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by

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About the equality between the p-module and the p-capacity in R^n

by Petru Caraman

In this paper, we establish that

$$(1) \qquad M_p\Gamma = \operatorname{cap}_p(D, E_0, E_1) ,$$

where D is an open set in the euclidean n-space R^n, $E_0, E_1 \subset R^n$ are 2 sets such that $d(E_0, E_1) > 0$ and $E_0 \cap \partial D$ or $E_1 \cap \partial D$ (∂D= the boundary of D) satisfies some additional conditions, $\Gamma = \Gamma(D, E_0, E_1)$ is the family of the arcs joining E_0 and E_1 in D, $M_p\Gamma$ is the p-module of Γ and $\operatorname{cap}_p(D, E_0, E_1)$ is the p-capacity of E_0, E_1 relative to D. In order to be able to do this, we use a very recent result established by us in [9] about the completion of the class of admissible functions involved in the definition of M_p, when they are supposed to be bounded in R^n, continuous in D and 0 in the complement CD of D. In some of the cases considered above, the relation still holds if D, E_0 and E_1 are assumed to be contained in the one point compactification $\overline{R^n}$ of R^n. In the particular case n=2, we obtain that

$$(2) \qquad M_pZ = \operatorname{cap}_p Z = \operatorname{cap}_p(Z, B_0, B_1) \qquad (p > 1) ,$$

where M_pZ is the p-module of a topological cylinder with respect to the euclidean metric. This result represents a generalization of the case of (2) for topological cylinders with respect to the relative metric.

As an application, we show that a certain exceptional set E^o of the unit sphere S(corresponding to a quasiconformal mapping f of the unit ball B) is of conformal capacity 0.

Now, let us precise the concepts contained in this Note.

Let χ be a family of continua γ and $F(\chi)$ the class of admissible functions ρ characterized by the following conditions: $\rho \geqq 0$ in R^n is Borel measurable and so that $\int \rho dH^1 \geqq 1$ $\forall \gamma \in \chi$ ($\forall$="for every"), where H^1 is the Hausdorff linear measure. Then the p-module of χ is given as

$$M_p\chi = \inf_{\rho \in F(\chi)} \int \rho^p dm,$$

where dm is the volum element with respect to Lebesgue n-dimensional measure and the integration is taken over the whole space R^n. If $F(\chi)=0$, then $M_p\chi=\infty$, while if $\chi=\emptyset$, then $M_p\chi=0$. In the particular case p=n, we write $M_n=M$ and call it the module.

Next, let us remind several equivalent definitions of the p-capacity of 2 closed sets $C_0, C_1 \subset \bar{D}$ and then, let us give a generalization of these definitions.

The p-capacity of 2 closed sets $C_0, C_1 \subset \bar{D}$ relative to a domain

D is defined as

$$cap_p(D,C_0,C_1) = \inf \int_{D-(C_0\cup C_1)} |\nabla u|^p dm ,$$

where $\nabla u=(\frac{\partial u}{\partial x^1},\ldots,\frac{\partial u}{\partial x^n})$ is the gradient of u and the infimum is taken over all u, which are continuous in $D\cup C_0\cup C_1$, locally Lipschitzian in $D-(C_0\cup C_1)$ and assume the boundary valuus 0 on C_0 and 1 on C_1. If D is contained in a fixed ball, then $cap=cap_n$ is said to be the <u>conformal capacity</u>.

A function $u:D\to R$ is said to be ACL (<u>absolutely continuous on lines</u>) in D if $\forall\ I=\{x;\alpha^i<x^i<\beta^i\ (i=1,\ldots,n)\}, I\subset\subset D$ (i.e. $\bar{I}\subset D$), u is AC (absolutely continuous in the classical sense) on a.e. (almost every) line segment parallel to the coordinate axes, which means that if $I_i=\{x;x\in\bar{I},x^i=\alpha^i\}$ is a face of I and E is the set of points $\xi\in I_i$ such that f is not AC on the segment $J_\xi=\{x;x=\xi+\lambda e_i, 0<\lambda<\beta^i-\alpha^i\}$, then $m_{n-1}E=0$, where m_{n-1} is the (n-1)-dimensional Lebesgue measure.

Now, we obtain 2 other definitions of $cap_p(D,C_0,C_1)$ if we change the condition on u of being in $D-(C_0\cup C_1)$ locally Lipschitzian, by being ACL and of class C^1 (continuously differentiable), respectively. The equivalence of the last 2 definitions is established by us in [7] (lemma 10). Hence and since the condition that u is locally Lipschitzian in $D-(C_0\cup C_1)$ is stronger that being ACL and weaker that $u\in C^1$, it follows that also the first definition of $cap_p(D,C_0,C_1)$

in equivalent to the other 2.

Now, let us remind

Proposition 1. If $C_0, C_1 \subset D$ are 2 disjoint, closed sets and u is admissible for $cap_p(D, C_0, C_1)$, then $\nabla u(x) = 0$ a.e. in $C_0 \cup C_1$ so that

$$\int_{D-(C_0 \cup C_1)} |\nabla u|^p dm = \int_D |\nabla u|^p dm ,$$

and then

$$cap_p(D, C_0, C_1) = \inf_u \int_{D-(C_0 \cup C_1)} |\nabla u|^p dm = \inf_u \int_D |\nabla u|^p dm$$

(see proposition 2 of our paper [7], the proof is similar as in Gehring's paper [11], lemma 3).

Arguing as in Gehring's paper [11], we obtain

Lemma 1. In the hypotheses of the preceding proposition, u is ACL in D.

From the preceding proposition and lemma, we deduce the following

Corollary. We obtain for $cap_p(D, C_0, C_1)$ ($C_0, C_1 \subset D$ closed disjoint sets) an equivalent definition by

$$(3) \qquad cap_p(D, C_0, C_1) = \inf_u \int_D |\nabla u|^p dm ,$$

where the functions u admissible for $cap_p(D, C_0, C_1)$ are supposed to be ACL in D, not only in $D-(C_0 \cup C_1)$.

Proposition 2. If D is bounded, $C_0, C_1 \subset \overline{D}$ are 2 disjoint, closed sets, then

$$\mathrm{cap}_p(D,C_0,C_1) = \inf_u \int_{D-(C_0\cup C_1)} |\nabla u|^p dm,$$

where the infimum is taken over all $u \in C^1$ in $D-(C_0 \cup C_1)$ and with boundary values 0 on C_0 and 1 on C_1 (our paper [7], lemma 10).

Following the general line of the argument of the preceding proposition, we have

Lemma 2. If $C_0, C_1 \subset \overline{D}$ are 2 disjoint, closed sets, then (3) holds, where the infimum is taken over all $u \in C^1$ ($0 \leqq u(x) \leqq 1$ in D) and with boundary values 0 on C_0 and 1 on C_1.

If we denote the new infimum by $\mathrm{cap}_p^*(D,C_0,C_1)$, then, evident, on account of the preceding corollary,

$$\mathrm{cap}_p(D,C_0,C_1) \leqq \mathrm{cap}_p^*(D,C_0,C_1) \; .$$

In order to obtain the opposite inequality, it is enough to prove that

$$(4) \qquad \mathrm{cap}_p^*(D,C_0,C_1) \leqq \int_D |\nabla u|^p dm$$

$\forall u$ admissible for $\mathrm{cap}_p(D,C_0,C_1)$. Given such a u, we may assume that $|\nabla u| \in L^p(D)$ (i.e. $\int_D |\nabla u|^p dm < \infty$) for otherwise there is nothing to prove.

Next, fix $0<a<\frac{1}{2}$, and let

$$(5)\qquad v(x)=\begin{cases}0 & \text{if}\quad u(x)<a\,,\\ \dfrac{u(x)-a}{1-2a} & \text{if}\quad a\leq u(x)\leq 1-a\,,\\ 1 & \text{if}\quad u(x)>1-a\,.\end{cases}$$

The set $E_a=\{x;a\leq u(x)\leq 1-a\}$ is a subset of D, closed relatively to D and lies at a distance b from $C_0\cup C_1$. Let $\varepsilon<\min[1,b,d(C_0,C_1)]$ and extend v to be 0 on a b-neighbourhood $C_0(b)$ of C_0 and 1 on a b-neighbourhood $C_1(b)$ of C_1. Next, set $\delta(x)\equiv 1$ if $\partial D-(C_0\cup C_1)=\emptyset$ (where by ∂D we mean the boundary of D) and $\delta(x)=\min\{1,d[x,\partial D-(C_0\cup C_1)]\}$ otherwise. By means of δ, let us define $y=x+\xi\delta(x)$, $\xi\in B(\varepsilon)$, where $B(\varepsilon)$ is a ball of radius ε centred at 0. For a fixed ξ, y maps E_a into $D-(C_0\cup C_1)$. The function

$$w(x,\varepsilon)=\frac{1}{\omega_m\varepsilon^m}\int_{B(\varepsilon)}v[x+\xi\delta(x)]\,dm\,,$$

where ω_m is the volume of the unit ball, is clearly continuous in $C_0\cup C_1$, taking boundary values 0 on C_0 and 1 on C_1. To see it is continuous in $D-(C_0\cup C_1)$ too, let $x,x'\in D-(C_0\cup C_1)$, then since $0\leq v(x)\leq 1$ in $D-[C_0(\varepsilon)\cup C_1(\varepsilon)]$ and arguing as in the preceding proposition, it follows that

$$|w(x',\varepsilon)-w(x,\varepsilon)|\leq\left|\frac{1}{\omega_m[\varepsilon\delta(x')]^m}-\frac{1}{\omega_m[\varepsilon\delta(x)]^m}\right|\int_{B[x'-x,\,\varepsilon\delta(x')]\cap B[\varepsilon\delta(x)]}v(x+y)\,dm+$$

$$\frac{1}{\omega_m[\varepsilon\delta(x')]^m}\int_{B[x'-x,\,\varepsilon\delta(x')]-B[\varepsilon\delta(x)]}v(x+y)\,dm+$$

$$\frac{1}{\omega_n[\varepsilon\delta(x)]^n}\int_{B[\varepsilon\delta(x)]-B[x'-x,\varepsilon\delta(x)]} v(x+y)\,dm \leq \omega_n\left|\frac{1}{\omega_n[\varepsilon\delta(x')]^n}-\frac{1}{\omega_n[\varepsilon\delta(x)]^n}\right| +$$

$$\frac{1}{\omega_n[\varepsilon\delta(x')]^n}\int_{B[x'-x,\varepsilon\delta(x')]-B[\varepsilon\delta(x)]} dm+\frac{1}{\omega_n[\varepsilon\delta(x)]}\int_{B[\varepsilon\delta(x)]-B[x'-x,\varepsilon\delta(x')]} dm,$$

which, on account of Radon-Nykodim's theorem, becomes arbitrarily small for $|x-x'|$ small enough.

From above, v is bounded, continuous and ACL in $D\cup C_0(\varepsilon)\cup C_1(\varepsilon)$, so that v exists a.e. in $D\cup C_0(\varepsilon)\cup C_1(\varepsilon)$. Let us extend now $\frac{\partial v}{\partial y^i}$ in $D\cup C_0(\varepsilon)\cup C_1(\varepsilon)$ by

$$\frac{\partial v(y)}{\partial y^i} = \begin{cases} \frac{\partial v(y)}{\partial y^i} & \text{if } \frac{\partial v(y)}{\partial y^i} \text{ exists ,} \\ 0 & \text{otherwise .}\end{cases}$$

And now, $|\nabla u|\in L^p(D)$ implies, by (5), that $|\nabla v|\in L^p(D)$. Then, for every compact set $F\subset D$, by Hölder inequality, we have

$$\int_F|\nabla v|\,dm \leq \left(\int_F|\nabla v|^p dm\right)^{\frac{1}{p}}(mF)^{\frac{p-1}{p}},$$

i.e. $|\nabla v|$ and then ∇v too (see S. Saks [20], theorem 12, p. 66) and <u>a fortiori</u> $|\frac{\partial v}{\partial x^i}|$ and $\frac{\partial v}{\partial x^i}$ are integrable over every compact set $F\subset D$.

Finally, arguing as in the preceding proposition, we conclude that

$$|\nabla w| \leq \frac{1+\varepsilon}{\omega_n\varepsilon^n}\int_{B(\varepsilon)}|\nabla v|\,dm ,$$

hence, applying Minkowski's inequality,

$$(6)\quad \begin{aligned}&\left[\int_D|\nabla w(x,\varepsilon)|^p dm(x)\right]^{\frac{1}{p}} \leq \frac{1+\varepsilon}{\omega_n\varepsilon^n}\left(\int_D\left\{\int_{B(\varepsilon)}|\nabla v[x+\xi\delta(x)]|\,dm(x)\right\}^p dm(x)\right)^{\frac{1}{p}} \leq \\ &\frac{(1+\varepsilon)}{\omega_n\varepsilon^n}\int_{B(\varepsilon)}\left\{\int_D|\nabla v[x+\xi\delta(x)]|^p dm(x)\right\}^{\frac{1}{p}}dm(\xi) ,\end{aligned}$$

where we denoted dm(x) and dm(ξ) for dm in order to point out the variable of integration. But, it is easy to see that δ is Lipschitzian with Lipschitz constant 1, hence, for instance,

$$\left|\frac{\partial v[x^1+\xi^1\delta(x),\ldots,x^n+\xi^n\delta(x)]}{\partial x^1}\right| = \left|\frac{\partial v(y^1,\ldots,y^n)}{\partial y^1}\right| \cdot \left|\frac{\partial [x^1+\xi^1\delta(x)]}{\partial x^1}\right| =$$

$$\left|\frac{\partial v(y)}{\partial y^1}\right| \cdot \left|1+\xi^1\frac{\partial \delta(x)}{\partial x^1}\right| \leq \left|\frac{\partial v(y)}{\partial y^1}\right|(1+|\xi^1|) \leq \left|\frac{\partial v(y)}{\partial y^1}\right|(1+\varepsilon) < \frac{\left|\frac{\partial v(y)}{\partial y^1}\right|}{1-\varepsilon},$$

hence

$$|\nabla v[x+\xi\delta(x)]| < \frac{|\nabla v(x)|}{1-\varepsilon},$$

so that, $\forall \xi \in B(\varepsilon)$,

$$\int_D |\nabla v[x+\xi\delta(x)]| dm < \frac{1}{(1-\varepsilon)^p}\int_D |\nabla v(x)|^p dm \leq \frac{1}{[(1-\varepsilon)(1-2a)]^p}\int_D |\nabla u|^p dm,$$

which, taking account of (6) and of the fact that $w(x,\varepsilon)$ is admissible for $\mathrm{cap}_p^m(D,C_0,C_1)$, yields

$$\mathrm{cap}_p^m(D,C_0,C_1) \leq$$

$$\int_D |\nabla w(x,\varepsilon)|^p dm < \frac{1}{[(1-\varepsilon)^2(1-2a)\omega_n\varepsilon^n]^p}\left\{\int_{B(\varepsilon)}\left[\int_D |\nabla u(x)|^p dm(x)\right]^{\frac{1}{p}} dm(\xi)\right\}^p =$$

$$\frac{1}{[(1-\varepsilon)^2(1-2a)]^p}\int_D |\nabla u|^p dm,$$

whence

$$\mathrm{cap}_p^m(D,C_0,C_1) < \frac{1}{[(1-\varepsilon)^2(1-2a)]^p}\int_D |\nabla u|^p dm$$

and, letting $\varepsilon \to 0$ and then $a \to 0$, we obtain (4), as desired.

From this lemma and the preceding corollary, we deduce

Corollary 1. The 3 definitions of the p-capacity $cap_p(D,C_0,C_1)$, where $C_0,C_1\subset\overline{D}$ are 2 disjoint, closed sets and the admissible functions u involved in the definitions are supposed to be in D: ACL, of class C^1 and locally Lipschitzian, respectively, are equivalent and also equivalent to the corresponding definitions, where the properties of being ACL, of class C^1 or locally Lipschitzian are supposed to hold only in $D-(C_0\cup C_1)$.

This corollary sugests us the following generalization of the preceding definitions:

The p-capacity of 2 sets E_0,E_1 relative to a domain D, where $E_0,E_1\subset\overline{D}$ and $d(E_0,E_1)>0$, is defined by

$$(7)\qquad cap_p(D,E_0,E_1) = \inf_{u}\int_D|\nabla u|^p dm ,$$

where the infimum is taken over all u, $(0\leq u(x)\leq 1)$ which are continuous in $D\cup E_0\cup E_1$, locally Lipschitzian in D and assume the boundary values 0 on E_0 and 1 on E_1.

We obtain the other 2 generalizations of $cap_p(D,E_0,E_1)$ if, instead of being locally Lipschitzian in D, they are ACL and of class C^1, respectively, there.

Remarks. 1. Clearly, in the particular case in which E_0,E_1 are closed, the preceding new 3 definitions come to the corresponding previous ones.

2. We had to suppose, in the last 3 definitions that u is ACL, locally

Lipschitzian or of class C^1 in D and not in

$D-(E_0\cup E_1)$ since the properties of being ACL, locally Lipschitzian or of class C^1 are meaningless in $D-(E_0\cup E_1)$ if this set is not open. Of course it is possible to try to extend the corresponding concepts for more general sets, but we preferred this way.

3. We had to introduce the condition $d(E_0,E_1)>0$ since u is supposed to be continuous in $D\cup E_0\cup E_1$ and to have boundary values 0 on E_0 and 1 on E_1 and then, if $\overline{E}_0\cap\overline{E}_1\cap D\neq\emptyset$, at such points, u has to be at the same time equal to 0 and to 1. Of course, it would be enough to assum only that $d(E_0\cap D, E_1\cap D)>0$, but the more restrictive condition is necessary in the proof of the equivalence of the last 3 new definitions.

Arguing, exactly as in the preceding lemma, we obtain

<u>Corollary 2. If</u> $E_0, E_1\subset\overline{D}$ <u>with</u> $d(E_0,E_1)>0$, <u>then for</u> $\mathrm{cap}_p(D,E_0,E_1)$ <u>in</u> (7), <u>we have the same value, no matter if the admissible functions</u> u, <u>involved in the definition are supposed to be</u> ACL, <u>locally Lipschitzian or of class</u> C^1 <u>in</u> D.

From each of the above definitions for the p-capacity, we obtain the corresponding definition for the conformal capacity if we take p=n and suppose that $\overline{D}$ is contained in a fixed ball.

Another generalization may be obtained if we get rid of the condition $E_0, E_1\subset\overline{E}$. In the particular case in which $E_0\cap C\overline{D}=\emptyset$ or $E_1\cap C\overline{D}=\emptyset$, we consider, as it is natural, $\mathrm{cap}_p(D,E_0,E_1)=0$ because, in the first case, the function u such that $u_{|D\cup E_0}=0$ and $u_{|E_1}=1$ is admissible, while in the second case, the function u such that $u_{|D\setminus E_1}=1$ and $u_{|E_0}=0$ is admissible too and then, in the 2 cases, $|\nabla u|_{|D}=0$.

Let us mention a last generalization by supposing that D is only open. In this case, we precise that, if there are components D_0 with $\overline{D}_0\cap E_1=\emptyset$, it is enough to consider only admissible functions u such that the restriction $u_{|\overline{D}_0}=0$ and if there are components D_1 such that $\overline{D}_1\cap E_0=\emptyset$ then, it is enough to consider only admissible functions such that $u_{|\overline{D}_1}=1$. In this 2 cases, $\nabla u_{|\overline{D}_0}=\nabla u_{|\overline{D}_1}=0$, so that, if we eliminate from the open set D all the components of these 2 kinds, the value of $\mathrm{cap}_p(D,E_0,E_1)$ remains unchanged.

<u>Lemma 3. If an open set</u> D <u>is a union of domains of the form</u> $D=(\bigcup_k D_k)\cup(\bigcup_m D_m^0)\cup(\bigcup_q D_q^1)$, <u>where</u> D_m^0 <u>and</u> D_q^0 <u>are of the type</u> D_0, D_1 <u>introduced above, while</u> $\overline{D}_k\cap E_0, \overline{D}_k\cap E_1\neq\emptyset$ $(k=1,2,\ldots)$, <u>then</u>

$$\mathrm{cap}_p(D,E_0,E_1) = \sum_k \mathrm{cap}_p(D_k,E_0,E_1).$$

<u>Remark</u>. From this lemma, it follows that it does not matter if 2 different components of D have common boundary points and if some of these common boundary points belong to E_0 or to E_1.

A crucial role in the generalization of Ziemer's relation(1) is played by

<u>Proposition 3. If</u> D <u>is a domain and</u> $\Gamma=\Gamma(D,E_0,E_1)$ <u>is the family of the arcs joining 2 disjoint sets</u> $E_0,E_1\subset\partial D$, <u>then</u>

$$(8)\quad M_p\Gamma = M_p^{\circ D}\Gamma = \inf_{\rho\in F^{\circ D}(\Gamma)} \int \rho^p dm ,$$

where $F^{\circ D}(\Gamma)$ <u>is the class of the admissible functions</u> $\rho\in F(\Gamma)$ <u>bounded in</u> R^n, <u>continuous in</u> D <u>and</u> 0 <u>in</u> CD(see our Note [9]).

Arguing as in the preceding proposition, we have also

<u>Lemma 4. In the hypotheses of the preceding proposition, where</u> D <u>is only an open set, then</u> (8) <u>still holds.</u>

<u>Lemma 5. If</u> D <u>is an open set</u>, $\Gamma=\Gamma(D,E_0,E_1)$ <u>where</u> $d(E_0,E_1)>0$ <u>and</u> $\Gamma_1=\Gamma(\Delta,E_0,E_1)$, <u>where</u> $\Delta=D-(\bar{E}_0\cup\bar{E}_1)$, <u>then</u>

$$(9)\quad M_p\Gamma = M_p\Gamma_1 .$$

If E_0 or E_1 is contained in $C\bar{D}$ or if there is no component of D whose closure contain simultaneosly points of E_0 and of E_1, then $\Gamma=\Gamma_1=\emptyset$ and (9) holds trivially, so that, without any loss of generality, we may suppose that there exists a domain $D_0\subset D$ such that $E_0\cap\bar{D}_0, E_1\cap\bar{D}_0\neq\emptyset$. Hence and since $d(E_0,E_1)>0$, it follows that $D_0-(\bar{E}_0\cup\bar{E}_1)\neq\emptyset$ and <u>a fortiori</u> $\Delta=D-(\bar{E}_0\cup\bar{E}_1)\neq\emptyset$.

Next, clearly, $\Gamma_1\subset\Gamma$, so that, on account of theorem 1 in Fuglede's paper [10],

$$(10)\quad M_p\Gamma_1 \leq M_p\Gamma .$$

Finally, Γ is minorized by Γ_1, i.e. $\forall\gamma\in\Gamma$, there exists a $\gamma_1\in\Gamma_1$ such that $\gamma_1\subset\gamma$. But then, by Fuglede's theorem quoted above, $M_p\Gamma\leq M_p\Gamma_1$, which together with (10), yields (9), as desired.

Hence, and taking into account Fuglede's theorem quoted above and the preceding proposition, we deduce the

Corollary. In the hypotheses of the preceding lemma, we have

$$M_p\Gamma = M_p^{e\Delta}\Gamma_1 .$$

Proposition 4. If χ is the set of all continua in R^n that intersect 2 disjoint closed sets C_0 and C_1, where C_0 is assumed to be contained in the complement of a ball, then

$$M_p\chi \leq cap_p(R^n, C_0, C_1) .$$

(For the proof, see W. Ziemer [13], lemma 3.1.)

Now, we remind that the families Γ_m $(m=1,2,\dots)$ are called separate if there exist disjoint Borel sets E_m $(m=1,2,\dots)$ such that $\gamma\in\Gamma_m$ imply $H^1(\gamma-E_m)=0$.

Theorem 1. If D is open, E_0, E_1 are 2 sets such that $d(E_0,E_1)>0$ and for each component D_k of D with $\overline{D}_k\cap E_i\neq\emptyset$ $(i=0,1)$, for $i=0$ or $i=1$, $\forall\xi\in\partial D_k\cap E_i$,

$$(11)\qquad \liminf_{\substack{x\to\xi\\ x\in D_k}} \inf_{\gamma} H^1[\gamma(E_i,x)] = 0 ,$$

where the infimum is taken over all $\gamma=\gamma(E_i,x)$ joining x and E_i in D_k, then (1) holds.

Using the notations of lemma 3, $D=(\bigcup_k D_k)\cup(\bigcup_m D_m^c)\cup(\bigcup_q D_q^1)$ and, clearly, the arc families $\Gamma_k=\Gamma(D_k,E_0,E_1),\Gamma^0_m=\Gamma(D^0_m,E_0,E_1),\Gamma^1_q=\Gamma(D^1_q,E_0,E_1)$ (k, m, q=1, 2,...) are separate so that, on account of lemma 2.1(c) of J.Väisälä's paper [12],

$$M_p\Gamma = \sum_k M_p\Gamma_k+\sum_m M_p\Gamma^0_m+\sum_q M_p\Gamma^1_q ,$$

where $\Gamma=(\bigcup_k\Gamma_m)\cup(\bigcup_m\Gamma^0_m)\cup(\bigcup_q\Gamma^1_q)$ and $\Gamma^0_m=\Gamma^1_q=\emptyset$ yielding $M_p\Gamma^0_m=M_p\Gamma^1_q=0$ (m, q=1, 2, ...). But them, taking into account lemma 3, it follows that we may suppose, without loss of generality, that D itself is a domain, since otherwise, we can establish (11) for each component of D separately.

The inequality

$$(12)\qquad M_p\Gamma \leqq \mathrm{cap}_p(D,E_0,E_1)$$

may be proved by the same argument as that used by W.Ziemer [13] for the preceding proposition since the additional conditions "C_0, C_1 closed and C_0 containing the complement of a ball" are not involved in the proof, while the use of Γ instead of γ rather simplifies things.

Next, in order to prove also the opposite inequality, it is sufficient to establish that

$$\mathrm{cap}_p(D,E_0,E_1) \leqq \int\rho^p dm ,$$

where, on account of the preceding corollary, ρ may be supposed to belong to $F^{\circ\Delta}(\Gamma_1)$, with $\Delta = D-(\overline{E}_0\cup\overline{E}_1)$ and $\Gamma_1=\Gamma(\Delta, E_0, E_1)$.

Now, assume that (11) holds for E_0 and $\forall x\in D$, let

$$u(x) = \inf_{\gamma} \int_{\gamma(E_0,x)} \rho dH^1 ,$$

where the infimum is taken over all $\gamma=\gamma(E_0,x)$ joining x and E_0 in $D\cup E_0\cup E_1$. Extend u to be 0 and 1 in the components of E_0 and E_1, respectively, which are disjoint of $\overline{D}$. Then, clearly $u_{|E_1}\geq 1$. Next, since $\rho\in F^{\circ\Delta}(\Gamma_1)$, it is easy to see that u has the boundary value 0 on $E_0\cap\overline{D}$. Indeed, if $\xi\in E_0\cap D$ and $\sup_{x\in R^n} \rho(x)=M<\infty$, then, by (11),

$$\lim_{\substack{x\to\xi\\ x\in D}} u(x) = \lim_{\substack{x\to\xi\\ x\in D}} \inf_{\gamma} \int_{\gamma(E_0,x)} \rho dH^1 \leq \lim_{\substack{x\to\xi\\ x\in D}} M \inf_{\gamma} \int_{\gamma(E_0,x)} dH^1 \leq$$

$$M \lim_{\substack{x\to\xi\\ x\in D}} \inf_{\gamma} H^1[\gamma(E_0,x)] = 0$$

and taking $u(x)=0$ also for $x\in E_0-\overline{D}$, it follows that $u_{|E_0}=0$.

We have also to show that u is locally Lipschitzian in D. Indeed, if $x_0\in D$, $U=B(x_0,r)\subset D$ is a spherical neighbourhood of x_0, $x_1, x_2\in U$, γ_1 is an arc joining E_0 and x_1 in $D\cup E_0$, while λ is the line segment joining x_1 and x_2, then

$$u(x_2) \leq \int_{\gamma_1} \rho dH^1 + \int_{\lambda} \rho dH^1 ,$$

and taking the infimum over all the arcs joining x_1 and E_0 in $D\cup E_0$, we obtain

$$u(x_2) \leq \inf_{\gamma_1} \int_{\gamma_1} \rho dH^1 + \int_{\lambda} \rho dH^1 \leq u(x_1) + M\int_{\lambda} dH^1 = u(x_1) + M|x_1 - x_2| \ ;$$

but then, $|u(x_1)-u(x_2)| \leq M|x_1-x_2|$, i.e. u is locally Lipschitzian in D, as desired.

Next, let us prove that

(13) $|\nabla u(x)| \leq \rho(x)$

a.e. in Δ. First, we observe that u is differentiable a.e. in D and, at a point of differentiability,

(14) $|\nabla u(x)| = \sup_{s} \left|\frac{\partial u(x)}{\partial s}\right|$,

where

$$\frac{\partial u(x)}{\partial s} = \lim_{t\to 0} \frac{u(x+te_s)-u(x)}{t}$$

is the directional derivative of u. Clearly,

$$u(x+te_s) \leq \int_{\gamma} \rho dH^1 + \int_{\lambda_t} \rho dH^1 ,$$

where $\gamma=\gamma(E_0,x)$ (from above), while λ_t is the line segment joining x and

$x+te_s$. This inequality yields

$$u(x+te_s) \leq \inf_\gamma \int_{\gamma(E_0,x)} \rho dH^1 + \int_{\lambda_t} \rho dH^1 = u(x) + \int_{\lambda_t} \rho dH^1 ,$$

hence, in the case $u(x) \leq u(x+te_s)$, we have

$$\left|\frac{u(x+te_s)-u(x)}{t}\right| = \frac{u(x+te_s)-u(x)}{t} \leq \frac{1}{t}\int_{\lambda_t} \rho dH^1 .$$

But, arguing as above, we get also

$$u(x) \leq \inf_\gamma \int_{\gamma(E_0,x+te_s)} \rho dH^1 + \int_{\lambda_t} \rho dH^1 = u(x+te_s) + \int_{\lambda_t} \rho dH^1 ,$$

so that, for $u(x+te_s) \leq u(x)$, we deduce

$$\left|\frac{u(x+te_s)-u(x)}{t}\right| = \frac{u(x)-u(x+te_s)}{t} \leq \frac{1}{t}\int_{\lambda_t} \rho dH^1 ,$$

and then, in the two cases,

$$\left|\frac{u(x+te_s)-u(x)}{t}\right| \leq \frac{1}{t}\int_{\lambda_t} \rho dH^1 ,$$

whence

$$\left|\frac{\partial u(x)}{\partial s}\right| = \lim_{t\to 0}\left|\frac{u(x+te_s)-u(x)}{t}\right| \leq \lim_{t\to 0}\frac{1}{t}\int_{\lambda_t} \rho dH^1 = \rho(x)$$

a.e. in Δ, since ρ is supposed to be continuous in Δ and the points of continuity are Lebesgue points implying the last part of the preceding relation, which, taking account of (14), yields (13), as desired.

And now, we remark that $h = \inf_{E_1} u(x) \geq 1$. Let us consider the truncation

$$u^*(x) = \begin{cases} h & \text{for } u(x) > h , \\ u(x) & \text{for } 0 \leq u(x) \leq h \end{cases}$$

and observe that $\frac{u^*}{h}$ is admissible for $\mathrm{cap}_p(D,E_0,E_1)$. Indeed, $0 \leq \frac{u^*(x)}{h} \leq 1$, $\frac{u^*}{h}\big|_{E_0} = 0$ since $u_{|E_0} = 0$ and $\frac{u^*}{h}\big|_{E_1} = 1$ because, from the definition of h, $u(x) \geq h$ $\forall x \in E_1$ and u^* is the truncation of u at the level h. But, since u

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locally is Lipshitzian in D with Lipschitz constant M (as it was proved above), it is easy to see that $\frac{u^*}{h}$ is locally Lipschitzian in D too with Lipschitz constant $\frac{M}{h}$. Indeed, if $x,y \in D_h = \{x \in D; u(x) < h\}$, then, clearly,

$$\left|\frac{u^*(x)}{h} - \frac{u^*(y)}{h}\right| = \frac{1}{h}|u(x)-u(y)| \leq \frac{M}{h}|x-y| \ ;$$

if $x,y \in D \cap CD_h$, then $u^*(x)=u^*(y)=h$, so that

$$\left|\frac{u^*(x)}{h} - \frac{u^*(y)}{h}\right| = 0 \leq \frac{M}{h}|x-y| \ ;$$

and if $x \in D_h$, $y \in D \cap CD_h$, then

$$\left|\frac{u^*(x)}{h} - \frac{u^*(y)}{h}\right| = \left|\frac{u(x)}{h} - 1\right| \leq \left|\frac{u(x)}{h} - \frac{u(y)}{h}\right| \leq \frac{M}{h}|x-y| \ .$$

Now, let us verify also that

$$(15) \quad |\nabla u^*(x)| = \sup_s \left|\frac{\partial u^*(x)}{\partial s}\right| \leq \rho(x)$$

at any point of differentiability of u^*, i.e. a.e. in D. Indeed, suppose x is such a point of differentiability. If $x \in D_h$, then, evident,

$$\frac{\partial u^*(x)}{\partial s} = \lim_{t \to 0} \frac{u^*(x+te_s)-u^*(x)}{t} = \lim_{t \to 0} \frac{u(x+te_s)-u(x)}{t} = \frac{\partial u(x)}{\partial s} \ ,$$

hence

$$|\nabla u^{*}(x)| = |\nabla u(x)| .$$

If $x\in C\overline{D_h}\cap D$, for t sufficiently small, $u^{*}(x+te_s)=u^{*}(x)=h$, yielding $\left|\frac{\partial u^{*}(x)}{\partial s}\right|=0$ $\forall s$ and then, $|\nabla u^{*}(x)|=0$. Now, if $x\in\partial D_h\cap D$, then

$$\left|\frac{\partial u^{*}(x)}{\partial s}\right| = \lim_{t\to 0}\left|\frac{u^{*}(x+te_s)-u^{*}(x)}{t}\right| = \lim_{t\to 0}\left|\frac{u^{*}(x+te_s)-u(x)}{t}\right| \leq \lim_{t\to 0}\left|\frac{u(x+te_s)-u(x)}{t}\right| \leq \left|\frac{\partial u(x)}{\partial s}\right| ,$$

hence $|\nabla u^{*}(x)|\leq|\nabla u(x)|$ a.e. in D and, on account of (13)

(16) $$|\nabla u^{*}(x)| \leq \rho(x)$$

a.e. in Δ.

Next, since $\rho\in F^{\circ\Delta}(\Gamma_1)$, it follows that $\rho_{|C\Delta}=0$ and then, <u>a fortiori</u>, $\rho_{|\overline{E}_0\cup\overline{E}_1}=\rho_{|(\overline{E}_0\cup\overline{E}_1)\cap D}=0$. In order to show that $|\nabla u^{*}(x)|\leq\rho(x)$ a.e. in D, it remains to prove that $|\nabla u^{*}(x)|=0$ a.e. in $(\overline{E}_0\cup\overline{E}_1)\cap D$. But u^{*} is continuous in $D\cup E_0\cup E_1$, and then, in particular, in $D\cap\overline{E}_0$ and in $D\cap\overline{E}_1$, so that $u^{*}_{|E_0}=0$ and $u^{*}_{|E_1}=1$ imply $u^{*}_{|\overline{E}_0\cap D}=0$ and $u^{*}_{|\overline{E}_1\cap D}=1$, respectively. And now, since u^{*} is differentiable a.e. in D, ∇u^{*} exists a.e. in D and then <u>a fortiori</u> a.e. in $\overline{E}_0\cap D$ and $\overline{E}_1\cap D$, which are measurable sets, so that almost all their points are of linear density in the direction of the coordinate axes (see S. Saks [20] p.298), implying that $|\nabla u^{*}(x)|=0$ a.e. in $\overline{E}_0\cap D$ and $\overline{E}_1\cap D$, hence $|\nabla u^{*}(x)|=0=\rho(x)$ a.e. in $(\overline{E}_0\cup\overline{E}_1)\cap D$, which, together with (16), allows us to conclude that (15) holds.

We remind that $x_0 \in R^n$ is said to be a point of linear density in the direction of the coordinate axes if

$$\lim_{r\to 0} \frac{m_1\{[B(x_0,r)\cap E]; x_0^1,\dots,x_0^{i-1},x_0^{i+1},\dots,x_0^n\}}{2r} = 1 \quad (i = 1,\dots,n),$$

where by $(E; x_0^1,\dots,x_0^{i-1},x_0^{i+1},\dots,x_0^n)$ was denoted the intersection of E with the axis X_i and m_1 is the linear Lebesgue measure.

Finally, since $\frac{u^*}{h}$ is admissible for $\mathrm{cap}_p(D,E_0,E_1)$ with $h\geq 1$ and taking account of (15), we obtain

$$\mathrm{cap}_p(D,E_0,E_1) \leq \frac{1}{h^p}\int_D |\nabla u^*|^p dm \leq \frac{1}{h^p}\int \rho^p dm \leq \int \rho^p dm ,$$

and since ρ is an arbitrary function of $F^{o\Delta}(\Gamma_1)$, taking the infimum over all such ρ, we deduce that

$$(17) \qquad \mathrm{cap}_p(D,E_0,E_1) \leq M_p^{o\Delta}\Gamma_1 = M_p\Gamma ,$$

which together with (12), yields (1) if (11) holds for E_0.

When E_1 satisfies condition (11), we repeat the above argument for E_1 instead of E_0 and consider finally the function $v=1-\frac{u^*}{h}$, which is admissible for $\mathrm{cap}_p(D,E_0,E_1)$.

<u>Remark</u>. We shall provide an example to show that it is possible to have a bounded simply connected domain D with 2 disjoint compact sets E_0,

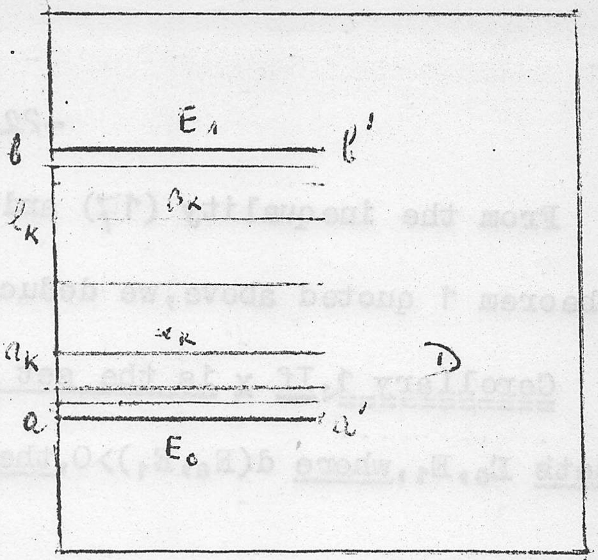

Fig.1

$E_1 \subset D$ such that all the points of $E_0 \cup E_1$ are accessible by rectifiable arcs, but they do not verify condition (11). Indeed, let $D \subset R^2$ be a square with the side $l=2$, let $E_0, E_1 \subset \partial D$ be 2 closed segments of length 1, parallel to a side of D, such that E_0 has the endpoints (a, a^*) and E_1 the endpoints (b, b^*), as in fig.1, and let $\{\alpha_k\}, \{\beta_k\}$ be 2 sequences of segments parallel to E_0, E_1 and with the endpoints a_k and b_k converging to a and b, respectively. Then all the points of $E_0 \cup E_1$ are accessible by rectifiable arcs, but for each of them, except a^* and b^*, (11) does not hold.

Now, let D be a bounded domain. For any 2 points $x, y \in D$, we shall define the <u>relative distance</u> $d_D(x,y)$ to be the greatest lower bound of the lengths of all polygonal lines joining x to y in D. It is clear that $d_D(x,y)$ is a metric and that $d_D(x,y) \geq |x-y|$, with equality iff x, y lie in some convex subset of D. If $x \in D$ and $\xi \in \partial D$, we define $d_D(x,\xi)$ to be the infimum of $\lim_{m\to\infty} d_D(x, x_m)$ on all sequences $\{x_m\}$ tending to ξ, with $x_m \in D$ $(m=1, 2, \ldots)$.

<u>Corollary 1</u>. <u>If</u> D <u>is open</u>, $d(E_0, E_1) > 0$ <u>and</u> $\forall D_k$ <u>with</u> $\overline{D}_k \cap E_i \neq \emptyset$ $(i=0,1)$ <u>for</u> $i=0$ <u>or</u> $i=1$, $\forall \xi \in \partial D_k \cap E_i$, $\lim_{x\to\xi} d_{D_k}(\xi, x) = 0$ (<u>i.e. the relative metric is continuous with respect to</u> D <u>on the corresponding set</u> $\partial D_k \cap E_i$).

From the inequality (17) and taking into account Fuglede's [10] theorem 1 quoted above, we deduce

<u>Corollary 2.</u> <u>If</u> χ <u>is the set of all continua in</u> R^n <u>that intersect the sets</u> E_0, E_1, <u>where</u> $d(E_0,E_1)>0$, <u>then</u>

$$M_p\chi = \mathrm{cap}_p(R^n, E_0, E_1) .$$

<u>Corollary 3.</u> <u>In the hypotheses of the preceding theorem,</u>

$$(18) \qquad M\Gamma = \mathrm{cap}(D, E_0, E_1) .$$

Now, in order to establish (1) in more general hypotheses, let us prove some properties of the p-capacity.

<u>Lemma 6.</u> <u>The</u> p-<u>capacity</u> $\mathrm{cap}_p(D,E_0,E_1)$, <u>where</u> $D\subset R^n$ <u>is a domain and</u> $E_0=\emptyset$ <u>or</u> $E_1=\emptyset$ <u>or</u> $E_0,E_1\subset R^n$ <u>are such that</u> $d(E_0,E_1)>0$, <u>satisfies the following conditions:</u>

(i) $\mathrm{cap}_p(D,\emptyset,E_1)=0$.

(ii) $E_0\subset E_0' \Rightarrow$ ("implies") $\mathrm{cap}_p(D,E_0,E_1)\leq \mathrm{cap}_p(D,E_0',E_1)$.

(iii) $E_0\subset \bigcup_{k=1}^{m} E_0^k \Rightarrow \mathrm{cap}_p(D,E_0,E_1)\leq \sum_{k=1}^{m} \mathrm{cap}_p(D,E_0^k,E_1)$.

(i') $\mathrm{cap}_p(D,E_0,\emptyset)=0$.

(ii') $E_1\subset E_1' \Rightarrow \mathrm{cap}_p(D,E_0,E_1)\leq \mathrm{cap}_p(D,E_0,E_1')$.

(iii') $E_1\subset \bigcup_{k=1}^{m} E_1^k \Rightarrow \mathrm{cap}_p(D,E_0,E_1)\leq \sum_{k=1}^{m} \mathrm{cap}_p(D,E_0,E_1^k)$.

Condition (i) is trivial since $u=1$ is an admissible function. It is easy to see that also (ii) holds since if $E_0 \subset E_0'$ and $\mathcal{U}, \mathcal{U}'$ are the 2 corresponding classes of admissible functions, then $\mathcal{U}' \subset \mathcal{U}$, hence

$$\mathrm{cap}_p(D,E_0,E_1) = \inf_{\mathcal{U}} \int_D |\nabla u|^p dm \leq \inf_{\mathcal{U}'} \int_D |\nabla u|^p dm = \mathrm{cap}_p(D,E_0',E_1) .$$

In order to establish (iii), let $\mathcal{U}_k$ be the class of admissible functions for $\mathrm{cap}_p(D,E_0,E_1)$ $(k=1,\dots,q)$ and let $u(x)=\min[u_1(x),\dots,u_q(x)]$, where $u_k \in \mathcal{U}_k$ $(k=1,\dots,q)$. Clearly, $0 \leq u(x) \leq 1$, $u_{|E_0}=0$ and $u_{|E_1}=1$.

Now, let us prove that u too is locally Lipschitzian in D. Indeed, given a point $x \in D$, let V_k be a neighbourhood of x where u_k is Lipschitzian and let us show that u is Lipschitzian in any neighbourhood $W \subset \bigcap_{k=1}^{m} V_k$. For this, let $y \in W$. Since u_k is Lipschitzian (let us precise: with Lipschitz constant M_k) in W, it follows, in particular, that $|u_k(x)-u_k(y)| < M_k|x-y|$ $(k=1,\dots,q)$, hence, if for instance $u(x) \leq u(y)$ then, since, by definition, there is an integer $k \in (1,q)$ such that $u(x)=u_k(x)$, it follows that

$$|u(x)-u(y)| = |u_k(x)-u(y)| \leq |u_k(x)-u_k(y)| \leq M_k|x-y| \leq M|x-y| ,$$

where $M = \max_{1 \leq k \leq q} M_k$, allowing us to conclude that, $\forall x \in D$, there exists a neighbourhood W_x of x, where the preceding inequality holds, i.e. u is locally Lipschitzian in D, allowing us to conclude that $u \in \mathcal{U}$, which means that u is admissible for $\mathrm{cap}_p(D,E_0,E_1)$.

Next, let us show that

$$(19) \quad |\nabla u(x)|^p \leq \sum_{k=1}^{q} |\nabla u_k(x)| \quad .$$

Let us consider a unit vector e_s of direction s and suppose first that

$$(20) \quad u(x) \leq u(x+|\Delta x|e_s).$$

Then, if $u_k(x)=\min_{1\leq i\leq q} u_i(x)=u(x)$, we have

$$\left[\frac{|u(x)-u(x+|\Delta x|e_s)|}{|\Delta x|}\right]^p=\left[\frac{|u_k(x)-u(x+|\Delta x|e_s)|}{|\Delta x|}\right]^p\leq\left[\frac{|u_k(x)-u_k(x+|\Delta x|e_s)|}{|\Delta x|}\right]^p$$

$$\leq \sum_{k=1}^{q}\left[\frac{|u_k(x)-u_k(x+|\Delta x|e_s)|}{|\Delta x|}\right]^p .$$

But, since u and u_k are Lipschitzian with Lipschitz constant M in a n-dimensional neighbourhood W_x of x, then they are Lipschitzian also in a linear neighbourhood of x contained in the axis X_s passing through x and having the direction s, so that u and u_k (k=1,...,q) (considered as functions of a real variable) have a directional derivative $\frac{\partial u}{\partial s}$ and $\frac{\partial u_k}{\partial s}$ a.e. in $W_x \cap X_s$. Assume that the point x from above is such a point. Then, letting $|\Delta x| \to 0$ in the preceding inequality, we obtain

$$\left|\frac{\partial u(x)}{\partial s}\right|^p= \lim_{|\Delta x|\to 0}\left[\frac{|u(x)-u(x+|\Delta x|e_s)|}{|\Delta x|}\right]^p\leq \overline{\lim_{|\Delta x|\to 0}}\left[\sum_{k=1}^{q}\left[\frac{|u_k(x)-u_k(x+|\Delta x|e_s)|}{|\Delta x|}\right]^p\right.$$

$$\leq \sum_{k=1}^{q} \overline{\lim_{|\Delta x|\to 0}} \left[\frac{|u_k(x)-u_k(x+|\Delta x|e_S)|}{|\Delta x|}\right]^p = \sum_{k=1}^{q} \left|\frac{\partial u_k(x)}{\partial s}\right|^p.$$

Next, since u and u_k $(k=1,\ldots,q)$ are Lipschitzian in W_x, they are differentiable a.e. in W_x. Let us suppose that x is such a point where u and u_k $(k=1,\ldots,q)$ are differentiable and then, where the relation (14) holds. But then, the preceding inequality yields

$$|\nabla u(x)|^p = \sup_S \left|\frac{\partial u(x)}{\partial s}\right|^p \leq \sup_S \sum_{k=1}^{q} \left|\frac{\partial u_k(x)}{\partial s}\right|^p \leq \sum_{k=1}^{q} \sup_S \left|\frac{\partial u_k(x)}{\partial s}\right|^p = \sum_{k=1}^{q} |\nabla u_k(x)|^p .$$

Thus, we established (13) in the hypothesis (10).

Now, assume that the opposite inequality holds, i.e. that $u(x) > u(x+|\Delta x|e_S)$. Then, if $u_k(x+|\Delta x|e_S) = \min_{1\leq i\leq q} u_i(x+|\Delta x|e_S) = u(x+|\Delta x|e_S)$, we have

$$\left[\frac{|u(x)-u(x+|\Delta x|e_S)|}{|\Delta x|}\right]^p = \left[\frac{|u(x)-u_k(x+|\Delta x|e_S)|}{|\Delta x|}\right]^p \leq \left[\frac{|u_k(x)-u_k(x+|\Delta x|e_S)|}{|\Delta x|}\right]^p \leq$$

$$\leq \sum_{k=1}^{q} \left[\frac{|u_k(x)-u_k(x+|\Delta x|e_S)|}{|\Delta x|}\right]^p$$

and arguing as above, in the hypothesis (10), we obtain (13) also in this case.

Finally, from (13), since $u\in\mathcal{U}$, we deduce that

$$\mathrm{cap}_p(D,E_0,E_1) \leq \int_D |\nabla u|^p dm \leq \sum_{k=1}^{q} \int_D |\nabla u_k|^p dm ,$$

hence, since each u_k was an arbitrary function of $\mathcal{U}_k$,

$$cap_p(D,E_0,E_1) \leq \sum_{k=1}^{q} cap_p(D,E_0^k,E_1) ,$$

as desired.

The same argument still holds for (iii′), but with $u(x)=\max[u_1(x), \ldots, u_q(x)]$.

(i′) is trivial since $u=0$ is an admissible function; arguing as (ii), we establish also (ii′).

Remark. In order to be able to obtain the subadditivity of the p-capacity in the case (iii) and (iii′) (i.e. the corresponding inequalities with $q=\infty$), we have, for instance to suppose that all u_k $(k=1,\ldots,q)$ are locally Lipschitzian with a fixed Lipschitz constant $K<\infty$, or at least that the set of all these constants is bounded (which is equivalent).

Let $q(x,y)=\dfrac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}$ be the spherical distance between x and y. If E_1,E_2 are 2 sets, then $q(E_1,E_2)=\inf\limits_{\substack{x\in E_1\\ y\in E_2}} q(x,y)$.

Proposition 5. For each $p>0$, p-almost every bounded curve is rectifiable (J. Väisälä [22], theorem 2.3).

This proposition means that if Γ_0 is the family of all bounded curves, which are not rectifiable, then $M_p\Gamma_0=0$.

Theorem 2. If D is open, E_0, E_1 are such that $q(E_0,E_1)>0$ (hence

<u>one of the sets</u> E_0, E_1 <u>is bounded) and for each component</u> D_k <u>of</u> D <u>with</u> $\overline{D}_k \cap E_i \neq \emptyset\ (i=0,1), \forall \xi \in \partial D_k \cap E_i$ (<u>where</u> E_i <u>is the bounded set), either</u> (11) <u>is satisfied or</u> ξ <u>is not accessible from</u> D_k <u>by rectifiable arcs,</u> <u>then</u> (1) <u>holds.</u>

Suppose that E_0 is bounded. As we observed in the proof of the preceding theorem, we may assume, without loss of generality, that D is a domain and $E_0 \cap \overline{D}, E_1 \cap \overline{D} \neq \emptyset$. Next, let us write $\overline{D} \cap E_0 = E' \cup E''$, where E' is the set of the points of $E_0 \cap \partial D$ inaccessible from D by rectifiable arcs. Since E_0 – and then <u>a fortiori</u> E' – is supposed to be bounded, then also $E'(r)$ will be so, where $E(r)$ is supposed to be the open set of points (of R^m), which lie within a distance r from E. Then, for $r < d(E_0, E_1)$, clearly, $\Gamma_1 = \Gamma[D \cap E'(r), E', E'(r)] < \Gamma(D, E', E_1)$, where, evident, $D \cap E'(r) \subset E'(r)$ is bounded. But then, theorem 1 of Fuglede's paper [10], combined with the preceding proposition yields

$$M_p \Gamma(D, E', E_1) \leq M_p \Gamma[D \cap E'(r), E', E'(r)] = 0$$

and

$$M_p \Gamma(D, E'', E_1) \leq M_p \Gamma(D, E_0, E_1) \leq M_p \Gamma(D, E', E_1) + M_p \Gamma(D, E'', E_1) = M_p \Gamma(D, E'', E_1),$$

hence

$$(21) \qquad M_p \Gamma(D, E_0, E_1) = M_p \Gamma(D, E'', E_1) .$$

Next, let us show that

(22) $\quad cap_p(D,E',E_1) = 0$.

First, let us denote $E(r_1,r_2)=\{x\in R^m, r_1<d(E,x)<r_2\}$ and $E(r_1,\infty)=\{x\in R^m; d(E,x)>r_1\}$. Again, on account of the preceding proposition, we have

(23) $\quad M_p\Gamma[D\cap E'(r_2),E',E'(r_1,r_2)] = 0$.

But, all the points $\xi\in\partial[D\cap E'(r_2)]\cap E'(r_1,r_2)$ verify a condition of the form (11), i.e.

$$\lim_{\substack{x\to\xi \\ x\in D\cap E'(r_2)}} \inf_{\gamma} H^1\{\gamma[E'(r_1,r_2),x]\} = 0$$

since, given $\xi\in\partial[D\cap E'(r_2)]\cap E'(r_1,r_2)\subset E'(r_1,r_2)$, every $x\in D\cap E'(r_2)$ sufficiently close to ξ will belong to $E'(r_1,r_2)$ (which is an open set) so that such an x may be joined to $E'(r_1,r_2)$ by an arc of length zero.

Thus, we are in the hypotheses of the preceding theorem, which on account of (23), yields

$$cap_p[D\cap E'(r_2),E',E'(r_1,r_2)] = M_p\Gamma[D\cap E'(r_2),E',E'(r_1,r_2)] = 0 .$$

Hence

$$cap_p[D,E',E'(r_1,\infty)]=\inf_u\int_D|\nabla u|^p dm=\inf_u\int_{D\cap E'(r_2)}|\nabla u|^p dm = \tag{24}$$
$$cap_p[D\cap E'(r_2),E',E'(r_1,r_2)] = 0 .$$

If we denote by $\mathcal{U}(D,E_0,E_1)$ the class of admissible functions for $cap_p(D,E_0,E_1)$, then, for $r_1<d(E_0,E_1)$, since $E_1\subset E'(r_1,\infty)$, evident, $\mathcal{U}[D,E_0,E'(r_1,\infty)]\subset\mathcal{U}(D,E',E_1)$, whence, taking into account (ii') of the preceding lemma and (22), we obtain that

$$cap_p(D,E',E_1) \leq cap_p[D,E',E'(r_1,\infty)] = 0 ,$$

implying (22).

And now, from the preceding lemma and arguing as above for the p-module, we deduce

$$\mathrm{cap}_p(D,E'',E_1)\leq \mathrm{cap}_p(D,E_0,E_1)\leq \mathrm{cap}_p(D,E',E_1)+\mathrm{cap}_p(D,E'',E_1)=\mathrm{cap}_p(D,E'',E_1),$$

so that

$$(25)\qquad \mathrm{cap}_p(D,E_0,E_1) = \mathrm{cap}_p(D,E'',E_1) .$$

Now, theorem 1 assure us that

$$\mathrm{cap}_p(D,E'',E_1) = M_p\Gamma(D,E'',E_1) ,$$

which together with (25) and (21), yields

$$\mathrm{cap}_p(D,E_0,E_1) = \mathrm{cap}_p(D,E'',E_1) = M_p\Gamma(D,E'',E_1) = M_p\Gamma(D,E_0,E_1) ,$$

as desired.

Corollary 1. If D is open, $E_0\cap E_1=\emptyset$, $q(D\cap E_0, D\cap E_1)>0$ and for i corresponding to the bounded set $\partial D\cap E_i$, the conditions of the preceding theorem are satisfied, then (1) holds.

Corollary 2. If D is open, $\partial D\cap E_0$ is bounded, $E_0\cap E_1=\emptyset$, $d(E_0\cap\bar{D}, E_1\cap\bar{D})>$ and $E_0\cap\partial D$ verifies the conditions of the preceding theorem, then (1) holds.

And now, in order to obtain (1) in some other hypotheses, let us remind first some concepts and preliminary results.

A domain $D\subset E^n$ is said to be m-*connected at a boundary point* $\xi\in\partial D$ if m is the least integer for which there is an arbitrary small neighbourhood U_ξ of ξ such that $U_\xi\cap D$ consists of m components.

We shall say that a domain D is m-*smooth at a boundary point* ξ if D is m-connected at ξ and there exists $\lambda_0>0$ and a neighbourhood U_ξ with the property that $U_\xi\cap D$ consists of m components $\Delta_1,\dots,\Delta_m$ and if V_ξ is any neighbourhood of ξ contained in U_ξ, there is a neighbourhood $V'_\xi\subset V_\xi$ of ξ so that $M\Gamma(V_\xi\cap\Delta_i,E_1,E_2)\geq\lambda_0$, whenever E_1,E_2 are disjoint connected sets ($i=1,\dots,m$), which meet both V_ξ and V'_ξ. If D is m-smooth at each point of a set $E\subset\partial D$, it is said to be m-*smooth on* E.

Remark. The above definition of the m-smoothness is a modified version of J. Hesse's definition, where D is not supposed to be m-connected. But then, the m-smoothness is no more a characteristic of the boundary point ξ and depends on the neighbourhood U_ξ considered. Thus, for instance, in fig. 2, ξ is 1-smooth according to our definition and 1-, 2-, 3-, and 4-smooth at the same time according to Hesse's definition, depending on the fact that the selected neighbourhood U_ξ of his

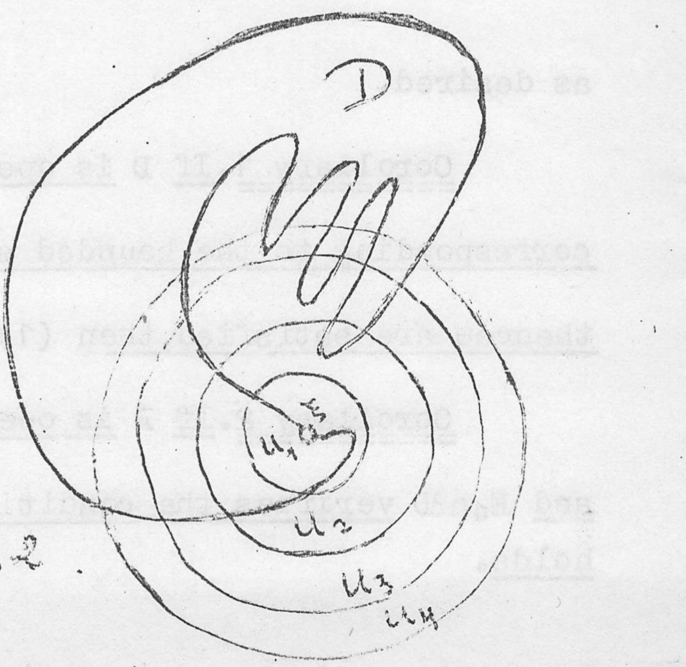

Fig. 2

definition is U_1, U_2, U_3 or U_4.

Let $E_0, E_1 \subset \overline{D}$ be 2 sets with $d(E_0,E_1)>0$. For $r \in (0,1)$, let $E_i(r)=\{x \in R^n; d(x,E_i) \leq r\}$ $(i=0,1)$. If $\rho \in F[\Gamma(D,E_0,E_1)]$, let $L(\rho,r)=\inf_{\gamma} \int_{\gamma} \rho dH^1$, where the infimum is taken over all locally rectifiable $\gamma \in \Gamma[D,E_0(r),E_1(r)]$. It is easy to verify that $0 \leq r_1 \leq r_2 \leq 1$ implies $L(\rho,r_2) \leq L(\rho,r_1)$. Then, let us define (following Hesse) $L(\rho)=\lim_{r \to 0} L(\rho,r)$.

Proposition 6. Let $D \subset \overline{R^n}$ be a domain and $\rho \in F[\Gamma(D,E_0,E_1)]$. Then, $L(\rho) \geq 1$ iff (= "if and only if") $\forall \varepsilon>0$ there exists a $\delta=\delta(\varepsilon) \in (0,1)$ such that $\frac{\rho}{1-\varepsilon} \in F\{\Gamma[D,E_0(r),E_1(r)]\}$ for all $0<r\leq\delta$ (J. Hesse [15] theorem 4.16).

Proposition 7. Let D be a domain in $\overline{R^n}$, E and F compact disjoint non-empty sets in $\overline{D}$ and suppose that at each point of $(E \cup F) \cap \partial D$, D is m-smooth for some m. Let $\Gamma=\Gamma(D,E,F)$ and $\mathcal{C}_0=\{\rho+\varepsilon\vartheta; \rho \in F(\Gamma) \cap L^n(R^n)$ and $\varepsilon \in (0,1)\}$, where

$$\vartheta(x) = \begin{cases} \dfrac{1}{|x| \log |x|} & \text{if} \quad |x| \geq e, \\ \dfrac{1}{e} & \text{if} \quad |x| \leq e. \end{cases}$$

Then $L(\rho) \geq 1$ $\forall \rho \in \mathcal{C}_0$ (J. Hesse [15], theorem 4.27).

Remark. In our opinion, the proof contains some inaccuracies. For instance, the author asserts that "since for all $i=1,2,\ldots, \int_{\gamma_i} \rho ds<2$ and since $\rho \geq \varepsilon\vartheta$ for some $\varepsilon \in (0,1)$, it follows that all the curves γ_i lie in some fixed closed euclidean ball and they are all rectifiable."

However, the preceding 2 inequalities may give

$$\frac{H^1(\gamma_i)\varepsilon}{|x_i|\log|x_i|} \leqq \theta(x_i)\varepsilon\int_{\gamma_i} ds \leqq \int_{\gamma_i} \theta\varepsilon ds \leqq \int_{\gamma_i} \rho ds < 2,$$

where $|x_i|=\sup_{x\in\gamma_i}|x|$. But, since $\gamma_i\subset\bar{D}\subset\bar{R}^n$, it is possible to have $x_i=\infty$ for all i or for some of them, or, at least, to have $\lim_{i\to\infty}|x_i|=\infty$, contradicting the boundedness and even the rectificability of the curves γ_i. Then, as a consequence of this mistake, he does not consider the case $F=\{\infty\}$, which is compatible with the hypotheses of the theorem, because his erronous conclusion that all γ_i lie in a closed ball implies the impossibility of such a case. We shall establish the preceding proposition in R^n changing slightly the hypotheses and we shall use some parts of the proof of the preceding proposition, but, for the completness sake - since (according to our knowledge) the mentioned proof is not published and it is to be found only in author's Ph.D. (1972) - we shall give a complete proof.

Let us denote first $L_1(\rho,r)=\inf_{\gamma}\int_{\gamma}\rho dH^1$, where the infimum is taken over all $\gamma\in\Gamma[D,E_0(r),E_1]$ and $L_1(\rho)=\lim_{r\to 0}L_1(\rho,r)$.

Arguing as in proposition 6, we have

<u>Lemma 7. If</u> $D\subset\bar{R}^n$ <u>is a domain and</u> $\rho\in F[\Gamma(D,E_0,E_1)]$, <u>then</u> $L_1(\rho)\geqq 1$ <u>iff</u> $\forall\varepsilon\in(0,1)$ <u>there exists a</u> $\delta(\varepsilon)\in(0,1)$ <u>such that</u> $\frac{\rho}{1-\varepsilon}\in F\{\Gamma[D,E_0(r),E_1]\}$ <u>for all</u> $0<r\leqq\delta(\varepsilon)$, <u>where</u> $E_0(r)=\{x\in\bar{R}^n; d(x,E_0)<r\}$.

<u>Proposition 8. Let</u> $A=\{x; r_1<|x|<r_2\}$ <u>and</u> $E_1, E_2\subset A$ <u>be</u> 2 <u>disjoint sets so that each sphere</u> $S(r)$ $(r_1<r<r_2)$ <u>contain at least one point of each</u> E_i $(i=1,$

2). Then,

$$M\Gamma[A-(E_1\cup E_2),E_1,E_2] \geq \frac{2^n}{A_0}\log\frac{r_2}{r_1},$$

where A_0 is a constant depending only on the dimension n (J.Väisälä [22] theorem 3.9 or our monograph [5] proposition 11, chap.5, part I).

Lemma 9. Let D be a domain, $E_0,E_1\subset\overline{D}\subset R^n$ with $d(E_0,E_1)>0$, E_0 compact and suppose that D is m-smooth on $E_0\cap\partial D$ for some m. Then $L_1(\rho)\geq 1$ $\forall\rho\in\mathcal{O}_0'=\{\rho\in F(\Gamma)\cap L^n(R^n);\rho(x)\geq\alpha_F>0\ \forall F\subset R^n \text{ compact}\}$.

Assume first that $E_0=\{\xi\}$. Then, suppose, to prove it is false, that $L_1(\rho)<1$. Fix and sequence $\{\eta_i\}$, $\eta_i\in(0,1)$ $(i=1,2,\ldots)$ so that $\sum_{i=1}^{\infty}\eta_i<\infty$. Since $\rho\in L^n$, the absolute continuity of the integral allows us to choose a strictly decreasing sequence $r_i\in(0,1)$ such that $\lim_{i\to\infty} r_i=0$ and

$$(26)\qquad \int_{B(\xi,r_i)}\rho^n dm < \eta_i^n\lambda_0 \quad (i = 1,2,\ldots).$$

We may assume also that $E_0(r_i)\cap E_1=\emptyset$. We observe that $E_0(r_i)=B(\xi,r_i)$ $(i=1, 2,\ldots)$. Choose a sequence of locally rectifiable arcs $\gamma_i\in\Gamma[D,E_0(r_i),E_1]$ so that

$$\int_{\gamma_i}\rho dH^1 < L_1(\rho,r_i)+\eta_i \quad (i = 1,2,\ldots).$$

Since, $\forall i\in N$, we have

$$\int_{\overline{\gamma}_i} \rho dH^1 = \int_{\gamma_i} \rho dH^1 < L_1(\rho, r_i)+\eta_i \leqq L_1(\rho)+\eta_i < 1+1 = 2$$

and since ρ is supposed to be bounded away from zero on compact sets and the closure $\overline{\gamma}_i$ of γ_i (obtained by adding to γ_i its endpoints) is compact, we obtain $0<\alpha_{\underline{\gamma_i}} H^1(\overline{\gamma}_i) \leqq \int_{\overline{\gamma}_i} \rho dH^1 < 2$, hence $H^1(\overline{\gamma}_i) < \frac{2}{\alpha_{\underline{\gamma_i}}} < \infty$, i.e. γ_i is rectifiable. Since one of the endpoints lies in $B(\xi, r_i)$, we can decompose γ_i into $\gamma_i = \alpha_i \cdot \alpha_i' \cdot \chi_i$, where $\alpha_i \in \Gamma[B(\xi, r_{i-1}), B(\xi, r_i), B(\xi, r_{i-1})]$, $\alpha_i' \in \Gamma[B(\xi, r_{i-2}), S(\xi, r_{i-1}), S(\xi, r_{i-2})]$, $\chi_i \in \Gamma[D, S(\xi, r_{i-2}), E_1]$. If ξ_i is the endpoint of γ_i contained in $F(r_i)$, then, clearly, $\lim_{i\to\infty} \xi_i = \xi$. Assume $\xi \in \partial D$. Since D is supposed to be m-smooth at ξ for some m, let U_ξ be the neighbourhood of ξ, involved in the definition of the smoothness. Then, it follows that $U_\xi \cap D = \bigcup_{k=1}^{m} \Delta_k$. We may assume (eventually choosing a subsequence) that all ξ_i are contained in one of the Δ_k, let us denote it by Δ. The m-smoothness of D at ξ implies the existence of a constant $\lambda_e > 0$ such that

$$(27) \qquad M\{\Gamma[B(\xi, r_{i-1}) \cap \Delta; \overline{\alpha_i'}, \overline{\alpha_{i-1}}]\} \geqq \lambda_e (i = 1,2,\ldots) .$$

If $\xi \in D$, then, the preceding inequality still holds on account of the preceding proposition (eventually choosing a subsequence of $\{\gamma_i\}$). Next, since (25) yields

$$(28) \qquad \int_{B(\xi, r_{i-2})} \left(\frac{\rho}{\eta_{i-2}}\right)^n dm < \lambda_e ,$$

it follows from (27) that there is a rectifiable arc β_i in $B(\xi, r_{i-2})$ connecting α_i' and α_{i-1} so that

$$(28)\quad \int_{\beta_i} \rho dH^1 < \eta_{i-2} ,$$

since otherwise, $\frac{\rho}{\eta_{i-2}} \in [B(\xi, r_{i-2}), \alpha_i', \alpha_{i-1}]$ and (28) would contradict (27). The arc $\alpha_i \cdot \alpha_i'$ contains a subarc α_i'' joining an endpoint of β_{i+1} with an endpoint of β_i $(i=3,4,\ldots)$. But then,

$$\int_{\alpha_i''} \rho dH^1 + L_1(\rho, r_{i-2}) \leqq \int_{\alpha_i \cdot \alpha_i'} \rho \; dH^1 + \int_{\chi_i} \rho dH^1 = \int_{\gamma_i} \rho dH^1 < L_1(\rho, r_i) + \eta_i ,$$

hence

$$(30)\quad \int_{\alpha_i''} \rho dH^1 \leqq L_1(\rho, r_i) - L_1(\rho, r_{i-2}) + \eta_i \quad (i = 3,4,\ldots) .$$

Define a locally rectifiable arc

$$\sigma_k = \ldots(\alpha_{i+1}'' \cdot \beta_{i+1}) \cdot (\alpha_i'' \cdot \beta_i) \ldots (\alpha_k'' \cdot \beta_k) \cdot \tau_{k-1} ,$$

where τ_{k-1} is a subarc of γ_{k-1} joining an endpoint of β_k with E_1. We have that $\sigma_k \in \Gamma(D, E_0, E_1)$ and

$$1 \leqq \int_{\sigma_k} \rho dH^1 \leqq \sum_{i=k}^{\infty} \int_{\alpha_i''} \rho dH^1 + \sum_{i=k}^{\infty} \int_{\beta_i} \rho dH^1 + \int_{\tau_{k-1}} \rho dH^1 \quad (k = 3,4,\ldots) ,$$

where

$$\int_{\tau_{k-1}} \rho dH^1 \leq \int_{\gamma_{k-1}} \rho dH^1 \leq L_1(\rho, r_{k-1})+\eta_{k-1} \quad (k = 3,4,\ldots) .$$

Hence, taking into account (28) and (30), we get

$$1 \leq \int_{\sigma_k} \rho dH^1 \leq \sum_{i=k}^{\infty} [L_1(\rho, r_i)-L_1(\rho, r_{i-2})]+ \sum_{i=k}^{\infty} \eta_i+ \sum_{i=k}^{\infty} \eta_{i-2}+L_1(\rho, r_{k-1})+\eta_{k-1} ,$$

whence

$$1 \leq \int_{\sigma_k} \rho dH^1 \leq L_1(\rho)+[L_1(\rho)-L_1(\rho, r_{k-2})]+2 \sum_{i=k-2}^{\infty} \eta_i \quad (k = 3,4,\ldots) .$$

For large k, the last part of the preceding inequality is strictly less than 1, which is absurde.

Now, let us consider the general case E_0 compact in R^n (and then bounded). Let $\rho \in \mathcal{O}_0'$ and suppose again, to prove it is false, that $L_1(\rho)<1$. By lemma 7, there exists an $\varepsilon \in (0,1)$, a strictly decreasing sequence $\{r_i\}$, $r_i \in (0,1) (i=1,2,\ldots)$ with $\lim_{i\to\infty} r_i = 0$ and a sequence of locally rectifiable arcs $\gamma_i \in \Gamma[D, E_0(r_i), E_1]$ so that

(31) $\int_{\gamma_i} \rho dH^1 < 1-\varepsilon \quad (i = 1,2,\ldots)$.

Hence and since ρ is bounded away from zero on compact sets, we deduce (arguing as above) that all γ_i are rectifiable. Next, since each endpoint ξ_i of γ_i lies in $E_0(r_i)$ and E_0 is compact, by considering a subsequence, we may assume that $\lim_{i\to\infty} \xi_i = \xi_0 \in E_0$ and, by the same argument as in the first part of the proof and taking into account (31), we obtain again a contradiction, as desired.

Proposition 9. *Let Γ be any curve family in $\overline{R^n}$, let $p\in(1,\infty)$ and assume $M_p\Gamma<\infty$. Let $\mathcal{A}_0''=\{\rho\in F(\Gamma);\rho$ is bounded away from zero on compact sets and $\rho\in L^p\}$. Then $\mathcal{A}_0''$ is a complete family for $M_p\Gamma$* (J. Hesse [15], lemma 4.40).

Lemma 9. *Let $D\subset R^n$ be a domain, $\Gamma=\Gamma(D,E_0,E_1)$, where $d(E_0,E_1)>0$, $M_p\Gamma<\infty$, $p\,(1,\infty)$ and $\mathcal{A}_p^{\Delta}$ is a family of admissible functions $\rho\in F(\Gamma)\cap L^p$ bounded in R^n, continuous in $\Delta=D-(\overline{E}_0\cup\overline{E}_1)$, bounded away from zero on compact sets $F\subset\overline{\Delta}$ and 0 in $C\overline{\Delta}$. Then $\mathcal{A}_p^{\Delta}$ is a complete family for $M_p\Gamma$.*

Indeed, if $\rho\in F(\Gamma)$ is supposed only to be bounded in R^n, continuous in Δ and 0 in $C\overline{\Delta}$, then, on account of the corollary of lemma 5, the corresponding subfamily of $F(\Gamma)$ is complete (that is yields the same value for $M_p\Gamma$); next, since $M_p\Gamma<\infty$, suppose $\rho\in L^p$ and, arguing as in the preceding proposition, $\forall\varepsilon>0$, let us consider $\tilde{\rho}=\rho+\varepsilon\phi$, where $\phi(x)=\dfrac{1}{1+|x|^{\frac{n+1}{p}}}$. Clearly, $\tilde{\rho}$ is bounded away from zero on compact sets in R^n and then a fortiori in $\overline{\Delta}$. By Minkowski inequality,

$$(\int\tilde{\rho}^p dm)^{\frac{1}{p}}=[\int(\rho+\varepsilon\phi)^p dm]^{\frac{1}{p}}\le(\int\rho^p dm)^{\frac{1}{p}}+\varepsilon(\int\phi^p dm)^{\frac{1}{p}}=(\int\rho^p dm)^{\frac{1}{p}}+\varepsilon(n\omega_n)^{\frac{1}{p}}[\int_0^{\infty}\frac{r^{n-1}dr}{(1+r^{\frac{n+1}{p}})^p}]^{\frac{1}{p}}$$

$$=(\int\rho^p dm)^{\frac{1}{p}}+\varepsilon(n\omega_n)^{\frac{1}{p}}[\int_0^1\frac{r^{n-1}dr}{(1+r^{\frac{n+1}{p}})^p}+\int_1^{\infty}\frac{r^{n-1}dr}{(1+r^{\frac{n+1}{p}})^p}]^{\frac{1}{p}}\le(\int\rho^p dm)^{\frac{1}{p}}+\varepsilon(n\omega_n)^{\frac{1}{p}}(\int_0^1 dr+$$

$$\int_1^{\infty}\frac{r^{n-1}dr}{r^{n+1}})^{\frac{1}{p}}=(\int\rho^p dm)^{\frac{1}{p}}+\varepsilon(n\omega_n)^{\frac{1}{p}}2^{\frac{1}{p}}<\infty,$$

where $n\omega_n$ is the area of the unit sphere in R^n.Hence $\tilde{\rho}\in L^p$ and then $\tilde{\rho}\in\mathcal{Q}_0^{\prime\prime}$.If $M=\inf_{\mathcal{Q}_0^{\prime\prime}}\int\tilde{\rho}^p dm$, then

$$M^{\frac{1}{p}}\leq\left(\int\tilde{\rho}^p dm\right)^{\frac{1}{p}}=\left[\int(\rho+\varepsilon\psi)^p dm\right]^{\frac{1}{p}}\leq\left(\int\rho^p dm\right)^{\frac{1}{p}}+\varepsilon\left(\int\psi^p dm\right)^{\frac{1}{p}},$$

where,from above,$\psi\in L^p$,and letting $\varepsilon\to 0$,we obtain $M\leq\int\rho^p dm$ and ρ being an arbitrary admissible function for $\hat{M}_p^{\circ\Delta}\Gamma=M_p\Gamma$,taking the infimum over all such admissible ρ,we are allowed to conclude,taking into account also the corollary of lemma 5,that $M\leq M_p^{\circ\Delta}\Gamma=M_p\Gamma$ and,since,evident,$M\geq M_p\Gamma$, also that $M=M_p\Gamma$,as desired.

Finally,let us consider the function

$$\rho_0(x)=\begin{cases}\tilde{\rho}(x) & \text{for } x\in\overline{\Delta},\\ 0 & \text{otherwise.}\end{cases}$$

Clearly,

$$\int_\gamma\rho_0(x)dH^1=\int_\gamma\tilde{\rho}(x)dH^1\geq 1,$$

since each open arc $\gamma\in\Gamma(\Delta,E_0,E_1)$ is contained in $\overline{\Delta}$.

Lemma 10.If a domain D is m-smooth at a point $\xi\in\partial D$,then,for $p\in[n,\infty)$,there exists $\lambda_p>0$ and a neighbourhood U_ξ with the property that $U_\xi\cap D$ consists of m components $\Delta_1,\ldots,\Delta_m$ and if V_ξ is any neighbourhood of ξ contained in U_ξ,there is a neighbourhood $V_\xi^{\prime}\subset V_\xi$ such that $M_p\Gamma(V_\xi\cap\Delta_i,E_1,E_2)\geq\lambda_p$,where E_1,E_2 are 2 disjoint connected sets in

Δ_i, <u>which meet both</u> ∂V_ξ <u>and</u> $\partial V'_\xi$.

Let U_ξ be the neighbourhood involved in the definition of the m-smoothness of D at ξ and suppose that $V_\xi \subset U_\xi$. If $M_p\Gamma(V_\xi \cap \Delta_i, E_1, E_2)=\infty$, then <u>a fortiori</u> $M_p\Gamma(V_\xi \cap \Delta_i, E_1, E_2) \geq \lambda_p$ for any $\lambda_p>0$. If $M_p\Gamma(V_\xi \cap \Delta_i, E_1, E_2)<\infty$, then, arguing as in the preceding lemma, ρ may be assumed to be bounded away from 0 on compact sets of $\bar{U}_\xi$ and 0 on $C\bar{U}_\xi$. For such a ρ, since $\bar{U}_\xi$ is compact, we have

$$\int_{\bar{U}_\xi \cap \Delta_i} \rho^p dm = \int_{E'_i} \rho^p dm + \int_{E''_i} \rho^p dm \geq \alpha^p_{\bar{U}_\xi} \int_{E'_i} dm + \int_{E''_i} \rho^n dm \geq \alpha^p_{\bar{U}_\xi} \int_{E'_i} \rho^n dm + \int_{E''_i} \rho^n dm \geq \tag{31}$$

$$\alpha^p_{\bar{U}_\xi} \int_{\bar{U}_\xi} \int_{\Delta_i} \rho^n dm \geq \alpha^p_{\bar{U}_\xi} M\Gamma(V_\xi \cap \Delta_i, E_1, E_2) \geq \alpha^p_{\bar{U}_\xi} \lambda_0 ,$$

where $E'_i=\{x \in U_\xi \cap \Delta_i ; \rho(x) \leq 1\}$, $E''_i=\{x \in U_\xi \cap \Delta_i ; \rho(x)>1\}$ and $\alpha_{\bar{U}_\xi}<1$; if $\alpha_{\bar{U}_\xi} \geq 1$, then $\alpha^p_{\bar{U}_\xi} \lambda_0 \geq \lambda_0$. Taking the infimum in (31) over all admissible ρ, we obtain that

$$M_p\Gamma(V_\xi \cap \Delta_i, E_1, E_2) \geq \alpha^p_{\bar{U}_\xi} \lambda_0 ,$$

so that, we may denote $\lambda_p=\alpha^p_{\bar{U}_\xi} \lambda_0$.

Arguing as in lemma 8, on account of the preceding lemma, we deduce

<u>Lemma 11</u>. <u>In the hypotheses of lemma</u> 8, $L_1(\rho) \geq 1$ $\forall \rho \in \mathcal{O}_p=\{\rho \in F(\Gamma) \cap L^p ; \rho$ <u>is bounded away from zero on compact sets in</u> $R^n\}$, <u>where</u> $p \in [n,\infty)$.

<u>Lemma 12</u>. <u>In the hypotheses of lemma</u> 8, <u>if</u> $p \in (1,\infty)$ <u>and</u> E_0 <u>is compact</u>.

then, $L_1(\rho)\geq 1\ \forall\rho\in\mathcal{A}_p^{\Delta}$.

We have only to show that each $\rho\in\mathcal{A}_p^{\Delta}$ belongs to $L^n(R^n)$. Indeed, let $E'=\{x\in\overline{\Delta};\rho(x)\leq 1\}$, $E''=\{x\in\overline{\Delta};\rho(x)>1\}$. Then, if $p\in(1,n)$, from the condition $\rho\in L^p$, it follows that

$$\infty>\int\rho^p dm=\int_{\overline{\Delta}}\rho^p dm=\int_{E'}\rho^p\, dm+\int_{E''}\rho^p dm\geq mE'',$$

hence

$$\int\rho^n dm=\int_{\overline{\Delta}}\rho^n dm=\int_{E'}\rho^n dm+\int_{E''}\rho^n dm\leq\int_{E'}\rho^p dm+M^n mE''<\infty,$$

where $M=\sup_{x\in R^n}\rho(x)$.

Finally, if $p\geq n$, the conclusion of the lemma follows on account of the preceding lemma.

<u>Proposition 10</u>. <u>Suppose that</u> D <u>is an open set, that</u> E_0, E_1 <u>are disjoint bounded continua in</u> D <u>and that</u> $\Gamma=\Gamma(D,E_0,E_1)$, $\Gamma_r=\Gamma[D,E_0(r),E_1(r)]$. <u>Then</u> $M\Gamma=\lim_{r\to 0}M\Gamma_r$ (F. Gehring and J. Väisälä [13], lemma 3.4).

Arguing as in the preceding proposition and taking into account the preceding lemma combined with lemma 7, we obtain

<u>Lemma 13</u>. <u>Suppose that</u> E_1 <u>is a set</u>, E_0 <u>is compact so that</u> $d(E_0,E_1)>0$, D <u>is a domain</u> m<u>-smooth on</u> $E_0\cap\partial D$ <u>for some</u> m, $\Gamma=\Gamma(D,E_0,E_1)$ <u>and</u> $\Gamma_r'=\Gamma[D,E_0(r),E_1]$. <u>Then</u>

(33) $$M_p\Gamma = \lim_{r\to 0} M_p\Gamma'_r \; .$$

Proposition 11. Let $E_1 \supset E_2 \supset \dots$ and $F_1 \supset F_2 \supset \dots$ be disjoint sequences of nonempty compact sets in the closure of a domain $D \subset \overline{R^n}$. Let $E = \bigcap_{m=1}^{\infty} E_m$, $F = \bigcap_{m=1}^{\infty} F_m$. Then

(34) $$\lim_{m\to\infty} \mathrm{cap}_p(D, E_m, F_m) = \mathrm{cap}_p(D, E, F) \; .$$

(For the proof, see J. Hesse [16], theorem 3.3)

Arguing as in the preceding proposition, we obtain

Lemma 14. Let $E_1 \supset E_2 \supset \dots$ and $F_1 \supset F_2 \supset \dots$ be 2 sequences of sets in the closure of a domain $D \subset R^n$ such that $E = \bigcap_{k=1}^{\infty} E_k \neq \emptyset$, $F = \bigcap_{k=1}^{\infty} F_k \neq \emptyset$ and $d(E_1, F_1) > 0$. Then (34) holds.

Theorem 3. If D is open, E_0, E_1 are 2 sets such that $d(E_0, E_1) > 0$ and each component D_k of D with $\overline{D}_k \cap E_i \neq \emptyset$ $(i=0,1)$, for $i=0$ or $i=1$, is m-smooth on the set $\partial D_k \cap E_i$, where $\overline{D}_k \cap E_i$ is compact, then (1) holds.

As we observed in the proof of the preceding theorem, we may suppose - without loss of generality - that D is a domain m-smooth on $\partial D \cap E_0$ for some m. Next, let us denote $C_0 = E_0 \cap \overline{D}$ and consider the sequence $C_0(r_1) \supset C_0(r_2) \supset \dots$, where $C_0(r)$ is the open set of points within a distance r of C_0, $\lim_{k\to\infty} r_k = 0$, $\bigcap_{k=1}^{\infty} C_0(r_k) = C_0$ and $d[E_1, C_0(r_k)] > 0$. Clearly, $\forall \xi \in [C_0(r_k) \cap \overline{D}]$,

$$\lim_{\substack{x\to\xi \\ x\in D}} \inf_{\gamma} H^1\{\gamma[C_0(r_k), x]\} = 0 \qquad (k = 1, 2, \dots) \; ,$$

where the infimum is taken over all the arcs $\gamma[C_0(r_k),x]$ joining $C_0(r_k)$ and x in D, since if $\xi\in C_0(r_k)\cap\overline{D}$, any $x\in D$ sufficiently close to ξ belongs to $C_0(r_k)$ and then, may be joined to $C_0(r_k)$ by an arc of length zero yielding

$$\inf_{\gamma} H^1\{\gamma[C_0(r_k),x]\} = 0 \qquad (k = 1,2,\dots) .$$

But then, we are in the hypotheses of theorem 1, allowing us to conclude that

$$M_p\Gamma[D,C_0(r_k),E_1] = \mathrm{cap}_p[D,C_0(r_k),E_1] \qquad (k = 1,2,\dots) ,$$

hence and taking into account the preceding 2 lemmas,

$$M_p\Gamma(D,E_0\cap\overline{D},E_1)=\lim_{k\to\infty} M_p\Gamma[D,C_0(r_k),E_1]=\lim_{k\to\infty}\mathrm{cap}_p[D,C_0(r_k),E_1]=\mathrm{cap}_p(D,E_0\cap\overline{D},E_1)$$

Finally, extending all the admissible functions for $\mathrm{cap}_p(D,E_0\cap\overline{D},E_1)$ to be 0 on $E_0-\overline{D}$ and arguing as in lemma 5, we obtain that

$$M_p\Gamma(D,E_0\cap\overline{D},E_1) = M_p\Gamma(D,E_0,E_1) ,$$

whence we deduce that

$$M_p\Gamma(D,E_0,E_1) = M_p\Gamma(D,E_0\cap\overline{D},E_1) = \mathrm{cap}_p(D,E_0\cap\overline{D},E_1) = \mathrm{cap}_p(D,E_0,E_1) ,$$

as desired.

Lemma 15. If D is a domain and E_i $(i=0,1)$ are 2 sets such that for at least one of them (say for E_0), we have $E_0 \cap \bar{D} = E' \cup E''$, where $\forall \xi \in E'$,

$$\liminf_{\substack{x \to \xi \\ x \in D}} \inf_{\gamma} H^1[\gamma(E', x)] = 0$$

and D is m-smooth on $E'' \cap \partial D$ for some m, where $E'' \cap \partial D$ is compact, then

$$(35)\; M_p\Gamma(D, E_0, E_1) = \lim_{k\to\infty} M_p\Gamma[D, E' \cup E''(r_k), E_1] .$$

Indeed, on account of lemma 8, if $L_1'(\rho, r_k) = \inf_{\gamma} \int \rho dH^1$, where the infimum is taken over all $\gamma \in \Gamma[D, E''(r_k), E_1]$ then $L_1'(\rho) = \lim_{k\to\infty} L_1'(\rho, r_k) \geqq 1$. Clearly, if $L_1(\rho, r_k) = \inf_{\gamma} \int_{\gamma} \rho dH^1$, where the infimum is taken over all $\gamma \in \Gamma[D, E' \cup E''(r_k), E_1] = \Gamma[D, E''(r_k), E_1] \cup \Gamma(D, E', E_1)$, then $L_1(\rho) = \lim_{k\to\infty} L_1(\rho, r_k) \geqq 1$, since all the additional arcs $\gamma \in \Gamma(D, E', E_1)$ with respect to which the infimum in $L_1(\rho, r_k)$ is taken satisfy the condition $\int_{\gamma} \rho dH^1 \geqq 1$, so that $\lim_{k\to\infty} L_1'(\rho, r_k) \geqq 1$ implies $L_1(\rho, r_k) \geqq 1$, and arguing as in lemma 13, we obtain (35), as desired.

Theorem 4. If D is open, $d(E_0, E_1) > 0$, $E_0 \cap \bar{D}$ is bounded and for every component D_k of D with $\bar{D}_k \cap E_i \neq \emptyset$ $(i=0,1)$, the set $\partial D_k \cap E_0$ may be written as $\partial D_k \cap E_0 = E' \cup E'' \cup E'''$, where E', E'' are as in the preceding lemma (with $D = D_k$) and all the points of E''' are not accessible from D_k by rectifiable arcs, then (1) holds.

This theorem is a consequence of the preceding lemma and theorem 3.

And now, in order to generalize some of the above results in $\overline{R^n}$, let us define the p-capacity in this case.

The p-capacity of 2 sets $E_0, E_1 \subset \overline{R^n}$ *with* $q(E_0,E_1)>0$ *relative to a domain* $D\subset\overline{R^n}$ is

$$\operatorname{cap}_p(D,E_0,E_1)=\inf_u\int_D|\nabla u|^p dm \ ,$$

where the infimum is taken over all u which are continuous in $D\cup E_0\cup E_1$, locally lipschitzian in $D-\{\infty\}$ and $u|_{E_0}=0, u|_{E_1}=1$.

Arguing as for the p-capacity in R^n, we have

Lemma 6'. In the hypotheses of lemma 6, the p-capacity in $\overline{R^n}$ *satisfies the conditions* (i) – (iii) and (i') – (iii').

Theorem 1'. If $D\subset\overline{R^n}$ *is open,* $E_0,E_1\subset\overline{R^n}$ *such that* $q(E_0,E_1)>0$, E_0 *is bounded and* $\forall D_k$ *of* D *with* $\overline{D}_k\cap E_i\neq\emptyset$ $(i=0,1)$, $\forall\xi\in\partial D_k\cap E_0$, (11) *is verified, where this time the linear Hausdorff measure* H^1 *is defined by means of the spherical distance* $q(x,y)$, *then* (1) *holds.*

Corollary 1. If χ *is the set of all continua in* $\overline{R^n}$ *that meet* E_0, E_1, *where* $q(E_0,E_1)>0$, *then*

$$M_p\chi=\operatorname{cap}_p(\overline{R^n},E_0,E_1) \ .$$

Corollary 2. In the hypotheses of the preceding theorem, (18) *holds.*

Theorem 2'. If $D\subset\overline{R^n}$ *is a domain,* $q(E_0,E_1)>0$, E_0 *is bounded and* $E_0\cap\partial D=E'\cup E''$, *where* $\forall\xi\in E'$, (11) *is verified, while every point of* E'' *is* inaccessible from D by rectifiable arcs, then (1) holds.

Now, let us remind 3 definitions of a topological cylinder: 2 of them with respect to the euclidean metric and the third with respect to the relative metric.

I. A triple (Z,B_0,B_1), where $Z\subset R^n$ is a domain and $B_0,B_1\subset\partial Z$ is called a topological cylinder with respect to the euclidean metric if there exists a homeomorphis $\varphi:\overline{Z_0}\rightleftarrows\overline{Z}$ such that $\varphi(B_k^0)=B_k\,(k=0,1)$, where $Z_0=\{x;(x^1)^2+\ldots+(x^{n-1})^2<1;0<x^n<1\}$ is the unit cylinder and $B_k^0\,(k=0,1)$ its bases. B_0,B_1 are the bases of the topological cylinder.

II. A triple (Z,B_0,B_1) (as above) is a topological cylinder with respect to the euclidean metric if there is a homeomorphism $\phi:Z_0\cup B_0^0\cup B_1^0\rightleftarrows Z\cup B_0\cup B_1$ such that $\phi(B_k^0)=B_k$ $(k=0,1)$.

III. A triple (Z,B_0,B_1) is said to be a topological cylinder with respect to the relative metric if there exists a bijection $\phi:Z_0\cup B_0^0\cup B_1^0\rightleftarrows Z\cup B_0\cup B_1$ so that, given $\varepsilon>0$ and a point $x_0\in Z_0\cup B_0^0\cup B_1^0$, there is a $\delta=\delta(\varepsilon,x_0)>0$ such that $x\in Z_0$ with $|x-x_0|<\delta$ imply $d_Z[\phi(x_0),\phi(x)]<\varepsilon$.

Remarks. 1. Clearly, the bijection ϕ of the preceding definition is a homeomorphism (with respect to the euclidean metric) of $Z_0\cup B_0^0\cup B_1^0$ onto $Z\cup B_0\cup B_1$; hence a topological cylinder with respect to the relative metric is also a topological cylinder with respect to the euclidean metric according to definition II, but not, in general, according to definition I.

2. All the points of the bases B_0,B_1 of a topological cylinder with respect to the relative metric are accessible from Z by rectifiable arcs.

The p-module M_pZ of a topological cylinder (Z,B_0,B_1)(according to the definitions I,II,III) is given by $M_pZ=M_p\Gamma_Z$, where $\Gamma_Z=\Gamma(Z,B_0,B_1)$.

In our note [9],we established

<u>Proposition 12</u>.<u>If</u> $Z=(Z,B_0,B_1)$ <u>is a topological cylider with repspect to the relative metric,then</u> (2) <u>holds</u>.

In the proof of this proposition,we established that $\forall\xi\in B_0, \lim_{\substack{x\to\xi\\ x\in Z}} u(x)=0$. It seems to us that it would be suitable to prove this assertion more in detail.In order to do it,it is enough to show that $\lim_{k\to\infty} u(x_k)=0$ for any sequence $\{x_k\}$ with $x_k\in Z$ and $x_k\to\xi\in B_0$ for $k\to\infty$.Indeed,let $y_k=\phi^{-1}(x_k)$ $(k=1,2,\ldots)$ and $\eta=\phi^{-1}(\xi)\in B_0$, where ϕ is the homeomorphism involved in the definition of Z.Since $\phi: Z_0\cup B_0^0\cup B_1^0 \rightleftarrows Z\cup B_0\cup B_1$ is a homeomorphism,from $x_k\to\xi$, it follows that $y_k\to\eta$, hence $d(y_k,\eta)\to 0$ as $k\to\infty$, and since ϕ is continuous with respect to the relative metric in $Z_0\cup B_0^0\cup B_1^0$ and then,in particular,at $\eta\in B_0^0$, it follows that $d_Z(x_k,\xi)\to 0$ as $k\to\infty$, i.e. in other words $\liminf_{k\to\infty}\limits_{\gamma} H^1[\gamma(x_k,\xi)]=\lim_{k\to\infty} d_Z(x_k,\xi)=0$, hence

$$\lim_{k\to\infty}\inf_{\gamma} H^1[\gamma(x_k,B_0)] \leqq \lim_{k\to\infty}\inf_{\gamma} H^1[\gamma(x_k,\xi)] = 0 ,$$

and since this relation holds for any sequence $\{x_k\}$ with $x_k\in Z$ and $x_k\to\xi$ we have that condition (11) is satisfied for any $\xi\in B_0$ implying (1) in this case, that is (2).

And now,we establish that,in the particular case n=2,relation (2) is true also for a topological cylinder with respect to the euclidean metric, i.e. for topological quadrilaterals.

<u>Theorem 5. If</u> $Z\subset R^2$ <u>is a topological quadrilateral with respect to the euclidean metric (according to definitions</u> I <u>and</u> II), <u>then</u> (2) <u>holds.</u>

Let us show first that $\forall\varepsilon>0$ there exists an $r>0$ such that $\forall$ $\rho\in F^{\circ Z}(\Gamma)$, we have $\frac{\rho}{1-\varepsilon}\in F^{\circ Z}(\Gamma_r')$, where $\Gamma=\Gamma_Z$ and $\Gamma_r'=\Gamma[Z,B_0(r),B_1]$. Let $\dot{B}_0$ be the basis B_0 without its endpoints, $\xi_0\in\dot{B}_0$ and $\gamma_0\in\Gamma$ an arc joining ξ_0 and B_1. $\partial Z-\dot{B}_0$ is a closed set; let $r<d(\xi_0,\partial Z-\dot{B}_0)$. The circumference $C(\xi_0,r)$ is disjoint of $\partial Z-\dot{B}_0$ and meets $\dot{B}_0$ in at least 2 points, hence, there is a circular subarc γ_r of $C(\xi_0,r)$ of length $l<\pi r$ joining the first point x_0 of the intersection $\gamma_0\cap C(\xi_0,r)$ (taken along γ_0 from B_1 toward B_0) and $\dot{B}_0$ in Z. Since $\rho\leqq M$ in R^2, it follows that $\int_{\gamma_r}\rho ds\leqq M\pi r<\varepsilon$ for $r<\frac{\varepsilon}{M\pi}$. But then, $\gamma_r''=\gamma_r\cup\gamma_r'$ – where γ_r' is the subarc of γ_0 joining B_1 and x_0 – belongs to Γ so that

$$1\leqq\int_{\gamma_r''}\rho dH^1\leqq\int_{\gamma_r}\rho ds+\int_{\gamma_r'}\rho dH^1<\int_{\gamma_r'}\rho dH^1+\varepsilon\ ,$$

whence

$$(36)\ \int_{\gamma_r'}\rho dH^1>1-\varepsilon\ .$$

Since any arc joining B_0 and $C(\xi_0,r)$ may be considered as a subarc of an arc joining B_1 and ξ_0, and then belonging to Γ_r', if we consider a closed subarc $\tilde{B}_0\subset\dot{B}_0$, we may take $r_0=\min[d(\tilde{B}_0,\partial Z-\dot{B}_0)\frac{\varepsilon}{4M},d(B_0,B_1)]$ and $\frac{\rho}{1-\varepsilon}$ will be admissible for all the arcs of Γ'_{r_0} joining $\tilde{B}_0(r_0)$ and B_1.

Now, let us consider a point $\xi\in B_0-\tilde{B}_0$ and the circumference $C(\xi,r_0)$. If $r_0\leq d(\xi,B_0-\dot{B}_0)$ with $C(\xi,r_0)\cap(\partial Z-B_0)=\emptyset$, then, arguing as above, any arc joining B_1 and $C(\xi,r_0)$ will satisfy (35) with $r=r_0$. Finally, if $r_0\geq d(\xi,\partial Z-B_0)$, then, one of the subarcs of $C(\xi,r_0)$ contained in Z will join $\partial Z-(B_0\cup B_1)$ and B_0, belonging to the boundary of a simply connected subdomain of Z, whose boundary contains also ξ. But then, all the arcs γ_0 joining B_1 and ξ will cross this arc. Arguing as above, we obtain (35) also in this case, and then, $\frac{\rho}{1-\varepsilon}\in F^{\circ Z}(\Gamma'_{r_0})$, so that, arguing as in proposition 10, we get (33).

Next, since $\Gamma[Z,B_0(r),B_1]$ satisfies relation (11), it follows, on account of theorem 1, that

$$M_p\Gamma[Z,B_0(r),B_1] = cap_p[Z,B_0(r),B_1] \quad \forall r < r_0 .$$

In particular, the preceding relation is verified for a sequence $\{r_k\}$ with $r_k\to 0$ as $k\to\infty$. But then, from (33) and lemma 14, we deduce that

$$M\Gamma_p(Z,B_0,B_1) = \lim_{k\to\infty} M_p\Gamma[Z,B_0(r_k),B_1] = \lim_{k\to\infty} cap_p[Z,B_0(r_k),B_1]=cap_p(Z,B_0,B_1)$$

as desired.

As a consequence of theorem 4, we have

Theorem 6. If $Z\subset R^n$ is a topological cylinder with respect to the euclidean metric (definition I or II) and ∂B_0 or ∂B_1 may be written as the union $E'\cup E''\cup E'''$ (with the same meaning as in theorem 4), then (2) holds.

Remark . .For the first time, J.Hersch [14] established (1) for the harmonic capacity of a ring in R^3 and (2) for the harmonic capacity of a domain $D \subset R^3$ homeomorphic to a ball and with 2 distinguished continua $B_0, B_1 \subset \partial D$, where he defines the harmonic capacity by the relation

$$cap(D,B_0,B_1) = \inf_u \int_{D-(B_0 \cup B_1)} \left|\frac{\partial u}{\partial \nu}\right|^n dm ,$$

where $\frac{\partial u}{\partial \nu}$ is the normal derivative of the admissible function u. B.Fuglede [10] established something similar to (1) in the particular case p=2 for the classical harmonic capacity $cap_2(D,K,\infty)$, where Γ is the family of the arcs joining, in the unbounded domain D, the point at infinity of R^n with the compact set K. F.Gehring [12] obtained (1) for the conformal capacity of a ring (according to C.Loewner's definition by means of Dirichlet integral and used also in this paper). Bagby [2] showed that $cap(\overline{R^n},C_0,C_1)=M\Gamma(\overline{R^n},C_0,C_1)$, where C_0,C_1 are disjoint compact sets. W.Ziemer indicated in [29] how to verify (1) in the case p=n if $D \subset R^n$ is a bounded domain and $C_0,C_1 \subset D$ are disjoint closed sets and asserted that this result is also valid for $C_0,C_1 \subset \overline{D}$ if certain conditions are imposed on the tangential behavior of $\partial D \cap (C_0 \cup C_1)$; in [24], he established $cap_p(R^n,C_0,C_1)=M_p\chi(R^n,C_0,C_1)$, where C_0 contains the complement of a ball and χ is the family of all continua joining C_0 and C_1, and in [25], the equality between the p-module ($1 \leq p < n$) of the family of all continua that join a Suslin set E to the point at infinity and the p-capacity of E, where the infimum involved in its definition is taken over all u ACL in $R^n - E$, $u_{|E} = 1$ and with compact support; in the case $p \geq n$, the support of each u is required to lie in some fixed ball containing E and the corresponding family of continua is supposed to join E to the complement of that ball. V.V.Krivov [17] and V.V.Aseev [1] tried to prove (1) for p=n and H.M.Reimann [19] (2) for p>1, but their proofs contain some inaccuracies (for some comments about this, see our papers [7,8]). Also A.V.Syčev's [21] proof for (1) - where p=n=3, D is a ring and E_0, $E_1 \subset \partial D$ are 2 simply connected sets of the 2 boundary components of D, respectively - is not correct; he claimed that given $\rho \in F(\Gamma)$, then, "according to Gehring [12], the function $u(x) = \min(1, \inf_\beta \int_\beta \rho ds)$ - where β is an arbitrary curve joining x and E_0 - is admissible for $cap(D, E_0, E_1)$", but Syčev did not observe that his hypotheses are different. Indeed, F.Gehring [12] considered the more particular case in which E_0, E_1 are not only subsets of the boundary components of the ring D, but coincide to them so that any rectifiable arc β joining E_0 and E_1 - no matter if $\beta \subset D$ or not - satisfies the condition $\int_\beta \rho ds \geq 1$ $\forall \rho \in F(\Gamma)$ since each such β contains a subarc $\beta_0 \subset D$ and joining its boundary components; however, it is easy to see that this is no more true in general if E_0 or E_1 or both of them are only subsets of the corresponding boundary component, and then, we are no more sure that $u_{|E_1} = 1$; nevertheless, this inaccuracy may be easily corrected if we suppose additionally that the arcs involved in Syčev's definition of u have to be contained in D. But, there is also something else: Syčev uses in his definition of u the expression $\inf_\beta \int_\beta \rho ds$, where $\rho \in F(\Gamma)$ is assumed to bounded only on compact subsets of D and satisfies the condition $\lim_{x\to\xi} \rho(x) < \infty\ \forall \xi \in \partial D-(E_0 \cup E_1)$, instead of the expression $\inf_\beta \int_\beta g ds$ (used by Gehring in his definition), where $g(x)=\frac{1}{mB}\int_B \rho(x+y)dm(y)$ is bounded and continuous in R^n, so that it is easy to see that, in the case considered by Syčev, it is possible not to have $u_{|E_0}=0$. This second mistake is of the same kind as in the papers of V.V.Krivov [17], H.M.Reimann [19] and V.V.Aseev [1] (for more detailed comments, see our papers [7,9]). In our paper [7], we established (2) for topological cylinders with respect to the relative metric if $\rho \in F(\Gamma)$ satisfies the additional condition $\int \rho dH^1 < \infty$ for every rectifiable $\gamma \in \Gamma_Z$, but, according to proposition 3 of this paper (established in [9]), it follows that the value of $M_p Z$ is not influenced by this condition. Finally, let us mention the extension of the equality between the p-module and the p-capacity in $\overline{R^n}$ considered by J.Hesse ([16], theorem 5.5) in the case $E_0, E_1 \subset D$ are compact, disjoint, non-empty sets. In his Ph.D. [15], he proves also that $M_p\Gamma(\overline{R^n}, E_0, E_1)=\mathrm{cap}_p(\overline{R^n}, E_0, E_1)$, where $E_0, E_1 \subset \overline{R^n}$ are suppposed to be disjoint, compact and non-empty. He asserted also that (18) holds if $D \subset \overline{R^n}$, and $E_0, E_1 \subset \overline{D}$ are disjoint, compact, non-empty sets and G is M m-smooth on $(E_0 \cup E_1) \cap \partial D$ for some m; however, this result is based on his theorem 4.27 (quoted in this paper as proposition 7), which is not correct (see our comment of proposition 7).

Now, let us mention that we established (18) in [8], but the corresponding proof contains a mistake. Indeed, we had to establish the relation

$$(37) \qquad \lim_{p\to\infty} \sup_q \int_{\gamma_p} \rho_j^q dH^1 = \sup_q \lim_{p\to\infty} \int_{\gamma_p} \rho_j^q dH^1 ,$$

where

$$\rho_j^q(x) = \begin{cases} q & \text{if} \quad \rho(x) \geq q , \\ \rho(x) & \text{if} \quad j^{-1} < \rho(x) < q , \\ j^{-1} & \text{if} \quad \rho(x) \leq j^{-1} \end{cases}$$

is the truncation of $\rho \in F(\Gamma)$. In order to do it, we used the following minimax theorem: "<u>If</u> $\{f_p\}$ <u>is a non-increasing sequence of real-valued upper semicontinuous functions on a compact set</u> A, <u>then</u>

$$(37) \quad \lim_{p\to\infty} \max_A f_p(x) = \max_A \lim_{p\to\infty} f_p(x)".$$

(For the proof, see V. Barbu and T. Precupanu [3], chap. 2, theorem 3.4, p. 141.) But, first, we had to make some changes in order that the hypotheses of the preceding minimax theorem be satisfied. Thus, we denoted $x_q = \frac{1}{q}$ (q=1,2,...), $A=\{0\}\cup\{x_1, x_2, \ldots\}$ and

$$f_p(x) = \begin{cases} \int_{\gamma_p} \rho_j^q dH^1 & \text{if} \quad x = x_q \\ \int_{\gamma_p} \rho_j^q dH^1 & \text{if} \quad x = 0 \end{cases} \qquad (p = 1, 2, \ldots) .$$

If, for an infinity of indices q, the sequences $\{f_p(x_q)\}$ are non-decreasing, then, it is easy to prove directly that (37) holds. If, for an infinity of indices q, $\{f_p(x_q)\}$ are non-increasing, then, on account of the preceding minimax theorem, (37) holds, which is correct. However, we deduced in [8] that this relation implies (37), which is wrong. Indeed, it is easy to see that

$$\sup_q \liminf_{p\to\infty} \int_{\gamma_p} \rho_1^q dH^1 \leq \lim_{p\to\infty} \sup_q \int_{\gamma_p} \rho_1^q dH^1 .$$

Next, since $\forall p \in N$, the numerical sequences $\{f_p(x_q)\}$ are non-decreasing, we have $f_p(0) = \max_A f_p(x) = \sup_q f_p(x_q) = \sup_q \int_{\gamma_p} \rho_1^q dH^1$, hence

$$\lim_{p\to\infty} \sup_q \int_{\gamma_p} \rho_1^q dH^1 = \lim_{p\to\infty} f_p(0) = \lim_{p\to\infty} \max_A f_p(x) .$$

On the other side, the numerical sequences $\{\liminf_{p\to\infty} f_p(x_q)\}$ are non-decreasing too, so that $\liminf_{p\to\infty} f_p(x_1) \leq \liminf_{p\to\infty} f_p(x_2) \leq \ldots \leq \liminf_{p\to\infty} f_p(0) = \max_A \liminf_{p\to\infty} f_p(x)$, but, we are not allowed to conclude that

$$\lim_{q\to\infty} \lim_{p\to\infty} f_p(x_q) = \sup_q \lim_{p\to\infty} f_p(x_q) = \lim_{p\to\infty} f_p(0) = \max_A \lim_{p\to\infty} f_p(x) .$$

It is easy to construct conterexamples showing that, in general, the implication is wrong. Thus, for instance, if

$$f_p(x_q) = \begin{cases} 1 & \text{if} \quad p \leq q , \\ 0 & \text{if} \quad p > q , \end{cases}$$

then $\forall q\in N, \lim_{p\to\infty} f_p(x_q)=0$, hence $\sup_q \lim_{p\to\infty} f_p(x_q)=0$, while $\forall p\in N, \sup_q f_p(x_q)=1$, hence $\lim_{p\to\infty} \sup_q f_p(x_q)=1$.

Finally, let us give the application (mentioned at the beginning of this paper) of theorem 1 in the theory of quasiconformal mappings. But first, we remind some concepts and preliminary results.

A homeomorphism $f:D\rightleftarrows D^*$ is said to be K-quasiconformal ($1\leq K<\infty$) if

$$\frac{M\Gamma}{K} \leq M\Gamma^* \leq KM\Gamma$$

$\forall\Gamma$ of D, where $\Gamma^*=f(\Gamma)$.

Let $f:D\rightleftarrows D^*$ be a quasiconformal mapping (i.e. a K-quasiconformal mapping with non-specified K) and E° the exceptional set of the points of S such that the image $f(\gamma)$ of any endcut γ of B from an arbitrary point $\xi\in E^\circ$ is unrectifiable (we remind that an endcut γ of B from $\xi\in S$ is an open arc $\gamma\subset B$ with an endpoint at ξ and the other one at a point of B).

Proposition 13. If Γ_0 is the family of the arcs of R^n with an endpoint belonging to E°, then $M\Gamma_0=0$ (our paper [6], lemma 3).

Now, let us prove that the conformal capacity of E° is zero.

Theorem 7. $\mathrm{cap}E^\circ=0$.

Clearly,

$$\mathrm{cap}[R^n, CE^\circ(r), E^\circ] \geq \mathrm{cap}E^\circ \geq \mathrm{cap}[R^n, CB(R), E^\circ] , \tag{38}$$

where B(R) is a fixed ball sufficiently large containing $E^\circ(r)$, since the class of admissible functions for $\mathrm{cap}[R^n, CE^\circ(r), E^\circ]$ is contained in that of $\mathrm{cap}E^\circ$. Next, let $\Gamma_r = \Gamma[R^n, E^\circ, CE^\circ(r)]$ and Γ_0 the arc family of the preceding proposition, then, evident, $\Gamma_r < \Gamma_0$ and the preceding proposition implies $M\Gamma_r \leq M\Gamma_0 = 0$ $\forall r > 0$, hence, by (38) and taking into account corollary 2 of theorem 1, we obtain

$$\mathrm{cap}E^\circ \leq \mathrm{cap}[R^n, CE^\circ(r), E^\circ] = M\Gamma_r \leq M\Gamma_0 = 0 ,$$

as desired.

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