INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

CONNECTIVITY OF DELETED PRODUCTS OF TOPOLOGICAL COUPLES

by

Jack WEINSTEIN

PREPRINT SERIES IN MATHEMATICS
Nr.105/1981

BUCURESTI

CONNECTIVITY OF DELETED PRODUCTS OF TOPOLOGICAL COUPLES

by

Jack WEINSTEIN *)

November, 1981

* Mathematical Seminar "A. Myller", Computer Center
University "Al. I. Cuza"
6600 - Iasi, Romania

CONNECTIVITY OF DELETED PRODUCTS OF TOPOLOGICAL COUPLES

by

JACK WEINSTEIN

<u>O. Introduction.</u> A nonvoid topological space Z is said to be nondegenerate, if it is not a singleton; it is said to be linearly ordered, if its topology is the interval topology defined by a linear order on the underlying set. The deleted product of Z is the subspace

$$R(Z) = \{(z_1, z_2) \in Z \times Z \mid z_1 \neq z_2\}$$

of $Z \times Z$. Obviously, $R(Z)$ is nonvoid if and only if Z is nondegenerate. For Z linearly ordered, define the "upper-left triangle"

$$T_{<}(Z) = \{(z_1, z_2) \in Z \times Z \mid z_1 < z_2\}$$

and the "lower-right triangle"

$$T_{<}(Z) = \{(z_1, z_2) \in Z \times Z \mid z_1 > z_2\}.$$

Then $R(Z) = T_{<}(Z) \cup T_{>}(Z)$ and $T_{<}(Z) \cap T_{>}(Z) = \emptyset$. In [1], Eilenberg proved the following result:

<u>Theorem O.</u> Let Z be a connected, nondegenerate space. Then:
(a) if Z is not linearly ordered, $R(Z)$ is connected;
(b) if Z is linearly ordered, $R(Z)$ has two components which are just $T_{<}(Z)$ and $T_{>}(Z)$.

In this paper we use Eilenberg's result to derive a generalization for the deleted product of an ordered couple of spaces with closed intersection. The work is motivated by its applications to the investigation of deleted products of polyhedra.

1. Statement of the Generalized Theorem. For any set E, let $\#(E)$ denote its cardinal number. If S is a topological space, let $\mathcal{C}(S)$ denote the set of its connected components; thus, $\#(\mathcal{C}(S))$ is the number of components of S. For any subspace T of S, $cl_S T$ denotes the closure of T in S.

If C is a nondegenerate, linearly ordered space, denote by $\inf(C)$ the least element of C (if this exists); similarly, denote by $\sup(C)$ the greatest element of C.

Let (X,Y) be an ordered couple of nonvoid spaces. The deleted product of (X,Y) is the subspace

$$R(X,Y) = \{(x,y) \in X \times Y \mid x \neq y\}$$

of $X \times Y$. Obviously, $R(X,Y) = X \times Y$ if and only if $X \cap Y = \emptyset$; $R(X,Y) = \emptyset$ if and only if $X = Y =$ a singleton.

Suppose now that $Z = X \cap Y$ is closed in both X and Y. For each $C \in \mathcal{C}(Z)$ define the connected subspace

$$X_C = C \cup \bigcup \{A \in \mathcal{C}(X-Z) \mid C \cap cl_X A \neq \emptyset\};$$

if, in addition, C is nondegenerate and linearly ordered, define

$$X_C^{<} = \bigcup \{A \in \mathcal{C}(X-Z) \mid C \cap cl_X A = \{\inf(C)\}\}$$

and

$$X_C^{>} = \bigcup \{A \in \mathcal{C}(X-Z) \mid C \cap cl_X A = \{\sup(C)\}\}.$$

Obviously, $X_C^{<} \cup X_C^{>} \subset X_C - C$ and $X_C^{<} \cap X_C^{>} = \emptyset$. Similarly, define the subspaces Y_C, $Y_C^{<}$, $Y_C^{>}$. A partition

$$\mathcal{C}(Z) = \mathcal{C}_1(Z) \cup \mathcal{C}_2^{+}(Z) \cup \mathcal{C}_2^{-}(Z)$$

is determined by the following definitions:

(+) $C \in \mathcal{C}_2^{+}(Z)$ if and only if C is nondegenerate, linearly ordered, and $X_C^{<} = X_C - C$, $Y_C^{>} = Y_C - C$;

(−) $C \in \mathcal{C}_2^{-}(Z)$ if and only if C is nondegenerate, linearly ordered, and $X_C^{>} = X_C - C$, $Y_C^{<} = Y_C - C$.

We prove the following generalization of Theorem 0:

Theorem 1. Let (X,Y) be an ordered couple of nonvoid connected spaces with $Z = X \cap Y$ closed in both X and Y. Suppose that $\mathcal{C}(X-Z)$, $\mathcal{C}(Y-Z)$, and $\mathcal{C}(Z)$ are finite sets. The following hold:

(a) if Z is a singleton and $Y = Z$, then $\mathcal{C}(R(X,Y)) = \{A \times Z \mid A \in \mathcal{C}(X-Z)\}$ and $\#(\mathcal{C}(R(X,Y))) = \#(\mathcal{C}(X-Z))$;

(b) if Z is a singleton and $X = Z$, then $\mathcal{C}(R(X,Y)) = \{Z \times B \mid B \in \mathcal{C}(Y-Z)\}$ and $\#(\mathcal{C}(R(X,Y))) = \#(\mathcal{C}(Y-Z))$;

(c) if Z is connected, nondegenerate, and linearly ordered, $Y = Z$, and $X - Z = X_Z^{<} \cup X_Z^{>}$, then $\mathcal{C}(R(X,Y)) = \{(X_Z^{<} \times Z) \cup T_{<}(Z), T_{>}(Z) \cup (X_Z^{>} \times Z)\}$ and $\#(\mathcal{C}(R(X,Y))) = 2$;

(d) if Z is connected, nondegenerate, and linearly ordered, $X = Z$, and $Y - Z = Y_Z^{<} \cup Y_Z^{>}$, then $\mathcal{C}(R(X,Y)) = \{T_{<}(Z) \cup (Z \times Y_Z^{>}), (Z \times Y_Z^{<}) \cup T_{>}(Z)\}$ and $\#(\mathcal{C}(R(X,Y))) = 2$;

(e) if $\mathcal{C}(X-Z) \neq \emptyset$, $\mathcal{C}(Y-Z) \neq \emptyset$, and $\mathcal{C}_2^{+}(Z) \cup \mathcal{C}_2^{-}(Z) \neq \emptyset$, then $\mathcal{C}(R(X,Y)) = \{T_{>}(C) \mid C \in \mathcal{C}_2^{+}(Z)\} \cup \{T_{<}(C) \mid C \in \mathcal{C}_2^{-}(Z)\} \cup \{R(X,Y) - \bigcup\{T_{>}(C) \mid C \in \mathcal{C}_2^{+}(Z)\} - \bigcup\{T_{<}(C) \mid C \in \mathcal{C}_2^{-}(Z)\}\}$ and $\#(\mathcal{C}(R(X,Y))) = \#(\mathcal{C}_2^{+}(Z)) + \#(\mathcal{C}_2^{-}(Z)) + 1$;

(f) in all other cases, $R(X,Y)$ is connected.

2. Proof of the Generalized Theorem. The proof will be split into two parts (Theorems 2.10, 2.11 and Theorems 2.12, 2.18). These exhaust all nontrivial cases. We first prove some technical preliminaries.

Let (X,Y) be an arbitrary ordered couple of spaces.

Lemma 2.1. If $A \subset X$, $B \subset Y$, and $A \cap B = \emptyset$, then

$$\mathrm{cl}_{R(X,Y)}(A \times B) = R(\mathrm{cl}_X A, \mathrm{cl}_Y B).$$

Proof. As $A \times B \subset R(X,Y)$, we have

$$\mathrm{cl}_{R(X,Y)}(A \times B) = \mathrm{cl}_{X \times Y}(A \times B) \cap R(X,Y) =$$
$$= (\mathrm{cl}_X A \times \mathrm{cl}_Y B) \cap R(X,Y) = R(\mathrm{cl}_X A, \mathrm{cl}_Y B).$$

Lemma 2.2. If $A \subset X$ and $B \subset Y$ are closed subspaces, then

$R(A,B)$ is closed in $R(X,Y)$.

Proof. It is sufficient to note that

$$R(A,B) = (A \times B) \cap R(X,Y)$$

and that $A \times B$ is closed in $X \times Y$.

Lemma 2.3. If a nondegenerate subspace $C \subset X \cap Y$ is closed in both X and Y, then $R(C,C)$ is closed in $R(X,Y)$. Moreover, if C is connected, linearly ordered, then $T_<(C)$ and $T_>(C)$ are also closed in $R(X,Y)$.

Proof. The first part follows from Lemma 2.2. For the second part, observe that, by Theorem 0 (b), $T_<(C)$ and $T_>(C)$ are closed in $R(C,C)$.

Let now $Z = X \cap Y$ be connected and closed in both X and Y. For Z nondegenerate, linearly ordered, define:

$$\mathscr{C}_{\inf}(X-Z) = \{A \in \mathscr{C}(X-Z) \mid Z \cap \mathrm{cl}_X A = \{\inf(Z)\}\}$$

and

$$\mathscr{C}_{\sup}(X-Z) = \{A \in \mathscr{C}(X-Z) \mid Z \cap \mathrm{cl}_X A = \{\sup(Z)\}\};$$

similarly, define $\mathscr{C}_{\inf}(Y-Z)$ and $\mathscr{C}_{\sup}(Y-Z)$. Obviously,

$$X_Z^< = \bigcup\{A \in \mathscr{C}_{\inf}(X-Z)\}$$

and

$$X_Z^> = \bigcup\{A \in \mathscr{C}_{\sup}(X-Z)\};$$

similar relations hold for $Y_Z^<$ and $Y_Z^>$. Further, define

$$\begin{aligned} V_<(X,Y) = T_<(Z) \cup {} & \\ & \cup \bigcup\{A \times Z \mid A \in \mathscr{C}(X-Z) - \mathscr{C}_{\sup}(X-Z)\} \cup \\ & \cup \bigcup\{Z \times B \mid B \in \mathscr{C}(Y-Z) - \mathscr{C}_{\inf}(Y-Z)\} \cup \\ & \cup \bigcup\{A \times B \mid (A,B) \in \mathscr{C}(X-Z) \times \mathscr{C}(Y-Z) - \\ & \qquad \mathscr{C}_{\sup}(X-Z) \times \mathscr{C}_{\inf}(Y-Z)\} \end{aligned}$$

and

$$\begin{aligned} V_>(X,Y) = T_>(Z) \cup {} & \\ & \cup \bigcup\{A \times Z \mid A \in \mathscr{C}(X-Z) - \mathscr{C}_{\inf}(X-Z)\} \cup \end{aligned}$$

$$\cup \bigcup\{Z \times B \mid B \in \mathcal{C}(Y-Z) - \mathcal{C}_{sup}(Y-Z)\} \cup$$
$$\cup \bigcup\{A \times B \mid (A,B) \in \mathcal{C}(X-Z) \times \mathcal{C}(Y-Z) - \mathcal{C}_{inf}(X-Z) \times \mathcal{C}_{sup}(Y-Z)\}.$$

Note that

$$R(X,Y) = V_{<}(X,Y) \cup V_{>}(X,Y)$$

and that $V_{<}(X,Y) \cap V_{>}(X,Y) \neq \emptyset$ if and only if

$$\mathcal{C}(X-Z) \times \mathcal{C}(Y-Z) - \mathcal{C}_{sup}(X-Z) \times \mathcal{C}_{inf}(Y-Z) - \mathcal{C}_{inf}(X-Z) \times \mathcal{C}_{sup}(Y-Z) \neq \emptyset.$$

<u>Lemma 2.4</u>. For each $(A,B) \in \mathcal{C}(X-Z) \times \mathcal{C}(Y-Z)$, the subspaces $(A \times Z) \cup (A \times B)$, $(Z \times B) \cup (A \times B)$, and $(A \times Z) \cup (Z \times B) \cup (A \times B)$ are connected.

<u>Proof</u>. From Lemma 2.1 we get

$$(A \times Z) \cap cl_{R(X,Y)}(A \times B) = A \times (Z \cap cl_Y B) \neq \emptyset$$

and

$$(Z \times B) \cap cl_{R(X,Y)}(A \times B) = (Z \cap cl_X A) \times B \neq \emptyset.$$

It follows that $(A \times Z) \cup (A \times B)$ and $(Z \times B) \cup (A \times B)$ are connected; therefore, $(A \times Z) \cup (Z \times B) \cup (A \times B)$ is also connected

<u>Lemma 2.5</u>. Suppose Z is nondegenerate, not linearly ordered. Then:

(a) $R(Z,Z) \cup (A \times Z)$ is connected for each $A \in \mathcal{C}(X-Z)$;

(b) $R(Z,Z) \cup (Z \times B)$ is connected for each $B \in \mathcal{C}(Y-Z)$;

(c) $R(Z,Z) \cup (A \times Z) \cup (Z \times B) \cup (A \times B)$ is connected for each $(A,B) \in \mathcal{C}(X-Z) \times \mathcal{C}(Y-Z)$.

<u>Proof</u>. Part (a) follows from the fact that, for $A \in \mathcal{C}(X-Z)$,

$$R(Z,Z) \cap cl_{R(X,Y)}(A \times Z) = R(Z \cap cl_X A, Z) \neq \emptyset.$$

Part (b) is derived in the same manner. Part (c) follows from (a) or (b) and Lemma 2.4.

Lemma 2.6. Suppose Z is nondegenerate, linearly ordered. Then:

(a_1) $T_<(Z)\cup(A\times Z)$ is connected for each $A\in\mathcal{C}(X-Z)-\mathcal{C}_{sup}(X-Z)$;

(a_2) $T_>(Z)\cup(A\times Z)$ is connected for each $A\in\mathcal{C}(X-Z)-\mathcal{C}_{inf}(X-Z)$;

(b_1) $T_<(Z)\cup(Z\times B)$ is connected for each $B\in\mathcal{C}(Y-Z)-\mathcal{C}_{inf}(Y-Z)$;

(b_2) $T_>(Z)\cup(Z\times B)$ is connected for each $B\in\mathcal{C}(Y-Z)-\mathcal{C}_{sup}(Y-Z)$;

(c_1) $T_<(Z)\cup(A\times Z)\ \ (A\ \ B)$ is connected for each $(A,B)\in$ $(\mathcal{C}(X-Z)-\mathcal{C}_{sup}(X-Z))\times\mathcal{C}(Y-Z)$;

(c_2) $T_>(Z)\cup(A\times Z)\cup(A\times B)$ is connected for each $(A,B)\in$ $(\mathcal{C}(X-Z)-\mathcal{C}_{inf}(X-Z))\times\mathcal{C}(Y-Z)$;

(d_1) $T_<(Z)\cup(Z\times B)\cup(A\times B)$ is connected for each $(A,B)\in$ $\mathcal{C}(X-Z)\times(\mathcal{C}(Y-Z)-\mathcal{C}_{inf}(Y-Z))$;

(d_2) $T_>(Z)\cup(Z\times B)\cup(A\times B)$ is connected for each $(A,B)\in$ $\mathcal{C}(X-Z)\times(\mathcal{C}(Y-Z)-\mathcal{C}_{sup}(Y-Z))$.

Proof. For each $A\in\mathcal{C}(X-Z)-\mathcal{C}_{sup}(X-Z)$, $Z\cap cl_X A$ contains at least one point $z_1\in Z$ different from $\sup(Z)$; thus, we can choose a point $z_2\in Z$ with $z_1<z_2$. This implies that

$$T_<(Z)\cap cl_{R(X,Y)}(A\times Z)=T_<(Z)\cap R(Z\cap cl_X A,Z)\neq\emptyset .$$

Therefore, part (a_1) is proved. Parts (a_2), (b_1), (b_2) are derived in the same manner. Parts (c_1), (c_2), (d_1), (d_2) follow from (a_1), (a_2), (b_1), (b_2), respectively, and from Lemma 2.4.

Lemma 2.7. If Z is nondegenerate and linearly ordered, then:

(a_1) $T_<(Z)\cup(A\times Z)$ is closed in $R(X,Y)$ for each $A\in\mathcal{C}_{inf}(X-Z)$;

(a_2) $T_>(Z)\cup(A\times Z)$ is closed in $R(X,Y)$ for each $A\in\mathcal{C}_{sup}(X-Z)$;

(b_1) $T_<(Z)\cup(Z\times B)$ is closed in $R(X,Y)$ for each $B\in\mathcal{C}_{sup}(Y-Z)$;

(b_2) $T_>(Z)\cup(Z\times B)$ is closed in $R(X,Y)$ for each $B\in\mathcal{C}_{inf}(Y-Z)$;

(c_1) $T_<(Z)\cup(A\times Z)\cup(Z\times B)\cup(A\times B)$ is closed in $R(X,Y)$ for each $(A,B)\in\mathcal{C}_{inf}(X-Z)\times\mathcal{C}_{sup}(Y-Z)$;

(c_2) $T_>(Z)\cup(A\times Z)\cup(Z\times B)\cup(A\times B)$ is closed in $R(X,Y)$ for each $(A,B) \in \mathscr{C}_{\sup}(X-Z) \times \mathscr{C}_{\inf}(Y-Z)$.

Proof. To prove part (a_1), note that, for $A \in \mathscr{C}_{\inf}(X-Z)$,

$$R(\mathrm{cl}_X A, Z) = R(\{\inf(Z)\}, Z) \cup (A \times Z) \subset T_<(Z) \cup (A \times Z);$$

then, applying Lemmas 2.1 and 2.3, we get

$$\mathrm{cl}_{R(X,Y)}(T_<(Z) \cup (A \times Z)) =$$
$$= T_<(Z) \cup R(\mathrm{cl}_X A, Z) \subset T_<(Z) \cup (A \times Z).$$

Parts (a_2), (b_1), (b_2) are derived in the same manner. To prove part (c_1), note that, for $(A,B) \in \mathscr{C}_{\inf}(X-Z) \times \mathscr{C}_{\sup}(Y-Z)$,

$$R(\mathrm{cl}_X A, \mathrm{cl}_Y B) = R(\{\inf(Z)\}, \{\sup(Z)\}) \cup (A \times \{\sup(Z)\}) \cup$$
$$\cup (\{\inf(Z)\} \times B) \cup (A \times B) \subset$$
$$\subset T_<(Z) \cup (A \times Z) \cup (Z \times B) \cup (A \times B);$$

then, again applying Lemmas 2.1 and 2.3, we get

$$\mathrm{cl}_{R(X,Y)}(T_<(Z) \cup (A \times Z) \cup (Z \times B) \cup (A \times B)) =$$
$$= T_<(Z) \cup R(\mathrm{cl}_X A, Z) \cup R(Z, \mathrm{cl}_Y B) \cup R(\mathrm{cl}_X A, \mathrm{cl}_Y B) \subset$$
$$\subset T_<(Z) \cup (A \times Z) \cup (Z \times B) \cup (A \times B).$$

For part (c_2), a similar reasoning applies.

Lemma 2.8. Suppose Z is nondegenerate, not linearly ordered. Then $R(X,Y)$ is connected.

Proof. Write $R(X,Y)$ as

$$R(X,Y) = R(Z,Z) \cup$$
$$\cup \bigcup\{R(Z,Z) \cup (A \times Z) \mid A \in \mathscr{C}(X-Z)\} \cup$$
$$\cup \bigcup\{R(Z,Z) \cup (Z \times B) \mid B \in \mathscr{C}(Y-Z)\} \cup$$
$$\cup \bigcup\{R(Z,Z) \cup (A \times Z) \cup (Z \times B) \cup (A \times B) \mid$$
$$(A,B) \in \mathscr{C}(X-Z) \times \mathscr{C}(Y-Z)\}$$

and apply Theorem 0 (a) and Lemma 2.5 (a,b,c).

Lemma 2.9. Suppose Z is nondegenerate, linearly ordered. Then the subspaces $V_<(X,Y)$ and $V_>(X,Y)$ of $R(X,Y)$ are connected.

Proof. First note that

$$\mathscr{C}(X-Z)\times\mathscr{C}(Y-Z) - \mathscr{C}_{sup}(X-Z)\times\mathscr{C}_{inf}(Y-Z) =$$
$$= (\mathscr{C}(X-Z) - \mathscr{C}_{sup}(X-Z))\times\mathscr{C}(Y-Z) \cup$$
$$\cup\,\mathscr{C}(X-Z)\times(\mathscr{C}(Y-Z) - \mathscr{C}_{inf}(Y-Z))$$

and

$$\mathscr{C}(X-Z)\times\mathscr{C}(Y-Z) - \mathscr{C}_{inf}(X-Z)\times\mathscr{C}_{sup}(Y-Z) =$$
$$= (\mathscr{C}(X-Z) - \mathscr{C}_{inf}(X-Z))\times\mathscr{C}(Y-Z) \cup$$
$$\cup\,\mathscr{C}(X-Z)\times(\mathscr{C}(Y-Z) - \mathscr{C}_{sup}(Y-Z)).$$

Write $V_<(X,Y)$ as

$$V_<(X,Y) = T_<(Z) \cup$$
$$\cup\bigcup\{T_<(Z)\cup(A\times Z) \mid A\in\mathscr{C}(X-Z) - \mathscr{C}_{sup}(X-Z)\}\cup$$
$$\cup\bigcup\{T_<(Z)\cup(Z\times B) \mid B\in\mathscr{C}(Y-Z) - \mathscr{C}_{inf}(Y-Z)\}\cup$$
$$\cup\bigcup\{T_<(Z)\cup(A\times Z)\cup(A\times B) \mid$$
$$(A,B)\in(\mathscr{C}(X-Z) - \mathscr{C}_{sup}(X-Z))\times\mathscr{C}(Y-Z)\}\cup$$
$$\cup\bigcup\{T_<(Z)\cup(Z\times B)\cup(A\times B) \mid$$
$$(A,B)\in\mathscr{C}(X-Z)\times(\mathscr{C}(Y-Z) - \mathscr{C}_{inf}(Y-Z))\}$$

and apply Theorem 0 (b) and Lemma 2.6 (a_1, b_1, c_1, d_1); similarly, write $V_>(X,Y)$ as

$$V_>(X,Y) = T_>(Z) \cup$$
$$\cup\bigcup\{T_>(Z)\cup(A\times Z) \mid A\in\mathscr{C}(X-Z) - \mathscr{C}_{inf}(X-Z)\}\cup$$
$$\cup\bigcup\{T_>(Z)\cup(Z\times B) \mid B\in\mathscr{C}(Y-Z) - \mathscr{C}_{sup}(Y-Z)\}\cup$$
$$\cup\bigcup\{T_>(Z)\cup(A\times Z)\cup(A\times B) \mid$$
$$(A,B)\in(\mathscr{C}(X-Z) - \mathscr{C}_{inf}(X-Z))\times\mathscr{C}(Y-Z)\}\cup$$
$$\cup\bigcup\{T_>(Z)\cup(Z\times B)\cup(A\times B) \mid$$
$$(A,B)\in\mathscr{C}(X,Z)\times(\mathscr{C}(Y-Z) - \mathscr{C}_{sup}(Y-Z))\}$$

and apply Theorem 0 (b) and Lemma 2.6 (a_2, b_2, c_2, d_2).

Theorem 2.10. Let (X,Z) be an ordered couple of nonvoid, connected spaces, with $Z\subset X$ and Z closed. Suppose that $\mathscr{C}(X-Z)$ is finite. The following hold:

(a) if Z is a singleton, then $\mathscr{C}(R(X,Z)) = \{A\times Z \mid A\in\mathscr{C}(X-Z)\}$ and $\#(\mathscr{C}(R(X,Z))) = \#(\mathscr{C}(X-Z))$;

(b) if Z is nondegenerate, linearly ordered, and $\mathcal{C}(X-Z) = \mathcal{C}_{inf}(X-Z) \cup \mathcal{C}_{sup}(X-Z)$, then $\mathcal{C}(R(X,Z)) = \{T_{<}(Z) \cup \bigcup\{A \times Z \mid A \in \mathcal{C}_{inf}(X-Z)\},\ T_{>}(Z) \cup \bigcup\{A \times Z \mid A \in \mathcal{C}_{sup}(X-Z)\}\}$ and $\#(\mathcal{C}(R(X,Z))) = 2$;

(c) in all other cases, $R(X,Z)$ is connected.

Proof. First note that

$$R(X,Z) = R(Z,Z) \cup \bigcup\{A \times Z \mid A \in \mathcal{C}(X-Z)\}.$$

If Z is a singleton, then $R(Z,Z) = \emptyset$ and $R(X,Z) = \bigcup\{A \times Z \mid A \in \mathcal{C}(X-Z)\}$; the subspaces $A \times Z$, $A \in \mathcal{C}(X-Z)$, are connected, closed in $R(X,Z)$, and pairwise disjoint. Suppose now that Z is nondegenerate. If Z is not linearly ordered, then, by Lemma 2.8, $R(X,Z)$ is connected. If Z is linearly ordered, then

$$V_{<}(X,Z) = T_{<}(Z) \cup \bigcup\{A \times Z \mid A \in \mathcal{C}(X-Z) - \mathcal{C}_{sup}(X-Z)\}$$

and

$$V_{>}(X,Z) = T_{>}(Z) \cup \bigcup\{A \times Z \mid A \in \mathcal{C}(X-Z) - \mathcal{C}_{inf}(X-Z)\}.$$

By Lemma 2.9, both $V_{<}(X,Y)$ and $V_{>}(X,Y)$ are connected. If $\mathcal{C}(X-Z) - \mathcal{C}_{inf}(X-Z) - \mathcal{C}_{sup}(X-Z) \neq \emptyset$, then $V_{<}(X,Z) \cap V_{>}(X,Z) \neq \emptyset$ and this implies that $R(X,Z)$ is connected. If $\mathcal{C}(X-Z) = \mathcal{C}_{inf}(X-Z) \cup \mathcal{C}_{sup}(X-Z)$, then $V_{<}(X,Z) \cap V_{>}(X,Z) = \emptyset$. On the other hand, in this case

$$V_{<}(X,Z) = T_{<}(Z) \cup \bigcup\{A \times Z \mid A \in \mathcal{C}_{inf}(X-Z)\}$$

and

$$V_{>}(X,Z) = T_{>}(Z) \cup \bigcup\{A \times Z \mid A \in \mathcal{C}_{sup}(X-Z)\};$$

Lemma 2.7 (a_1, a_2) and the finiteness of $\mathcal{C}(X-Z)$ imply that both $V_{<}(X,Z)$ and $V_{>}(X,Z)$ are closed in $R(X,Z)$. It follows that $V_{<}(X,Z)$ and $V_{>}(X,Z)$ are the two components of $R(X,Z)$.

The following theorem describes a completely similar situation and will be stated without proof.

Theorem 2.11. Let (Z,Y) be an ordered couple of nonvoid, connected spaces, with $Z \subset Y$ and Z closed. Suppose that $\mathscr{C}(Y-Z)$ is finite. The following hold:

(a) if Z is a singleton, then $\mathscr{C}(R(Z,Y)) = \{Z \times B \mid B \in \mathscr{C}(Y-Z)\}$ and $\#(\mathscr{C}(R(Z,Y))) = \#(\mathscr{C}(Y-Z))$;

(b) if Z is nondegenerate, linearly ordered, and $\mathscr{C}(Y-Z) = \mathscr{C}_{inf}(Y-Z) \cup \mathscr{C}_{sup}(Y-Z)$, then $\mathscr{C}(R(Z,Y)) = \{T_{<}(Z) \cup \bigcup\{Z \times B \mid B \in \mathscr{C}_{sup}(Y-Z)\}, T_{>}(Z) \cup \bigcup\{Z \times B \mid B \in \mathscr{C}_{inf}(Y-Z)\}\}$ and $\#(\mathscr{C}(R(Z,Y))) = 2$;

(c) in all other cases, $R(Z,Y)$ is connected.

Now we proceed to the case when the complements of Z in both X and Y are nonvoid.

Theorem 2.12. Let (X,Y) be an ordered couple of nonvoid, connected spaces, with $Z = X \cap Y$ nonvoid, connected, and closed in both X and Y. Suppose that $\mathscr{C}(X-Z)$ and $\mathscr{C}(Y-Z)$ are both nonvoid and finite. The following hold:

(a) if Z is nondegenerate, linearly ordered, and $\mathscr{C}(X-Z) = \mathscr{C}_{inf}(X-Z)$, $\mathscr{C}(Y-Z) = \mathscr{C}_{sup}(Y-Z)$, then $\mathscr{C}(R(X,Y)) = \{T_{>}(Z), R(X,Y) - T_{>}(Z)\}$ and $\#(\mathscr{C}(R(X,Y))) = 2$;

(b) if Z is nondegenerate, linearly ordered, and $\mathscr{C}(X-Z) = \mathscr{C}_{sup}(X,Y)$, $\mathscr{C}(Y-Z) = \mathscr{C}_{inf}(Y-Z)$, then $\mathscr{C}(R(X,Y)) = \{T_{<}(Z), R(X,Y) - T_{<}(Z)\}$ and $\#(\mathscr{C}(R(X,Y))) = 2$;

(c) in all other cases, $R(X,Y)$ is connected.

Proof. First note that

$$R(X,Y) = R(Z,Z) \cup \bigcup\{(A \times Z) \cup (Z \times B) \cup (A \times B) \mid (A,B) \in \mathscr{C}(X-Z) \times \mathscr{C}(Y-Z)\}.$$

If Z is a singleton, then $R(Z,Z) = \emptyset$ and

$$R(X,Y) = \bigcup\{R_B \mid B \in \mathscr{C}(Y-Z)\},$$

where

$$R_B = \bigcup\{(A\times Z)\cup(Z\times B)\cup(A\times B)\mid A\in\mathcal{C}(X-Z)\},$$

for each $B\in\mathcal{C}(Y-Z)$. By Lemma 2.4, R_B is connected for each $B\in\mathcal{C}(Y-Z)$. As $A\times Z\subset R_B$ for each $(A,B)\in\mathcal{C}(X-Z)\times\mathcal{C}(Y-Z)$, it follows that $R(X,Y)$ is also connected. Suppose now that Z is nondegenerate. If Z is not linearly ordered, Lemma 2.8 shows that $R(X,Y)$ is connected. If Z is linearly ordered, then $R(X,Y) = V_<(X,Y)\cup V_>(X,Y)$ and, by Lemma 2.9, the subspaces $V_<(X,Y)$ and $V_>(X,Y)$ are connected. In the case when

$$\mathcal{C}(X-Z)\times\mathcal{C}(Y-Z) - \mathcal{C}_{inf}(X-Z)\times\mathcal{C}_{sup}(Y-Z) - \mathcal{C}_{sup}(X-Z)\times\mathcal{C}_{inf}(Y-Z) \neq \emptyset,$$

it is easily seen that $V_<(X,Y)\cap V_>(X,Y)\neq\emptyset$; this implies that $R(X,Y)$ is connected. If, on the contrary,

$$\mathcal{C}(X-Z)\times\mathcal{C}(Y-Z) = \mathcal{C}_{inf}(X-Z)\times\mathcal{C}_{sup}(Y-Z)\cup \mathcal{C}_{sup}(X-Z)\times\mathcal{C}_{inf}(Y-Z),$$

then either

$$\mathcal{C}(X-Z) = \mathcal{C}_{inf}(X-Z) \quad\text{and}\quad \mathcal{C}(Y-Z) = \mathcal{C}_{sup}(Y-Z),$$

or

$$\mathcal{C}(X-Z) = \mathcal{C}_{sup}(X-Z) \quad\text{and}\quad \mathcal{C}(Y-Z) = \mathcal{C}_{inf}(Y-Z).$$

When the first situation arises, we have $\mathcal{C}_{inf}(X-Z) = \mathcal{C}(X-Z)\neq\emptyset$ and $\mathcal{C}_{sup}(Y-Z) = \mathcal{C}(Y-Z)\neq\emptyset$ and we can write

$$V_<(X,Y) = \bigcup\{T_<(Z)\cup(A\times Z)\cup(Z\times B)\cup(A\times B)\mid (A,B)\in\mathcal{C}(X-Z)\times\mathcal{C}(Y-Z)\};$$

Lemma 2.7 (c_1) and the finiteness of $\mathcal{C}(X-Z)$ and $\mathcal{C}(Y-Z)$ imply that $V_<(X,Y)$ is closed in $R(X,Y)$. On the other hand,

$$V_>(X,Y) = T_>(Z);$$

by Theorem 0 (b), $V_>(X,Y)$ is also closed in $R(X,Y)$. We now see that $V_<(X,Y)\cap V_>(X,Y) = \emptyset$ and this shows that $V_<(X,Y)$ and $T_>(Z)$ are the two components of $R(X,Y)$. When the second situation arises, a similar reasoning applies; it follows that $T_<(Z)$ and $V_>(X,Y)$ are the two components of $R(X,Y)$.

We now remove the condition on Z to be connected and prove a generalization of the preceding result. We still need a few lemmas. Let Z still be closed in both X and Y and assume that $\mathscr{C}(X-Z)$, $\mathscr{C}(Y-Z)$, and $\mathscr{C}(Z)$ are nonvoid and finite.

Lemma 2.13. For each $C \in \mathscr{C}(Z)$, the subspace X_C is open in X and the subspace Y_C is open in Y.

Proof. Note that if $A \in \mathscr{C}(X-Z)$ and $C \cap \mathrm{cl}_X A = \emptyset$, then $\mathrm{cl}_X A \subset A \cup (Z-C)$. On the other hand,

$$X - X_C = \bigcup\{A \in \mathscr{C}(X-Z) \mid C \cap \mathrm{cl}_X A = \emptyset\} \cup (Z-C).$$

The finiteness of $\mathscr{C}(Z)$ implies that $Z-C$ is closed in Z and this implies that $Z-C$ is closed in X; by the finiteness of $\mathscr{C}(X-Z)$ we have

$$\mathrm{cl}_X(X-X_C) \subset \bigcup\{A \in \mathscr{C}(X-Z) \mid C \cap \mathrm{cl}_X A = \emptyset\} \cup (Z-C) = X - X_C ,$$

therefore X_C is open in X. A similar argument shows that Y_C is open in Y.

Lemma 2.14. For each $C \in \mathscr{C}(Z)$, the subspaces $R(X,Y_C)$ and $R(X_C,Y)$ are open in $R(X,Y)$.

Proof. This is an immediate consequence of Lemma 2.13.

Lemma 2.15. For each $C \in \mathscr{C}(Z)$, the subspaces $X_C - C$ and $Y_C - C$ are nonvoid.

Proof. Assume $X_C = C$; then, by Lemma 2.13, C is both open and closed in X, but this contradicts the connectedness of X. In the same way it follows that $Y_C - C \neq \emptyset$.

Lemma 2.16. For each $C \in \mathscr{C}(Z)$, the subspace $R(X_C,Y_C)$ is nonvoid.

Proof. Assume $R(X_C,Y_C) = \phi$; then $X_C = Y_C$ = a singleton. As $C \neq \phi$, this implies $X_C = C = Y_C$ and contradicts Lemma 2.15.

Lemma 2.17. Let $C \in \mathscr{C}(Z)$. The following hold:

(a) if $C \in \mathscr{C}_1(Z)$, then $R(X,Y_C) \cup R(X_C,Y)$ is connected;

(b) if $C \in \mathscr{C}_2^+(Z)$, then $\mathscr{C}(R(X,Y_C) \cup R(X_C,Y)) = \{T_>(C),\ R(X,Y_C) \cup R(X_C,Y) - T_>(C)\}$;

(c) if $C \in \mathscr{C}_2^-(Z)$, then $\mathscr{C}(R(X,Y_C) \cup R(X_C,Y)) = \{T_<(C),\ R(X,Y_C) \cup R(X_C,Y) - T_<(C)\}$.

Proof. First note that

$$X \cap Y_C = C = X_C \cap Y$$

and that C is closed in both X_C and Y_C. If $C \in \mathscr{C}_1(Z)$, then, by Theorem 2.12 (c), both $R(X,Y_C)$ and $R(X_C,Y)$ are connected. By Lemma 2.16,

$$R(X,Y_C) \cap R(X_C,Y) = R(X_C,Y_C) \neq \phi .$$

Therefore, $R(X,Y_C) \cup R(X_C,Y)$ is connected. Suppose now $C \in \mathscr{C}_2^+(Z)$. By Theorem 2.12 (a),

$$\mathscr{C}(R(X,Y_C)) = \{T_>(C),\ R(X,Y_C) - T_>(C)\}$$

and

$$\mathscr{C}(R(X_C,Y)) = \{T_>(C),\ R(X_C,Y) - T_>(C)\};$$

as

$$(R(X,Y_C) - T_>(C)) \cap (R(X_C,Y) - T_>(C)) \supset T_<(C),$$

it follows that $R(X,Y_C) \cup R(X_C,Y) - T_>(C)$ is connected. On the other hand, by Lemma 2.3, $T_>(C)$ is closed in $R(X,Y)$, and by Theorem 2.12 (a) and Lemma 2.14, $T_>(C)$ is open in $R(X,Y)$. It turns out that $T_>(C)$ and $R(X,Y_C) \cup R(X_C,Y) - T_>(C)$ are the two components of $R(X,Y_C) \cup R(X_C,Y)$. A similar argument applies for $C \in \mathscr{C}_2^-(Z)$.

Theorem 2.18. Let (X,Y) be an ordered couple of nonvoid, connected spaces, with $Z = X \cap Y$ nonvoid and closed in both X and Y. Suppose that $\mathscr{C}(X-Z)$, $\mathscr{C}(Y-Z)$, and $\mathscr{C}(Z)$ are nonvoid and finite. The following hold:

(a) if $\mathscr{C}_2^+(Z) \cup \mathscr{C}_2^-(Z) \neq \emptyset$, then $\mathscr{C}(R(X,Y)) = \{T_>(C) \mid C \in \mathscr{C}_2^+(Z)\} \cup \{T_<(C) \mid C \in \mathscr{C}_2^-(Z)\} \cup \{R(X,Y) - \bigcup\{T_>(C) \mid C \in \mathscr{C}_2^+(Z)\} - \bigcup\{T_<(C) \mid C \in \mathscr{C}_2^-(Z)\}\}$ and $\#(\mathscr{C}(R(X,Y))) = \#(\mathscr{C}_2^+(Z)) + \#(\mathscr{C}_2^-(Z)) + 1$;

(b) in all other cases, $R(X,Y)$ is connected.

Proof. First note that, if $C \in \mathscr{C}_2^+(Z)$, then $T_>(C)$ is closed and open in $R(X,Y)$; therefore, $T_>(C)$ is a component of $R(X,Y)$. Similarly, for each $C \in \mathscr{C}_2^-(Z)$, $T_<(C)$ is a component of $R(X,Y)$. The finiteness of $\mathscr{C}(Z)$ then implies that

$$V = R(X,Y) - \bigcup\{T_>(C) \mid C \in \mathscr{C}_2^+(Z)\} - \bigcup\{T_<(C) \mid C \in \mathscr{C}_2^-(Z)\}$$

is also open and closed in $R(X,Y)$. It still remains to show that V is connected. Denote

$$U_C = R(X,Y_C) \cup R(X_C,Y)$$

for each $C \in \mathscr{C}(Z)$ and

$$V_C = \begin{cases} U_C, & \text{for } C \in \mathscr{C}_1(Z), \\ U_C - T_>(C), & \text{for } C \in \mathscr{C}_2^+(Z), \\ U_C - T_<(C), & \text{for } C \in \mathscr{C}_2^-(Z). \end{cases}$$

Note that, for $C, C' \in \mathscr{C}(Z)$, $C \neq C'$, we have

$$U_C \cap U_{C'} \supset (X_C \times Y_{C'}) \cup (X_{C'} \times Y_C)$$

and

$$U_C \cap R(C',C') = \emptyset.$$

Therefore, $V_C \cap V_{C'} = U_C \cap U_{C'} \neq \emptyset$, for each $(C,C') \in \mathscr{C}(Z) \times \mathscr{C}(Z)$ with $C \neq C'$. On the other hand, the same argument shows that

$$V = \bigcup\{U_C \mid C \in \mathscr{C}(Z)\} - \bigcup\{T_>(C) \mid C \in \mathscr{C}_2^+(Z)\} - \bigcup\{T_<(C) \mid C \in \mathscr{C}_2^-(Z)\} = \bigcup\{V_C \mid C \in \mathscr{C}(Z)\}.$$

By Lemma 2.17, each V_C , $C \in \mathscr{C}(Z)$, is connected. It follows that V is also connected.

Bibliography:

1. Eilenberg, Samuel - Ordered topological spaces, Amer. J.Math. 63 (1941), 39-45.

Seminarul Matematic
"A.Myller"
Universitatea "Al.I.Cuza"
6600 - Iaşi
România