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INTRODUCTION

Under the name of "Combinatorial Decomposition" Garsia and Baclawski ([2]) proved a certain almost canonical decomposition for the elements of a noetherian, connected graded algebra over a field.

In fact, although not stated in the same form, such a decomposition, reflecting the connection between the depth and the Krull dimension of the ring, was long ago familiar to algebraists (see, for instance, Serre[4], ch.IV, §4, pp.14-16) and finer results are known, using the local cohomology description of the depth (see A.Grothendieck, [3]).

The aim of this note is to give an algebraic proof to this decomposition theorem and to show its connection with the structure of the Hilbert series of a noetherian graded algebra (Th.7 and Cor.8, § 2).

For the sake of completeness, we included, in § 1, some classical result on rational formal power series.

Using § 1, one can easily translate the results on the Hilbert series in terms of the Hilbert function. The main result of the paper and some comment are contained in § 2. The numbering of the results is sequential.

1. RATIONAL FORMAL POWER SERIES

Let $\mathcal{S}$ be the set of all the integer-valued sequences: $s = (r_n)_{n\in\mathbb{Z}}$ such that $r_n=0$, for $n<0$. The algebra structure of $\mathbb{Z}[[t]]$ is transfered on $\mathcal{S}$ by the converse of the "generating series" operator $\varphi : \mathcal{S} \to \mathbb{Z}[[t]]$, $\varphi((r_n)_n) := \sum_{n\geq 0} r_n t^n$. With this structure on $\mathcal{S}$, φ becomes a $\mathbb{Z}$- algebra isomorphism, identifying the subring $\mathcal{F} = \{(r_n) \in \mathcal{S} \mid r_n=0 \text{ for } n \gg 0\}$ of $\mathcal{S}$ with the polynomial subring $\mathbb{Z}[t]$ of $\mathbb{Z}[[t]]$. If $\mathbb{Z}(t)$ and $\mathbb{Z}((t))$ denote the quotient fields of $\mathbb{Z}[t]$ and $\mathbb{Z}[[t]]$ respectively, then $\mathbb{Z}(t)$ is a subfield of $\mathbb{Z}((t))$. The elements of the subring $\mathcal{R} := \mathbb{Z}(t) \cap \mathbb{Z}[[t]]$ of $\mathbb{Z}((t))$ will be called "rational" formal power series.
The subring $\mathcal{L} := \varphi^{-1}(\mathcal{R})$ consists of the usual linear reccurent sequences. $\mathcal{L}$ may be directly defined by

$$\mathcal{L} = \{(r_n) \in \mathcal{S} \mid (\exists)\ e>0,\ h\geq 0 \text{ and } a_o,\dots,a_e \in \mathbb{Z} \text{ (not all zero), such that: } a_o r_n + a_1 r_{n-1} + \dots + a_e r_{n-e} = 0, \text{ for } n \geq h\}.$$

For any $c \in \mathbb{Z}[t]$, the multiplication by $c: m_c : \mathbb{Z}[[t]] \to \mathbb{Z}[[t]]$ invariates $\mathcal{R}$ and, if $c \in 1 + t\mathbb{Z}[t]$, $c^{-1} \in \mathbb{Z}[[t]]$ and $m_{c^{-1}}$ also invariates $\mathcal{R}$. Then $\nabla_c := \varphi^{-1}(m_c)$ and $\sigma_c := \varphi^{-1}(m_{c^{-1}})$ (for $c \in 1 + t\mathbb{Z}[t]$) are $\mathbb{Z}$- automorphisms of $\mathcal{L}$. The ∇_c's and the σ_c's commute in $\mathrm{Aut}_{\mathbb{Z}}(\mathcal{L})$ and generate a subgroup G, with the property that $\mathcal{L}$ is the G-orbit of the unit sequence $u = (\delta_{on})_{n\geq 0}$ (Kronecker delta). ∇_c and σ_c are sometimes called "umbral" operators. In case $c=1-t$, they are the usual operators of the finite differences calculus.

Let Q be the field of the rational numbers. A simple linear algebra argument shows that, if $g_j = \sum_{n\geq 0} r_{jn} t^n \in Q[[t]]$, $j=1,\ldots,e$ are Q-linear dependent, then, for any sequence $0\leq n_1 \leq n_2 \leq \ldots \leq n_e$ the determinant $\det(r_{jn_i})_{1\leq i,j\leq e}$ must vanish. Conversely, the vanishing of <u>all</u> these determinants implies the Q(resp. $\mathbb{Z}$)-linear dependence of $g_1, g_2, \ldots, g_e$. The non-degenerate case is obtained when we suppose that there are (e-1) between $g_1,\ldots,g_e$, which are Q (resp. $\mathbb{Z}$)-linear independent.

Now, a direct computation shows that a series $f \in \mathbb{Z}[[t]]$ belongs to $\mathcal{R}$ iff there are integers $e>0$, $h\geq 0$ such that, for $n\geq h$, the series: $t^{-n-j}(f - T_{n+j-1})$, $j=0,1,\ldots,e$ are Q (resp. $\mathbb{Z}$)-linear dependent (where T_ℓ denotes the $(\ell+1)$st partial sum f, $l\geq 0$). (The minimal degree denominator for f is obtained in the non-degenerate situation).

Combining these two arguments, one gets the following characterization of $\mathcal{R}$ (resp. $\mathcal{L}$): $f = \sum_{n\geq 0} r_n t^n$ belongs to $\mathcal{R}$ (resp. $s=(r_n)_n$ belongs to $\mathcal{L}$) iff the infinite matrix: $(r_{i+j})_{i,j\geq 0}$ has finite rank. This means that all the Hankel determinants of $(r_n)_{n\geq 0}$ vanish, from a certain order on.

By further extending the scalars from Q to $\mathbb{C}$ (the field of the complex numbers), the denominator of any $f=pq^{-1}$ splits into linear factors. If $q \in 1 + t\mathbb{Z}[t]$, then $q = \prod_{i=1}^{s} (1-\alpha_i t)^{m_i}$, with $s\geq 1$, $\alpha_i \in \mathbb{C}$, $m_i \in \mathbb{N}$ and $f=pq^{-1}$ may be developed into simple fractions ($p \in \mathbb{Z}[t]$). The binomial formula gives:

$$(\forall)\ n\geq 0,\ r_n = \sum_{j=1}^{s} \sum_{i=1}^{m_j} \beta_{ji} \binom{i+n-1}{i-1} \alpha_j^n + \gamma_n,$$

where $f = \sum_{n\geq 0} r_n t^n$, $\beta_{ji}, \gamma_j \in \mathbb{C}$ and $\gamma_n = 0$ for $n \gg 0$.

In case t=1 is the only root of q, r_n is a polynomial in n, for n large and it is a polynomial for $n\geq 0$ (i.e. the perturbations γ_n disappear) if the degree of f as a rational function (i.e. deg p-deg q) is negative.

2. NOEHTERIAN GRADED ALGEBRAS OVER A FIELD

By k we shall denote a fixed field. A "gradation" on k is a constant function $\delta : k^* \longrightarrow \mathbb{Z}$ (sometimes extended to k by $\delta(0)=-\infty$). If $s=\delta(k^*)$, we shall denote by $\underline{k}(s)$ the pair (k,δ) and by $\underline{k}$ the gradation $\underline{k}(0)$. A $\underline{k}$-module is a k-vector space V, together with a $\mathbb{Z}$ - decomposition $V=\bigoplus_{n\in\mathbb{Z}} V_n$, where V_n are k-vector subspaces of V. A $\underline{k}$-homomorphism is a k-linear application: $f:V=\bigoplus_n V_n \longrightarrow U=\bigoplus_n U_n$, such that $f(V_n)\subseteq U_n$, for all $n\in\mathbb{Z}$ and a "homogeneous" $\underline{k}$-homomorphism (of degree s) is a $\underline{k}$-homomorphism $V\longrightarrow U\otimes_{\underline{k}} \underline{k}(s)$, $s\in\mathbb{Z}$.

The category so-obtained is denoted by Mod$(\underline{k})$. Let $\mathcal{H}$ be the full subcategory whose objects are: $V=\bigoplus_{n\in\mathbb{Z}} V_n \in$ Mod $(\underline{k})$ such that $\dim_k V_n<\infty$, $(\forall)\ n\in\mathbb{Z}$. For $V=\bigoplus V_n$, the function $h_V:\mathbb{Z}\longrightarrow\mathbb{Z}$, given by $h_V(n)=\dim_k V_n$ is called "the Hilbert function" of V and the generating series of $(h_V(n))_n$: $H_V(t)=\sum_{n\in\mathbb{Z}} h_V(n)t^n \in \mathbb{Z}((t))$ is called "the Hilbert series" of V.

For $V,U\in\mathcal{H}$ we have the formulas: $H_{V\oplus U}=H_V+H_U$ and $H_{V\otimes_{\underline{k}} U}=H_V.H_U$.

An object $V\in\mathcal{H}$ is called "positively graded" if $V_n=0$ for $n<0$.

In case $V=\bigoplus V_n$ is a k-algebra, we call it a "$\underline{k}$-algebra" if $V\in$ Mod$(\underline{k})$ and, when $V=\bigoplus V_n$, $V_nV_m\subseteq V_{n+m}$ for any $n,m\in\mathbb{Z}$. An ideal of V is called "homogeneous" if it is generated by elements of $\bigcup_n V_n$.

A V-module M is a "graded" module if $M\in$ Mod$(\underline{k})$ and, if $M=\bigoplus M_n$, then $V_nM_m\subseteq M_{n+m}$, $(\forall)\ n,m\in\mathbb{Z}$.

If $V=\bigoplus V_n$, we call the elements of V_n $(n\in\mathbb{Z})$, "homogeneous of degree n".

From now on, we shall denote by R a commutative, unitary, positively graded k-algebra, such that

(I) R is connected, i.e. $R_o=k$

(II) R is noetherian, i.e. R is finitely generated over k or, equivalently, the ideal $R_+:=\bigoplus_{n>0} R_n$ has (homogeneous) finite basis in R.

In this case, $R\in\mathcal{H}$ and, more, any graded, finitely generated R-module belongs to $\mathcal{H}$. It is classical ([1], ch.11, Th.11.1) that, for a noetherian, graded R-module M, $H_M(t)$ is rational and $H_M(t)=p(t)\prod_{i=1}^{e}(1-t^{d_i})^{-1}$, with $d_1,\dots,d_e$, $e>0$ and $p\in\mathbb{Z}[t]$. When $p(1)\neq 0$, e=dim M= the order of the pole (t=1) in $H_M(t)$.

We shall give below a more precise expression of H_R in lowest terms, showing its relevance to the algebraic invariants of R.

We begin with the most important of them, the (Krull) dimension of R. Parallely, we shall treat the depth of R.

As $k\subset R$, dim $R=\text{tr.deg}_k R$. Let $\tilde{k}$ be the algebraic closure of k in R. Then $\tilde{k}$ is a graded subalgebra of R. As a homogeneous element of R is algebraic over k iff it is nilpotent, it follows that $\tilde{k}=k\oplus r(R)$, where r(R) is the nilradical of R.

In general, $\tilde{k}$ is not noetherian. $\tilde{k}$ is noetherian iff it is artinian and, in this case, it is a finitely dimensional k-vector space. Thus, R is artinian iff $R=\tilde{k}$ and we see that the artinian k-algebras are characterized by the fact that their Hilbert series is a polynomial (with non-negative integral coefficients).

Thus, let us suppose $n=\dim R>0$. Then R has an homogeneous element $x\in R_d$, $d>0$, transcendental over k: this means that $k[x]$ (with the induced gradation) is k-isomorphic to $k[T]$, where T is a variable of degree d, over k.

When x is not a zerodivisor in R, we say that x is "pure" transcendental over k. The reason for this name comes from the following:

Proposition 1

Let $x\in R_d$, $d>0$ be a homogeneous non-zerodivisor in R. Then there is a canonical k-isomorphism:

$$R = k[x]\otimes_k R/_{xR},$$

where $k[x]$ has the induced gradation and $R/_{xR}$ the quotient one.

Proof.

For any $n\geq 0$, the descending sequence of k-vector spaces

$$R_n \supset xR_{n-d} \supset x^2R_{n-2d} \supset \ldots \supset (0)$$

gives the isomorphism of k-vector spaces:

(a) $R_n = \bigoplus_{i=0}^{r} x^iR_{n-id}/x^{i+1}R_{n-(i+1)d}$, where r is the residue of n modulo d.

If x does not divide 0, in R, $x^iR_{n-id}/x^{i+1}R_{n-(i+1)d} \cong$

$\cong R_{n-id}/xR_{n-(i+1)d}$, for any n and $i=0,1,\ldots,r$.

However, this last isomorphism of k-vector spaces is not compatible with the gradations; tensorizing over k the right-hand side of it by kx^i, then replacing in (a), we get the conclusion.

q.e.d.

Corollary 2

If $\{x_1,\dots,x_p\}$ is a homogeneous R-sequence in R_+, then $R \cong_{\underline{k}} k[x_1,\dots,x_p] \otimes_{\underline{k}} R/_{(x_1,\dots,x_p)R}$. In particular:

$\dim R = p + \dim R/_{(x_1,\dots,x_p)R}$ and:

$$H_R = H_{R/(x_n,\dots,x_p)R} \cdot \prod_{j=1}^{e} (1-t^{\deg x_j})^{-1}.$$

Proof

$k[x_1] \otimes_{\underline{k}} k[x_2] \otimes_{\underline{k}} \dots \otimes_{\underline{k}} k[x_p] \cong_{\underline{k}} k[x_1,\dots,x_p]$ and

$$H_{k[x_1,\dots,x_p]}(t) = \prod_{j=1}^{e} (1-t^{\deg x_j})^{-1}.$$

q.e.d.

Remark 3

If $\{x_1,\dots,x_p\} \subset R_+$ is a homogeneous R-sequence, then $k[x_1,\dots,x_p]$, with the induced gradation, is $\underline{k}$-isomorphic to $k[T_1,\dots,T_p]$, where $T_1,\dots,T_p$ are variables over k and $\deg T_j = \deg_R x_j$, $j=1,2,\dots,p$.

The condition $\dim R>0$, as we saw above, ensures only the existence of some transcedental homogeneous element $x \in R$. In general, however, x is not pure transcendental.

The algebraic object measuring the "purity" of the transcedental element in R is the classical local cohomology homogeneous ideal:

$$\Gamma(R) = \bigcup_{n\geq 1} Ann_R(R_+^n).$$

For the sake of completeness, we reprove here the fundamental property of $\Gamma(R)$, under the following rude form (see also [3], exp.III, Exemple III-1, pp.11):

Proposition 4

R has a pure transcendental, homogeneous element in R_+ iff $\Gamma(R)=(0)$.

Proof

If R has no non-zerodivisors (in R_+), then R_+ is associated with zero: this means there is some nonzero $x \in R_+$, homogeneous, and $R_+ = Ann_R(xR) \Rightarrow \Gamma(R) \neq (0)$.

Conversely, $\Gamma(R) \neq 0 \Rightarrow$ depth $R=0$; cf. [3], loc.cit., i.e. R has no non-zerodivisor in R_+.

q.e.d.

Proposition 5

In the above notations and hypotheses:

(i) $R \cong \Gamma(R) \oplus R/_{\Gamma(R)}$ as k-modules

(ii) dim R = dim $R/_{\Gamma(R)}$ and depth $R \leq$ depth $R/_{\Gamma(R)}$.

Proof

(i) This property holds for any homogeneous ideal of R, in particular for $\Gamma(R)$.

(ii) Indeed, for any $x \in \Gamma(R)$, Rx is a finitely dimensional k-vector space and $\Gamma(R)$ has a finite basis, R being noetherian. Thus $\Gamma(R) \subseteq r(R)$ and the equality for the dimensions holds.

The strict inequality for the depths holds when $\Gamma(R)=0$, by Proposition 4 above.

q.e.d.

Corollary 6

$H_{\Gamma(R)}(t)$ is a polynomial with non-negative integral coefficients and $H_R=H_{\Gamma(R)}+H_{R/\Gamma(R)}$.

Now, we are in measure to prove the

Theorem 7

Let R be a noetherian, connected, graded algebra over k. Then there is an isomorphism of k-modules:

$$(*)\quad R\cong\Gamma_o\oplus\Gamma_1\otimes_{\underline{k}} k[T_1,\dots,T_{1n_1}]\oplus\dots\oplus\Gamma_h\otimes_{\underline{k}} k[T_1,\dots,T_{1n_1},\dots,T_{h1},\dots,T_{hn_h}]$$

where h is an integer, $h\geq 0$ and:

(1) $\Gamma_o,\Gamma_1,\dots,\Gamma_{h-1}$ are R-modules, which are graded in degrees ≥ 1, and $\dim_k\Gamma_j<\infty$, $j=1,\dots,h-1$. They are all zero if $h=0$.

(2) Γ_h is an artinian graded R-algebra

(3) $n_1,n_2,\dots,n_h$ are positive integers, such that

$$n_1+n_2+\dots+n_h=\dim R \text{ and } n_1=\text{depth } R, \text{ when } \Gamma_o=(0).$$

(4) $\{T_{ij}\}_{\substack{1\leq i\leq h\\ 1\leq j\leq n_1+n_2+\dots+n_i}}$ are indeterminates over k and $\deg T_{ij}>0$ for any $1\leq i\leq h$ and $1\leq j\leq n_1+n_2+\dots+n_i$.

Proof

We take $\Gamma_o=\Gamma(R)$ then we use inductively the Proposition 5 and the Corollary 2, until we get rid of dim R, which is finite, R being noetherian.

q.e.d.

Corollary 8

In the above notations and hypotheses:

$$(**)\quad H_R(t)=\sum_{j=0}^{h}\gamma_j(t)\cdot\prod_{i=1}^{n_1+\dots+n_j}(1-t^{d_{ji}})^{-1},$$

where $\gamma_1,\dots,\gamma_{h-1}\in t\mathbb{Z}_+[t]$, $\gamma_h\in 1+t\mathbb{Z}_+[t]$ and n_j, d_{ji} are positive integers for $j=0,1,\dots,h$ and $i=1,2,\dots,n_1+\dots+n_j$, (for $j=0$, $n_1+\dots+n_j$ means 0 and "$\prod_1^0$" means 1. $\mathbb{Z}_+$ denotes the non-negative integers).

Proof

We pass to the Hilbert series in the k-isomorphism (*).

q.e.d.

We shall make, now, some comment on the above results.

(1) In (*), it follows that each term $\Gamma_j\otimes_k P_j$, $j=0,\dots,h$ (where $P_o=k$ and $P_j=k[T_1,\dots,T_{1n_1};\dots;T_{jn_j}]$ for $j>0$ is a Cohen-Macaulay k-module. Then we obtain:

Corollary 9

R is Cohen-Macaulay k-algebra iff, in (*),

~~$\Gamma_o=\Gamma_1=\dots=\Gamma_{h-1}=0$.~~ $h=1$ and $\Gamma_o=0$

This corollary is equivalent to the statement that the decomposition (*) of R reduces to its last term, i.e. $R \underset{k}{\simeq} \Gamma_h \otimes_k k[T_{11},\dots,T_{1n_1};\dots;T_{h_1},\dots,T_{hn_h}]$, in which case the artinian algebra Γ_h must be a factor-ring of R, modulo some homogeneous ideal, generated by a system of parameters which is an R-sequence.

For R Cohen-Macaulay, it follows, then, that:

$$H_R(t)=p(t).\prod_{i=1}^{\dim R}(1-t^{d_i})^{-1}, \text{ where } d_1,\dots,d_{\dim R} \geq 0$$

and $p \in 1+t\mathbb{Z}_+[t]$ (see also [5], Corollary 3.11).

This is a necessary condition for the Macaulayness of a noetherian algebra R, in terms of its Hilbert series.

(2) For the sake of completeness, let us remind the criterium for the Gorensteinness of a Macaulay, noetherian algebra R, in terms of its Hilbert series ([5]):

R is Gorenstein iff $H_R \cdot \prod_{j=1}^{\dim R}(1-t^{d_j})$ is a reciprocical polinomial in $1+t\mathbb{Z}_+[t]$.

This criterium is valid with the caution that R must be a domain.

(3) The decomposition (**) of $H_R(t)$, as given in the Corollary 8 above, is relevant for the algebraic structure of R only with respect to dim R and depth R (and, as is easily seen, to $\mu(R)$ (the multiplicity of R), which is p(1), p(t) being the numerator in H_R, after we write it as a rational function; more, in case R is Cohen-Macaulay, the usual "type" of the Cohen-Macaulay R-module R is given by the greatest degree coefficient in the numerator of H_R).

However, (**) does not reveal more of the structure of R because of the following reasons:

(a) the decomposition (*) of R, although unique, takes place only in Mod($\underline{k}$), not in Mod ($\underline{R}$). Indeed, with respect to the multiplication of R, we see that: first, the terms $\Gamma_j \otimes_{\underline{k}} P_j$ in (*) may have various positions between themselves, the only restriction being that each Γ_j must annihilate great powers of the irrelevant ideal of $\bigoplus_{i>j} \Gamma_i \otimes_{\underline{k}} P_i$ (thus, one can put the trivial multiplication on the decomposition (*), preserving the Hilbert series); secondly, as is easy to see, on each finitely dimensional k-vector space Γ_j there are, in general, various non-isomorphic multiplicative structures, with the same Hilbert series for Γ_j.

(b) for a given rational series $H(t) = \dfrac{p(t)}{\prod_{i=1}^{e}(1-t^{d_i})}$,

with $p \in 1+t\mathbb{Z}_+[t]$ and $e, d_1, \ldots, d_e > 0$, one has various decompositions of the type (**).

For instance, let $H(t) = \dfrac{1+t+t^2-t^3}{1-t}$ (by (1) above, we directly see that H can not represent a Cohen-Macaulay noetherian graded algebra). Then H has the decompositions:

(i) $H = t^2 + \dfrac{1+t}{1-t}$, corresponding to $R = \dfrac{k[X,Y]}{(X^3, X^2Y)}$ (with deg X= = deg Y =1), with the (*) - decomposition:

$$R \cong {}_{\underline{k}}kX^2 \oplus (k+kX) \otimes_{\underline{k}} k[Y]$$

(ii) $H=(t+t^2)+\frac{1+t^2}{1-t}$, corresponding to:

$$R=\frac{k[X_1,X_2;Y_1,Y_2]}{(X_1^2,X_1X_2,X_1Y_1,X_1Y_2;Y_1X_2,Y_1^2,Y_1Y_2,Y_2^2)},$$

(with deg X_1=deg X_2=1 and deg Y_1=deg Y_2=2), having the following (*)-decomposition:

$$R\cong_{\underline{k}}(kX_1+kY_1)\oplus(k+kY_2)\otimes_{\underline{k}}k[X_2]$$

Thus, we see that the result in Theorem 7 may be interpreted only as a "closure" theorem, saying that the class of the connected, noetherian graded k-algebras is recovered from the class of artinian k-algebras and the class of the homogeneous polynomial k-algebras, by finitely operating with (graded) direct sums and tensor products.

With respect to the relevance of the Hilbert series to the algebraic structure, all we can positively say is that the expression (**) of H_R, when written as a rational function, must have the property:

(P) <u>all the terms with positive coefficients, in the numerator of H_R, must "precede" the terms with negative coefficients.</u>

In order to make more precise this statement, we give the following example.

Let $H(t)=\frac{1-t+t^2}{(1-t)^2}$. Then H does not have a (**)- decomposition, because the term "-t" precedes the term "t^2". Indeed, we find that H is the Hilbert series of the following <u>non-noetherian</u> k-algebra:

$$R=k[x_1,x_2^2,x_3^3,\dots,x_n^n,\dots]$$

which is graded with the induced gradation by the envelopping algebra $k[x_1,x_2,\dots,x_n,\dots]$, where $x_1,\dots,x_n,\dots$ is a numerable family of indeterminates over k, all of degree +1.

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