrightarrow \tilde{A}$ be the canonical embedding afore considered. Then i is analytically T"-existentially complete.

Proof. Let $f=(f_1,\dots,f_e) \in A[[Z]][Y]^e$ and $G=(G_1,\dots,G_e)$, where $G_i=(g_{i1},\dots,g_{ie}) \in A[[Z]][Y, U]^e$ for $i=1,\dots,e$, $Z=(Z_1,\dots,Z_n)$, $Y=(Y_1,\dots,Y_m)$, $U=(U_1,\dots,U_s)$. Assume that there exist $(b,b') \in \underline{m}{}^*A^n \times {}^*A^m$ such that the following conditions are satisfied.

i) $f_i(p(b),p(b'))=0$ for $i=1,\dots,e$;

ii) The systems $g_{i1}(p(b),p(b'),U)=\dots=g_{ie}(p(b),p(b'),U)=0$ for $i=1,\dots,e$ have no solutions in $\tilde{A}^s$.

Since $\tilde{A}$ is a noetherian complete local ring, and hence an AP-ring, it follows that there exists a natural number c such that for each natural number $d \geq c$ ($b \bmod \underline{m}^d{}^*A$, $b' \bmod \underline{m}^d{}^*A$) satisfies the previous conditions i) and ii) over $\tilde{A}/_{\underline{m}^d\tilde{A}}={}^*A/_{\underline{m}^d{}^*A}$. On the other hand, by (3.7) there exists $(a,a') \in \underline{m}{}^*A^n \times {}^*A^m$ such that ${}^*f_i(a,a')=0$

for $i=1,\dots,e$, and $(a,a')\equiv(b,b')\bmod m^{c\,*}A$. It follows that the systems:

$$^*g_{i1}(a,a',U)=\dots={}^*g_{ie}(a,a',U)=0 \qquad i=1,\dots,e$$

have no solutions on $^*A$. By permanence principle we conclude that there exists $(\bar{a},\bar{a}')\in mA^n \times A^m$ such that $f_i(\bar{a},\bar{a}')=0$ for $i=1,\dots,e$, and the systems $g_{i1}(\bar{a},\bar{a}',U)=\dots=g_{ie}(\bar{a},\bar{a}',U)=0$, $i=1,\dots,e$, have no solutions in A^s.

Q.E.D.

Moreover we have the following more general result.

(5.5) Theorem. Let A be a noetherian complete local ring, m its maximal ideal, and B a Cohen A-algebra. Then the following assertions are equivalent:

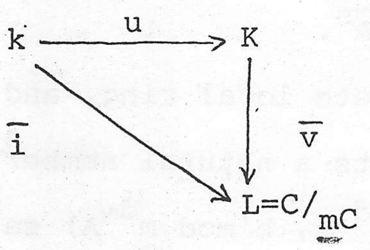
i) The structure morphism $u:A\longrightarrow B$ is analytically T"-existentially complete.

ii) u is T"-existentially complete.

iii) The residue extension $k=A/_m \hookrightarrow K=B/_{mB}$ is $\exists\forall$-complete.

Proof. The implication i) $\longrightarrow$ ii) is trivial.

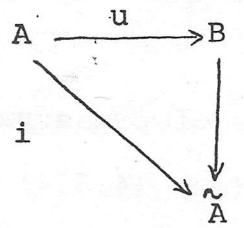
ii) $\longrightarrow$ iii). If u is T"-existentially complete then there exists an A-algebra morphism $v:B\longrightarrow C$ such that v is algebraically pure and C is an elementary extension of A. By base change, we obtain the commutative diagram of field extensions



where $\bar{i}$ is an elementary embedding and $\bar{v}$ is existentially complete. We conclude that $\bar{u}$ is $\exists\forall$-complete.

iii) $\longrightarrow$ i) Assume that the residue extension K/k is $\exists\forall$-complete. Then K can be embedded over k into an ultrapower $k^I/_D$ such that K is existentially complete in $k^I/_D$.

Then we proceed as in (5.3) and consider the higher order structure $\underline{M}$ and an $\aleph_1$-saturated nonstandard model $^*\underline{M}$ of $\underline{M}$. Thus we have the field extensions $k \hookrightarrow K \hookrightarrow {}^*k$, where K is existentially complete in $^*k$. This diagram induces in the same way as in (5.3) a commutative diagram of (faithfully) flat local morphisms:



Since K is existentially complete in $^*k$ it follows that $\tilde{A}$ is a Cohen B-algebra with residue field $^*k$. Moreover, by (5.2), v is algebraically pure. On the other hand, by (5.4), i is analitically T"-existentially complete **and hence u is analytically T"-existentially complete too.** Q.E.D.

(5.6) <u>Corollary</u>. Let A be an AP-ring, m its maximal ideal, B a noetherian local ring, and $u: A \longrightarrow B$ a local morphism such that mB is the maximal ideal of B, and u is flat. Then a necessary and sufficient condition for u to be T"-existentially complete is that the residue field extension $k = A/_m \longrightarrow K = {}^B/_{mB}$ is $\exists\forall$-complete.

<u>Proof</u>. The necessary part follows as in (5.5), ii) $\longrightarrow$ iii). Conversely, suppose that the field extension K/k is $\exists\forall$-complete. Let us consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_B} \\ \hat{A} & \xrightarrow{\hat{u}} & \hat{B} \end{array}$$

where $\hat{u}$ is the completion morphism of u.

By (5.2) all morphisms are algebraically pure. Since, by (5.1) and (5.5), i_A and $\hat{u}$ are T"-existentially complete, we conclude that u is T"-existentially complete too.

Q.E.D.

(5.6.1) <u>Remark</u>. Using (5.6), (4.6) and (4.7.2) we get some properties, which are preserved from A to $\tilde{A}$ (cf. 3.5). So we get as corollaries some results from [21], i.e. Proposition 2.10, Remark 2.11, Proposition 2.12.

Now we consider another case of preservation of T′-existential completeness by base change (see (4.5.4) and (4.6.1)ii)).

(5.7) <u>Proposition</u>. Let $(A,\underline{m})$ be an AP-ring and $u:A\longrightarrow B$ be a T′-existentially complete ring morphism of noetherian local rings. Then the induced ring morphism $u':A[X]_{(m,X)}\longrightarrow B[X]_{(m,X)}$ is T′-existentially complete, $X=(X_1,\ldots,X_n)$ being some indeterminates.

<u>Proof</u>. The induced field morphism $k:=A/_m \hookrightarrow B/_{mB}$ is existentially complete. So as in (5.3) there exist a noetherian complete local ring C and a local morphism $w:B\longrightarrow C$ such that w is flat, n=mC, $k'=C/_n$ is an $\aleph_1$-saturated nonstandard model of k, and the composite residue field extensions $k\hookrightarrow B/_{mB}\hookrightarrow k'$ induced by u and w is an elementary extension. By (5.5) v:=w u is T"-existentially complete morphism. As the induced morphism $B[X]_{(m,X)}$ $C[X]_{(m,X)}$ is faithfully flat (base change), it is enough to show that v induces a T′-existentially complete morphism $v':A[X]_{(m,X)}\longrightarrow C[X]_{(m,X)}$.

Now, by (4.1.1) there exist an elementary extension $i:A \longrightarrow {}^*A \simeq A^I/_D$ and an algebraic pure morphism $j:C \longrightarrow {}^*A$ such that $i=jv$. Consider the composite morphism h

$$C[X]_{(m,X)} \xrightarrow{j'} A[X]_{(m,X)} \longrightarrow {}^*(A[X]_{(\underline{m},X)}) \cong (A[X]_{(\underline{m},X)})^I/_D$$

where j' is induced by j. Clearly hv' is an elementary morphism and thus it is enough to show that h is a pure morphism. Let $p: {}^*(A[X]_{(\underline{m},X)}) \longrightarrow \tilde{A}[[X]]$ be the surjective map defined in (3.5) and $\tilde{h}:C[[X]] \longrightarrow \tilde{A}[[X]]$ the canonical extension of ph to $C[[X]]$. As $\tilde{h}$ and j induce the some map on the residue fields we deduce that $\tilde{h}$ is T"-existentially complete (5.5). So h is a pure morphism because ph is so.

(5.7.1) Corollary. Let $(A,\underline{m})$ be an AP-ring and $u:A \longrightarrow B$ be a T'-existentially complete ring morphism of noetherian local rings. Let $(C,\underline{n})$ be a local A-algebra essentially of finite type such that its structure morphism induces on the residue fields the identity automorphism of $A/\underline{m}$. Then $\underline{n}':=\underline{n}(B \otimes_A C)$ is a prime ideal in $B \otimes_A C$ and the morphism $C \longrightarrow (B \otimes_A C)_{\underline{n}'}$ induced by u is T'-existentially complete.

The proof is a consequence of (5.7) and (4.6.1)ii).

In the rest of this section we shall apply the previous results to the case when the morphism $u:A \longrightarrow B$ is a local formally smooth morphism.

(5.8) Theorem. Let $u:A \longrightarrow B$ be a local formally smooth morphism between noetherian complete local rings with residue fields k respectively K. Then the following assertions are equivalent:

i) u is analytically pure

ii) u is algebraically pure

iii) The residue field extension K/k is algebraically pure.

<u>Proof</u> The implication i)$\longrightarrow$ii) is trivial. On the other hand, B is of the type $B=A'[[X]]$ where A' is a Cohen A-algebra with residue field k and $X=(X_1,\ldots,X_n)$ are variables (see [12],IV,0, theorem 19,7,2).Thus u admits the following decomposition

$$A \xrightarrow{u'} A' \xrightarrow{u''} A'[[X]]=B$$

Since u" admits the A'-retract $X_i \mapsto 0$,u" is analytically pure.

ii)$\longrightarrow$iii). If u is algebraically pure then clearly u' is algebraically pure too. By base change (2,3) ii), the residue field extension $k=A/\underline{m} \hookrightarrow K=A'/\underline{m}A'$ induced by u' is algebraically pure.

iii)$\longrightarrow$i) If k is algebraically pure in K, then by (5.3) u' is analytically pure. Since u" is analytically pure, we conclude that the composite morphism u=u"u' is analytically pure too.

Q.E.D.

(5.9) <u>Remark</u>. Let $u:A\longrightarrow B$ be as in (5.8). If the residue extension K/k is algebraically pure, but B is not a Cohen A-algebra, it follows by (4.6.2) i) and (4.6.8) iii) that u is not T'-existentially complete. Moreover, if B is not a Cohen A-algebra the morphism u is not in general existentially complete. For instance, the morphism $k\hookrightarrow k[[X]]$,where k is a finite field, is not existentially complete. Let us give another example. Consider the formally smooth local morphism $u:\mathbb{Q}\longrightarrow \mathbb{Q}[[T]]$,where T is a variable The ring morphism is not existentially complete. Indeed let φ denote the following existential sentence

$$(\exists x)\ (\exists y)\ (\exists z)\ x^3+y^3=z^3 \wedge xyz\neq 0.$$

It is well known that φ is not true on $\mathbb{Q}$. However φ is satisfied by $\mathbb{Q}[[T]]$. Indeed, let us take x=T,y=1. By Hensel's lemma the polynomial $Z^3-(1+T^3)$ has a solution $z(T)\in \mathbb{Q}[[T]]$ such that z(0)=1.

(5.10) <u>Theorem</u> Let K be a field and T a variable. The following statements are equivalent:

i) The field extension $K \hookrightarrow K((T))$ is algebraically pure.

ii) The field extension $K \hookrightarrow Q(K\langle T\rangle)$ is algebraically pure ($K\langle T\rangle$ denotes the ring of algebraic power series with coefficients in K).

iii) For each AP-ring A with residue field K, the morphism $A \longrightarrow A[\![T]\!]$ has a weaker form of T'-existential completeness: the only modification is that the matrices G_i from (4.5.2) have entries in A.

iv) For each ring A as above, the morphism $A \longrightarrow A[\![T]\!]$ is existentially complete.

<u>Proof</u> Clearly i) implies ii), and the implication ii)$\longrightarrow$i) is a consequence of (2.3) i),ii) and of the fact that $K\langle T\rangle$ is, according to [1], an AP-ring. The implication iii)$\longrightarrow$ iv) is trivial

i)$\longrightarrow$iii) Let A be an AP-ring, $\underline{m}$ its maximal ideal, and $A/\underline{m} = K$. Denote by B the localization $A[\![T]\!]_{\underline{m}\, A[\![T]\!]}$. Clearly B satisfies the hypothesis of (5.2). Since, by i), K is algebraically pure in K((T)), the residue field of B, we conclude by (5.2) that the morphism $A \longrightarrow B$ is T'-existentially complete.

Let $f_i \in A[X]$, $F_i=(F_{i1},\ldots,F_{in}) \in A[X]^n$, $G_i=(G_{i;j,k})_{1\leq j,k\leq n}$ matrices with entries in A, where $i=1,\ldots,e$ and $X=(X_1,\ldots,X_m)$ are variables. Suppose that $b \in A[\![T]\!]^m$ satisfies the conditions:

1) $f_1(b)=\ldots=f_e(b)=0$

2) $F_i(b)$ does not belong to the submodule of $A[\![T]\!]^n$ generated by $(G_{i;1,k},\ldots,G_{i;n,k}) \in A^n$ $(k=1,\ldots,n)$, for $i=1,\ldots,e$.

Since A is T'-existentially complete in B, it suffices to show that 2) remains valid on B. Let $\varphi_i : A^n \longrightarrow A^n$ be the map associated to the matrix $(G_{i;j,k})_{j,k=1,\ldots,n}$. Using [18], Theorem 12, page 58, we get :

$$\mathrm{Ass}_{A[[T]]}(\mathrm{Coker}\ \varphi_i \otimes_A A[[T]]) = \{pA[[T]] \mid p \in \mathrm{Ass}_A \mathrm{Coker}\ \varphi_i\}.$$

Thus the prime ideals associated to Coker $\varphi_i \otimes_A A[[T]]$ does not intersect the multiplicative system $S := A[[T]] \setminus mA[[T]]$ and so the $\mathrm{Im}(\varphi_i \otimes A[[T]])$ is S-saturated as a sub-$A[[T]]$-modul of $A[[T]]^n$. Consequently, if $d_i := (F_{i,1}(b), \ldots, F_{i,n}(b))$ is contained in $\mathrm{Im}(\varphi_A B)$ then $d_i \in \mathrm{Im}(\varphi_i \otimes B) \cap A[[T]]^n = \mathrm{Im}(\varphi_i \otimes A[[T]])$, which contradicts 2).

iv) $\longrightarrow$ i) Applying iv) to the particular case A=K, it follows that $K[[T]]$ can be embedded over K into an elementary extension $^*K$ of K Therefore K((T)) can be identified with an intermediate field between K and $^*K$, i.e. K is existentially complete in K((T)).

Q.E.D.

(5.11) <u>Corollary</u> Let $u: A \longrightarrow B$ be a local formally smooth morphism between noetherian complete local rings with residue fields k respectively K. Suppose that K is algebraically pure in K((T)), where T is a variable. Then the following assertions are equivalent:

i) u has the weaker form of T'-existential completeness from (5.10) iii).

ii) u is existentially complete.

iii) The residue field extension K/k is algebraically pure.

<u>Proof</u> By (5.8) we have the implications i)$\longrightarrow$ii)$\longrightarrow$iii).

iii)$\longrightarrow$i). With the notations from (5.8), u' is T'-existentially complete by (5.3), and u" satisfies the condition from (5.10) iii) using (5.10) applied n-times succesively.

Q.E.D.

(5.12) <u>Corollary</u> Let A be an AP-ring, k its residue field, and $u: A \longrightarrow B$ a local formally smooth morphism into a notherian local ring B with residue field K. Assume that K is algebraically pure in K((T)), where T is a variable and k is algebraically pure in K. Then u has the weaker form of T'-existential completeness from (5.10) iii).

<u>Proof</u> By [12], IV, 0; Proposition 19,3,6 ,the induced morphism

$\hat{u}:\hat{A} \longrightarrow \hat{B}$ is still formally smooth. Let us consider the commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\quad u \quad} & B \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_B} \\ \hat{A} & \xrightarrow{\quad \hat{u} \quad} & \hat{B} \end{array}$$

By (5.1), i_A is T'-existentially complete, and by (5.11), $\hat{u}$ has the weak form of T'-existential completeness from (5.10) iii), and hence the composite morphism $\hat{u}i_A = i_B u$ has the same property. We conclude that u has the desired property.

Q.E.D.

(5.13) Remark If we leave the case of formally smooth morphisms, then the last results are not generally true. For instance the morphism $\mathbb{C}[[X]] \longrightarrow \mathbb{C}[[X]][Y]_{(X,Y)}/(Y^2-X)$ is not algebraically pure, but if we change the base from $\mathbb{C}[[X]]$ to $\mathbb{C}[[X]]/(X)=\mathbb{C}$ it becomes so.

(5.14) Examples when the condition i) from (5.10) is fulfilled:

i) if K is separable closed, by (4.4.6) c).

ii) if K is pseudo-algebraically closed, by (4.4.6) b).

iii) if K is real closed, by (4.4.6) d), because K((T)) is formally real having a valuation with a formally real residue field.

iv) if K is a p-adically closed field (see (4.4.6) e)) with finite absolute ramification index. Indeed K((T)) is a formally p-adic field extension of K.

v) more generally, if K is a p-adically closed field in the sense of [4]; for instance we can take K the field of Puiseaux series with coefficients in the field $\mathbb{Q}_p$ of p-adic numbers, or in the field $\mathbb{R}$ of reals, or in a pseudo-algebraically closed field k of characteristic zero with bounded corank (i.e. k has only finitely many algebraic extensions L subject to $[L:k]=n$ for each positive integer n).

(5.15) Remark Inthe particular case when A and B are complete discrete valuation rings of characteristic zero and $u:A \longrightarrow B$

is an unramified valuation ring morphism, (5.3) and (5.5) are immediate consequences of more general results of the model theory of henselian valued fields (see [2], [11], [14], [28], [30], [3]). Moreover, in this case, u is an elementary embedding iff the residue field extension is an elementary embedding.

For its particular interest, we end the present paper with a purely algebraic proof of (5.3) in the case of complete discrete valuation ring of characteristic zero.

(5.16) Theorem Let $u:A \longrightarrow B$ be an unramified extension of complete discrete valuation rings of characteristic zero. Then the following statements are equivalent:

i) u is analytically T'-existentially complete.

ii) The residue extension is algebraically pure.

Proof Let $f_i \in A[[Z]][Y]$, $F_i=(F_{i1},\dots,F_{ie}) \in A[[Z]][Y]^e$, $G=(G_{i;j,k})_{1\le j,k\le e}$ matrices with entries in $A[[Z]][Y]$, where $i=1,\dots,e$, and $Z=(Z_1,\dots,Z_n)$, $Y=(Y_1,\dots,Y_m)$ are variables. Assume that there exists $(b,b')\in \pi B^n \times B^m$ (π denotes the local parameter of A) such that the following conditions are fulfilled:

1) $f_i(b,b')=0$ for $i=1,\dots,e$

2) The linear systems

$$(S_i) \quad \sum_{\alpha=1}^{e} G_{i;j,\alpha}(b,b')\, U_\alpha = F_{i,j}(b,b') \; ; j=1,\dots,e$$

have no solutions in B^e.

We have to show that there exists $(a,a')\in \pi A^n \times A^m$ subject to the same conditions on A, if the residue extension $k=A/\pi A \longrightarrow K=B/\pi B$ is algebraically pure. First we show that the incompatibility of each linear system (S_i) is equivalent to the existence of a solution of some finite system of equations, which can be added to 1).

Let us fix an index i and let r be the rank of the matrix $\|G_{i;j,k}(b,b')\|$, and $\Delta(b,b')$ a non-zero rxr-minor of minimal valuation. This situation can be described by writing that all $(r+1)\times(r+1)$-minors of $\|G_{i;j,k}(b,b')\|$ are zero (if $r<e$), and that the equations

$\Delta(b,b')-\pi^t T_1=0$ (where t is the valuation of $\Delta(b,b')$), $T_1T_2-1=0$, $\Delta'(b,b')-\pi^t T'=0$ for each other rxr-minor $\Delta'(b,b')$ have solutions, where T_1,T_2,T', etc are new variables.

There exist two kinds of incompatibility for (S_i):

a) (S_i) is incompatible in the fraction field of B. This happens when there exists a non-zero $(r+1)\times(r+1)$-minor $H(b,b')$ of the matrix $\| G_{i;j,k}(b,b')/F_{i,j}(b,b')\|$. Let $s=\mathrm{ord}(H(b,b'))$. This fact can be described by equations as above: $H(Z,Y)-\pi^s W=0$, $WW'-1=0$.

b) (S_i) is compatible in the fraction field of B, but not in B. Clearly (S_i) is equivalent with a system of the form:

$$(+)\quad \Delta(b,b')U_j+\sum_{\alpha=r+1}^{e}\Delta_{j\alpha}(b,b')U_\alpha=F_j^*(b,b')\ ,\ j=1,\ldots,r$$

where $\Delta_{j\alpha},F_j^*\in A[[Z]][Y]$. Remark that $\mathrm{ord}\,(\Delta(b,b'))\leq \mathrm{ord}\,(\Delta_{j\alpha}(b,b'))$ since $\Delta(b,b')$ has the minimal valuation, and thus (+) is incompatible iff there exists $j_o\in\{1,\ldots,r\}$ such that $\mathrm{ord}\,(\Delta(b,b'))>$ $\mathrm{ord}\,(F_{j_o}^*(b,b'))$. Let $s=\mathrm{ord}(F_{j_o}^*(b,b'))$. Then we add the equations $F_{j_o}(Z,Y)-\pi^s V=0$, $VV'-1=0$ where V and V' are new variables.

Thus we succeded in replacing the situation described by 1) and 2) with the compatibility of a finite system of equations of type 1). Then it suffices to prove that u is analytically pure,which is equivalent by (2.6) with the fact that u is algebraically pure. So we have to show that u is algebraically pure if the residue extension K/k is algebraically pure. Let $F=(F_1,\ldots,F_m)$ be a system of polynomials in $A[Y]$, $Y=(Y_1,\ldots,Y_n)$ and $b\in B^n$ be such that $F(b)=0$. Let q be the kernel of the A-morphism $A[Y]\longrightarrow B\ :Y\longmapsto b$ and denote $r=\mathrm{ht}(q)$. Adding some polynomials to F we may suppose that F generates q. Since A is of characteristic zero, the field extension $Q(A)\hookrightarrow Q(B)$ is separable. By [18] Theorem 64 ,and Remark 2, p 219-221, there exist $h_1,\ldots,h_r\in q$ subject to:

i) $(h_1,\ldots,h_r)\,A[Y]_q=q\,A[Y]_q$.

ii) The jacobian matrix $\left(\frac{\partial h_i}{\partial Y_j}\right)_{\substack{1\leq i\leq r\\ 1\leq j\leq n}}$ has an rxr-minor M, which is not in q. Clearly we can choose the polynomials $h_1,\dots,h_r$ from the system F, because F generates q. Thus we may assume that $h_i=F_i$ for $i=1,\dots,r$. Since $M(b)\neq 0$, we may suppose according to Néron's p-desingularization (see [1] or [21]) that M(b) is invertible in B. Remark that b induces a solution of the system

$$F(Y)=0,\ M(Y)U=1$$

in K. The extension $k\subset K$ being algebraically pure, it follows that there exists $\tilde{a}\in A^n$ such that $F(\tilde{a})\equiv 0 \bmod \pi$ and $M(\tilde{a})\neq 0 \bmod \pi$. By the implicit function theorem, there exists $a\in A^n$ such that $F_1(a)=\dots=F_r(a)=0$ and $a\equiv\tilde{a} \bmod \pi$.

It remains to show that a is a solution of the whole system F. Let $\sqrt{(F_1,\dots,F_r)}=\bigcap_{i=1}^{t} q_i$ be the reduced primary decomposition of $\sqrt{(F_1,\dots,F_r)}$ in $A[Y]$; $q_1,\dots,q_t$ are prime ideals. Since $q=(F)\supset (F_1,\dots,F_r)$, q contains some q_i, say $q\supset q_1$. We have $(F_1,\dots,F_r)R[Y]_q \subset q_1R[Y]_q\subset qR[Y]_q$. Since $(F_1,\dots,F_r)R[Y]_q=qR[Y]_q$, we conclude that $q=q_1$. If t=1 then clearly a is a solution for the whole system F. If $t>1$, then $M\in\sqrt{q+\underline{a}}$, where $\underline{a}=\bigcap_{i=2}^{t} q_i$. Indeed if $\underline{b}\supset q+\underline{a}$ is a prime ideal which does not contain M, then $C:=(A[Y]/(F_1,\dots,F_r))_{\underline{b}}$ is not an integral domain. Let us show that C must be an integral domain. Suppose $M=\det\left(\frac{\partial F_i}{\partial Y_j}\right)_{i,j=1,\dots,r}$. Then the morphism $R[Y_{r+1},\dots,Y_n]\longrightarrow C$ is etale. As the normality is going up by etale morphisms (see [23], VII, § 2, proposition 2), it results that C is a normal ring and in particular an integral domain because C is local. Contradiction!

Thus there exists an integer $d\geq 1$ such that $M^d=M_1+M_2$, $M_1\in q$, $M_2\in\underline{a}$. As $F(a)=F(\tilde{a})\equiv 0 \bmod \pi$ and $M(a)\equiv M(\tilde{a})\neq 0 \bmod \pi$ it results $M_2(a)\neq 0$. Consequently $F(a)=q(a)=0$ because $M_2q\subset\sqrt{(F_1,\dots,F_r)}$.

Q.E.D.

REFERENCES

[1] M.Artin, Algebraic approximation of structures over complete local rings, Publ. Math. IHES,36(1969),23-58

[2] J.Ax and S.Kochen, Diophantine problems over local fields I,II, Amer. J. Math. 87 (1965),605-648; III, Annals Math. 83 (1966), 437-456

[3] Ş.Basarab, A model-theoretic transfer theorem for henselian valued fields, J. reine angew. Math.,311/312,(1979), 1-30.

[4] Ş.Basarab, Towards a general theory of formally p-adic fields, to appear in Manuscripta Math.,30,(1980),279-327.

[5] J.Becker, J.Denef, L.Lipshitz, and L.van den Dries, Ultraproducts and approximation in local rings I, Inventiones Math.,51,(1979),189-203.

[6] J.L.Bell, and A.B.Slomson, Models and ultraproducts: An introduction, North-Holland 1971.

[7] N. Bourbaki, Algèbre commutative, Hermann, Paris, 1961-1965.

[8] J.W.S.Cassels, Diophantine equations with special reference to elliptic curves, J. London Math. Soc, 41,(1966), 193-291.

[9] J.Denef, and L.Lipshitz, Ultraproducts and approximation in local rings II, Preprint.

[10] L.van der Dries, Model theory of fields, Thesis (1978),Utrecht.

[11] J.Ershov, On elementary theories of maximal normed fields I,II ,III (Russian),Algebra Logica Sem. 4,Nr. 3(1965) 31-70; 5,Nr. 1,(1966),5-40;6,Nr. 3,(1967),39-49.

[12] A.Grothendieck,and J.Dieudonné, E.G.A.,III,IV,Publ. Math. IHES (1964).

[13] M.Jarden, and P.Roquette, The Nullstellensatz over p-adically closed fields, to appear in J.Math. Soc. Japan

[14] S.Kochen, The model theory of local fields, Logic Conference Kiel, 1974, Lecture Notes in Math. 499, Springer-Verlag, Berlin, 1975, 384-425.

[15] H.Kurke,T.Mostowski,G.Pfister,D.Popescu and M.Roczen, Die Approximationseigenschaft lokaler Ringe, Lecture Notes in Math.,634,Springer-Verlag, Berlin,1978.

[16] D.Lazard, Autour de la platitude, Bull.Soc. Math.,France,97, (1969),81-128.

[17] W.A.J.Luxemburg, A general theory of monads, in "Contributions to Non-standard Analysis", Ed. W.A.J.Luxemburg, and A.Robinson, North-Holland,1972,18-86.

[18] H.Matsumura, Commutative Algebra, W.A.Benjamin Co.,New-York,1970

[19] V.Nica,and D.Popescu, Approximation Properties of the formally smooth morphisms, INCREST, Preprint 46,1979.

[20] G.Pfister,and D.Popescu, Die strenge Approximationseigenschaft lokaler Ringe, Inventiones Math.,30,(1975),145-174.

[21] D.Popescu, Algebraically pure morphisms, Rev.Roum.Math.Pures et Appl.,XXIV,6(1979),947-977.

[22] A.Prestel,and M.Ziegler, Model theoretic methods in the theory of topological fields, J.reine angew.Math. 299/300 (1978),318-341.

[23] M.Raynaud, Anneaux locaux henséliens ,Lecture Notes in Math., 169, Springer-Verlag, Berlin,1970.

[24] A.Robinson, Non-standard Analysis,North-Holland,1966.

[25] A.Robinson, Nonstandard arithmetic, Bull.A.M.S. 73(1967),818-843

[26] A.Robinson,and P.Roquette, On the finiteness theorem of Siegel and Mahler concerning diophantine equations, J. Number Th.,7 (1975),121-176.

[27] G.E.Sacks, Saturated Model Theory, W.A.Benjamin Co.,New-York 1972.

[28] V.Weispfenning, On the elementary theory of Hensel fields,Annals Math. Logic 10 (1976),59-93.

[29] W.Wheeler, Model-complete theories of pseudo-algebraically closed fields, Preprint.

[30] M.Ziegler, Die elementare Theorie der Henselschen Körper, Dissertation, Köln, 1972.