INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

CONTINUOUS DEPENDENCE ON DATA IN ABSTRACT CONTROL PROBLEMS

by

Constantin ZALINESCU

PREPRINT SERIES IN MATHEMATICS
Nr.109/1981

BUCURESTI

CONTINUOUS DEPENDENCE ON DATA IN ABSTRACT CONTROL PROBLEMS

by

Constantin ZALINESCU

November, 1981

*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii 220, 79622 Bucharest, Romania

CONTINUOUS DEPENDENCE ON DATA IN ABSTRACT CONTROL PROBLEMS

Constantin Zălinescu

Faculty of Mathematics, University of Iaşi

6600 - Iaşi, Romania

Key words : perturbations of abstract control problems, convergence in the sense of Mosco, uniformly convex function, duality mappings, G-convergence.

1. Introduction. Zolezzi [1] was interested in the following problem : How to perturbe the coefficients of a plant in order to the corresponding solutions converge to the solution of the initial problem. He solved this problem and gave many characterizations((A) - (K) in our paper). Some of this characterizations are given in terms of convergence of sets and functions in the sense of Mosco, or G-convergence. After that Lucchetti and Mignanego [2] and Bennati [3] were interested in generalizing the results of Zollezzi. So, Lucchetti and Mignanego obtained that (C) $\Longleftrightarrow$ (D) for strongly smooth E-spaces and $F(x,y) = (\|x\|^p + \|y\|^p)/p$, $p > 1$ (instead of $p = 2$ as in [1]), while Bennati showed some equivalences from [1] remain valid, in Hilbert spaces, for a larger class of functions. In this paper it is shown that all the equivalences established by Zolezzi remain valid for more general spaces and functions. Our results do not cover the ones in [2] (see also Remark 4), but show that the essentially new implications proved by Bennati are valid in a more general context. We also give a characterization of the convergence of sets in the sense of Mosco which

is just Theorem 1 in [4] for Hilbert spaces and also generalizes the one of Sonntag [5].

2. Notations, Definitions, Preliminary Results.

Throughout this paper X, Y denote real Banach spaces, X^*, Y^* are their topological duals, L(X,Y) is the Banach space of continuous linear operators from X into Y; if $L \in L(X,Y)$, L^* denotes the adjoint of L and gph L denotes the graph of L, i.e., gph L $= \{(x, Lx) \in X \times Y : x \in X\}$. If $x^* \in X^*$ and $x \in X$ then $\langle x, x^* \rangle$ denotes $x^*(x)$. $\rightarrow$ and $\rightharpoonup$ denote the strong and weak convergence, respectively. We shall also use the following notations : $S(x,r) = \{y \in X : \|y-x\| < r\}$, $\bar{S}(x,r) = \{y \in X : \|y-x\| \le r\}$; $S_X = \{x \in X : \|x\| = 1\}$. The Banach space X is strictly convex iff $\forall x, y \in S_X$, $x \neq y$: $\|x+y\| < 2$, X is locally uniformly convex iff $\forall x \in S_X$ $\forall \varepsilon > 0$ $\exists \delta > 0$ $\forall y \in S_X$, $\|y-x\| \geq \varepsilon$: $\|y+x\| \le 2(1-\delta)$, X is uniformly convex iff $\forall \varepsilon > 0$ $\exists \delta > 0$ $\forall x, y \in S_X$, $\|y-x\| \ge \varepsilon$: $\|y+x\| \le 2(1-\delta)$, X is smooth if $\forall x \in S_X$ $\exists x^* \in S_{X^*}$ unique : $\langle x, x^* \rangle = 1$ ($\Leftrightarrow$ the norm of X is Gateaux differentiable on $X \sim \{0\}$), and X is strongly smooth iff the norm of X is Fréchet differentiable on $X \sim \{0\}$, where $M \sim N = \{x \in M : x \notin N\}$. X has property (h) if $x_n \rightharpoonup x$, $\|x_n\| \rightarrow \|x\|$ then $x_n \rightarrow x$, and X is an E-space if X is reflexive, strictly convex and has property (h). R, R_+, $\bar{R}_+$ denote the reals, the nonnegative reals and $R_+ \cup \{\infty\}$ respectively, while N and N^* denote the nonnegative and positive integers, respectively.

Let $f : X \rightarrow R \cup \{\infty\}$. The domain of f is dom f $= \{x \in X : f(x) < \infty\}$, the epigraph of f is the set epi f $= \{(x, \alpha) \in X \times R : f(x) \le \alpha\}$; f is proper if dom f $\neq \phi$ (ϕ - the empty set). The proper function f is convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \quad \forall x, y \in \text{dom } f,\ x \neq y,\ \forall \lambda \in]0,1[,$$

and f is <u>strictly convex</u> if in the above inequality $\leq$ is replaced by $<$. The <u>subdifferential</u> of the convex function f at $\bar{x} \in \mathrm{dom}\, f$ is the set

$$\partial f(\bar{x}) = \{x^* \in X^* : \langle x-\bar{x}, x^* \rangle \leq f(x) - f(\bar{x})\ \forall x \in X\},$$

while for $\bar{x} \notin \mathrm{dom}\, f$, $\partial f(\bar{x}) = \phi$. It is known (see [6]) that $\partial f(\bar{x})$ is a w^*-closed convex subset of X^* which is nonempty and w^*-compact if f is continuous and finite at $\bar{x}$. The <u>conjugate</u> of the proper function $f : X \to R \cup \{\infty\}$ is

$$f^* : X^* \to R \cup \{\infty\},\ f^*(x^*) = \sup\{\langle x, x^* \rangle - f(x) : x \in X\}.$$

If $f : X \to R \cup \{\infty\}$ is a proper function then $f^{**} = f$ iff f is a lower semicontinuous (l.s.c.) convex function. The <u>indicator</u> of the set $U \subset X$ is the function $I_U : X \to R \cup \{\infty\}$, $I_U(x) = 0$ if $x \in U$ and $I_U(x) = \infty$ if $x \notin U$. I_U is l.s.c. and convex iff U is closed and convex.

Let $\varphi : R_+ \to \bar{R}_+$; φ is <u>non decreasing</u> if $t_1, t_2 \in R_+$, $t_1 \leq t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2)$ and φ is <u>increasing</u> if $t_1, t_2 \in \mathrm{dom}\,\varphi$, $t_1 < t_2 \Rightarrow \varphi(t_1) < \varphi(t_2)$. For a nondecreasing function $\varphi : R_+ \to R_+$ we put $\varphi_+(t) = \lim_{\tau \downarrow t} \varphi(\tau)$, $\varphi_-(t) = \lim_{\tau \uparrow t} \varphi(\tau)$ and $\varphi_-(o) = 0$. Let

$$\Delta = \{\delta : R_+ \to R_+ : \delta \text{ is nondecreasing, } \delta(t) = 0 \Rightarrow t = 0\}.$$

Note that if $\delta \in \Delta$ and $\delta(t_k) \to 0$ then $t_k \to 0$.

Let now $f : X \to R \cup \{\infty\}$ be a proper l.s.c. convex function; f is <u>uniformly convex at</u> $\bar{x} \in \mathrm{dom}\, f$ if there exists $\delta \in \Delta$ such that

$$f\left(\frac{x+\bar{x}}{2}\right) \leq \frac{1}{2} f(x) + \frac{1}{2} f(\bar{x}) - \delta(\|x-\bar{x}\|),\ \forall x \in \mathrm{dom}\, f,$$

f is <u>uniformly convex</u> if there exists $\delta \in \Delta$ such that

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2} f(x) + \frac{1}{2} f(y) - \delta(\|x-y\|)\quad \forall x, y \in \mathrm{dom}\, f,$$

and f is <u>uniformly convex on the convex set</u> $U \subset X$ if $f + I_U$ is uniformly convex.

Let $\varphi : R_+ \to \overline{R}_+$ a nondecreasing function such that

$$a = \sup\{x : \varphi(x) < \infty\} > 0. \qquad (1)$$

We associate to φ the following mappings :

$$\psi : R_+ \to \overline{R}_+ , \ \psi(x) = \int_0^x \varphi(t)\,dt, \qquad (2)$$

$$f : X \to \overline{R}_+ , \quad f(x) = \psi(\|x\|), \qquad (3)$$

$$\Phi : X \to 2^{X^*}, \ \Phi(x) = \{ x^* \in X^* : \langle x, x^* \rangle = \|x\| \, \|x^*\|, \ \varphi_-(\|x\|) \le \|x^*\| \le \varphi_+(\|x\|) \}. \qquad (4)$$

It is known (see [7]) that ψ is a l.s.c. convex function, ψ is increasing iff $\psi(t) = 0 \Leftrightarrow t = 0$ ($\Leftrightarrow \varphi(t) = 0 \Rightarrow t = 0$); therefore f is l.s.c. convex function. Φ is the <u>duality mapping</u> associated to φ . The conjugate of ψ is $\psi^* : R_+ \to \overline{R}_+$,
$\psi^*(x) = \sup\{ tx - \psi(t) : t \in R_+ \}$.

Concerning the above mappings we have the following results we shall need in the sequel.

<u>Theorem A</u>. Let X be a Banach space and φ, ψ, f, Φ as above. Then :

(i) the right hand side derivative of ψ^* is given by
$\psi^{*\prime}_+(t) = \max\{\tau \ge 0 : \varphi_-(\tau) \le t\}$;

(ii) $[\psi(t) = 0 \Leftrightarrow t = 0] \Leftrightarrow [\varphi(t) = 0 \Rightarrow t = 0] \Leftrightarrow \psi^{*\prime}_+(0) = 0$;

(iii) $\lim_{t\to\infty} \frac{\psi(t)}{t} = \infty \Leftrightarrow \lim_{t\to\infty} \varphi(t) = \infty \Leftrightarrow \psi^{*\prime}_+(t) < \infty \ \forall t \in R_+ \Leftrightarrow \operatorname{dom}\psi^* = R_+$;

(iv) φ is increasing $\Leftrightarrow \psi^{*\prime}_+$ is continuous ;

(v) $\partial f(x) = \Phi(x) \ \forall x \in X$;

(vi) $f^*(x^*) = \psi^*(\|x^*\|) \ \forall x^* \in X^*$;

(vii) f is Gateaux differentiable on int(dom f) = S(o,a) $\Leftrightarrow$ Φ is single-valued on S(o,a) $\Leftrightarrow$

φ is continuous on $[o,a[$ and X is smooth $\Leftarrow\Rightarrow$ (if X is reflexive)

φ is continuous on $[o,a[$ and X^* is strictly convex ;

If X is reflexive and one of the above conditions is verified then

$$x_n,\ x\in S(o,a),\quad x_n \to x \Rightarrow \phi(x_n) \to \phi(x),$$

so that, if X^* is an E-space then ϕ is continuous on $S(o,a)$.

(viii) f is strictly convex $\Leftarrow\Rightarrow$

φ is strictly convex and X is strictly convex $\Leftarrow\Rightarrow$

φ is increasing and X is strictly convex;

(ix) ϕ is onto $\Leftarrow\Rightarrow$ $\lim\limits_{t\to\infty} \varphi(t) = \infty$ and X is reflexive ;

(x) f is uniformly convex at any $x\in S(o,a) \Leftarrow\Rightarrow$

φ is increasing and X is locally uniformly convex;

(xi) f is uniformly convex on $S(o,M)$, $\forall M \in\ ]o,a[\ \Leftarrow\Rightarrow$

φ is increasing and X is uniformly convex;

(xii) suppose $\varphi : R_+ \to R_+$ is increasing with $\lim\limits_{t\to\infty} \varphi(t) = \infty$; then

ϕ^{-1} is single-valued and uniformly continuous on bounded sets $\Leftarrow\Rightarrow$

X is uniformly convex.

(xiii) Let $g : X \to R\cup\{\infty\}$ be uniformly convex on the convex set $U\subset X$; then there exists $\delta\in\Delta$ such that

$$f(y) \geqslant f(x) + \langle y-x, x^*\rangle + \delta(\|y-x\|)\ \forall x,y\in U,\ x^*\in\partial f(x).$$

For the proof of this theorem see [8].

Consider now X,Y Banach spaces, $L\in L(X,Y)$, $F : X \times Y \to R\cup\{\infty\}$ a l.s.c. convex functions and the following optimization problems :

$$\text{(P)}\qquad \min_{x\in X} F(x, L\,x),$$

$$\text{(D)}\qquad \max_{y^*\in Y^*} -F^*(L^*y^*, -y^*).$$

called the primal and dual problem, respectively. Denote the value of problem (P) by inf P, and, when it is attained, by min P. Analogously for (D).

<u>Theorem B</u>. Suppose for some $x_o \in X$, $(x_o, L x_o) \in \mathrm{dom}\, F$ and $F(x_o, .)$ is continuous at $L x_o$. Then

$$\inf P = \max D. \qquad (5)$$

Moreover, $\bar{x}$ is a solution for (P) and $\bar{y}^*$ is a solution for (D) iff

$$(L^* \bar{y}^*, -\bar{y}^*) \in \partial F(\bar{x}, L\bar{x}). \qquad (6)$$

In the case $F(x,y) = f(x) + g(y)$, f,g being l.s.c. convex functions (therefore $F^*(x^*,y^*) = f^*(x^*) + g^*(y^*)$), if there exists $x_o \in \mathrm{dom}\, f$ such that g is finite and continuous at $L x_o$, then (5) is valid and $\bar{x}$ is solution for (P) and $\bar{y}^*$ is solution for (D) iff

$$L^* \bar{y}^* \in \partial f(\bar{x}) \text{ and } -\bar{y}^* \in \partial g(L\bar{x}). \qquad (7)$$

For the proof see [9] , even in a more general setting.

We remember that $f(\bar{x}) \leq f(x)\ \forall x \in X \iff 0 \in \partial f(\bar{x})$.

Let now $S_n \subset X$, $n \in N$; we say that $(S_n)_{n \in N^*}$ <u>converges in the sense of Mosco</u> at S_o, written $S_n \xrightarrow{M} S_o$, if

$$\forall x \in S_o \quad \forall n \in N^* \quad \exists x_n \in S_n : x_n \to x, \qquad (8)$$

and

$$x_{n_k} \in S_{n_k},\ x_{n_k} \to x \implies x \in S_o. \qquad (9)$$

If $f_n : X \longrightarrow R \cup \{\infty\}$, $n \in N$, we say that $(f_n)_{n \in N^*}$ <u>converges in the sense of Mosco</u> at f_o, written $f_n \xrightarrow{M} f_o$, if $\mathrm{epi}\, f_n \xrightarrow{M} \mathrm{epi}\, f_o$. This one turns to be equivalent to the followings :

$$\forall x \in \mathrm{dom}\, f_o \quad \forall n \in N^* \quad \exists x_n \in \mathrm{dom}\, f_n : f_n(x_n) \to f_o(x), \qquad (10)$$

and

$$x_{n_k} \in \operatorname{dom} f_{n_k},\ x_{n_k} \rightharpoonup x \Rightarrow \liminf_{k\to\infty} f_{n_k}(x_{n_k}) \geq f_o(x). \quad (11)$$

We say that $(f_n)_{n\in N^*}$ is G-convergent at f_o, written $f_n \xrightarrow{G} f_o$, if

$$\inf(f_n + \langle .,x^*\rangle) \to \inf(f_o + \langle .,x^*\rangle) \quad \forall x^* \in X^*. \quad (12)$$

For these definitions see [1].

3. Results. Throughout this section $(L_n)_{n\in N} \subset L(X,Y)$ is a sequence satisfying

$$\|L_n\| \leq C \quad \forall n \in N, \quad (13)$$

$\varphi_1 : R_+ \to \overline{R}_+$ is an increasing function such that $\varphi_{1-}(a_1) = \infty$, where $a_1 = \sup(\operatorname{dom}\varphi_1)$, and $\varphi_2 : R_+ \to R_+$ is a nondecreasing function. In establishing the results we shall also use other assumptions on $X, Y, \varphi_1, \varphi_2$, which will be made later. To φ_1 and φ_2 we associate $\psi_1, \psi_2, f_1 : X \to \overline{R}_+$ and $f_2 : Y \to R_+$, $\Phi_1 : X \to 2^{X^*}$ and $\Phi_2 : Y \to 2^{Y^*}$, defined by (2), (3), (4) respectively. Let define for every $(\hat{x},\hat{y}) \in X \times Y$

$$F_{\hat{x},\hat{y}} = F : X \times Y \to \overline{R}_+,\ F(x,y) = f_1(x-\hat{x}) + f_2(y-\hat{y}), \quad (14)$$

and

$$I_{n,\hat{x},\hat{y}} = I_n : X \to \overline{R}_+,\ I_n(x) = F(x, L_n x). \quad (15)$$

For the pair $(\hat{x},\hat{y})$ we consider the problems

$$(P_n) \quad \min_{x\in X} I_n(x);$$

$$(D_n) \quad \max_{y^*\in Y^*} -J_n(y^*),$$

where

$$J_n(y^*) = F^*(L_n^* y^*, -y^*) = f_1^*(L_n^* y^*) + f_2^*(-y^*) + \langle L_n\hat{x}-\hat{y}, y^*\rangle. \quad (16)$$

Suppose X is strictly convex and reflexive; then (P_n) has an unique solution $\overline{x}_n = \overline{x}_n(\hat{x},\hat{y})$ which is characterized by

$$0 \in \Phi_1(\overline{x}_n - \hat{x}) + L_n^* \Phi_2(L_n\overline{x}_n - \hat{y}). \quad (17)$$

(For this one take into account that $\overline{x}_n$ is solution iff $0 \in \partial I_n(\overline{x}_n)$ and the rules for the calculus of subdifferentials in [9] . From Theorem B we know that (D_n) has at least one solution. If Y^* is strictly convex and φ_2 is continuous (so that ψ_2^* is strictly convex) then (D_n) has an unique solution $\overline{y}_n^* = \overline{y}_n^*(\hat{x},\hat{y})$ given by

$$\overline{y}_n^* = -\Phi_2(L_n\overline{x}_n - \hat{y}). \qquad (18)$$

With the above notations consider the following possible conditions upon $(L_n)_{n\in N}$ and $(I_n)_{n\in N}$.

(A) $\text{gph}\, L_n \xrightarrow{M} \text{gph}\, L_o$;

(B) $L_n x \longrightarrow L_o x \quad \forall x \in X$ and $x_n \rightharpoonup x \Rightarrow L_n x_n \rightharpoonup L_o x$;

(C) $L_n x \to L_o x \quad \forall x \in X$ and $L_n^* y^* \to L_o^* y^* \quad \forall y^* \in Y^*$;

(D) $\overline{x}_n \to \overline{x}_o \quad \forall(\hat{x},\hat{y})$;

(E) $I_n \xrightarrow{M} I_o \quad \forall(\hat{x},\hat{y})$;

(F) $I_n(x_n) - \min I_n \to 0 \Rightarrow x_n \to \overline{x}_o \quad \forall(\hat{x},\hat{y})$;

(G) $F(x_n,y_n) - \min I_n \to 0,\ y_n - L_n x_n \to 0 \Rightarrow x_n \to \overline{x}_o \quad \forall(\hat{x},\hat{y})$;

(H) $I_n \xrightarrow{G} I_o$ and $I_n(x) \longrightarrow I_o(x) \quad \forall x \in X,\ \forall(\hat{x},\hat{y})$;

(I) $I_n \xrightarrow{G} I_o$ and $I_n(\overline{x}_o) \to I_o(\overline{x}_o) \quad \forall(\hat{x},\hat{y})$;

(J) $\liminf \min I_n \geq \min I_o,\ \limsup I_n(\overline{x}_o) \leq I_o(\overline{x}_o),\ \forall(\hat{x},\hat{y})$;

(K) $\overline{x}_n \rightharpoonup \overline{x}_o,\ L_n\overline{x}_n \rightharpoonup L_o\overline{x}_o$ and $\limsup \min I_n \leq$
$\leq \min I_o,\ \forall(\hat{x},\hat{y})$;

(L) $\overline{y}_n^* \to \overline{y}_o^*,\ \forall(\hat{x},\hat{y})$;

(M) $I_n \xrightarrow{G} I_o,\ L_n x \to L_o x \quad \forall x \in X,\ \forall(\hat{x},\hat{y})$;

(N) $\min I_n \to \min I_o,\ L_n x \to L_o x \quad \forall x \in X, \forall(\hat{x},\hat{y})$.

For $(\hat{x},\hat{y})$ fixed we denote by (D') the condition $\overline{x}_n(\hat{x},\hat{y}) \longrightarrow \overline{x}_o(\hat{x},\hat{y})$. Analogously for conditions (E') - (N'),

$(\hat{x}, \hat{y})$ being the same in all these conditions.

Before seeing the verious implications between conditions (A) - (N), let us give a characterization of the convergence of sets in the sense of Mosco.

Theorem 1. Let $S_n \subset X$, $n \in N$ be closed convex sets and suppose X and X^* are E-spaces. Then the following conditions are equivalent :

(i) $S_n \xrightarrow{M} S_o$,

(ii) $p_n(x) \longrightarrow p_o(x) \quad \forall x \in X$,

(iii) $\mathrm{dist}(x, S_n) \longrightarrow \mathrm{dist}(x, S_o) \; \forall x \in X$,

where $p_n(x) \in S_n$ is the element of best approximation for x by elements of S_n (which exists and is unique in our case) and $\mathrm{dist}(x, S_n) = \|x - p_n(x)\|$.

Proof. Consider Φ the duality mapping corresponding to $\varphi(t) = t$. By Theorem A, in the present conditions, Φ and Φ^{-1} are single-valued and continuous. Then for $x \in X$, $p_n(x)$ is characterized by

$$\langle p_n(x) - y, \Phi(x - p_n(x))\rangle \geqslant 0 \quad \forall y \in S_n. \tag{19}$$

(ii) $\Rightarrow$ (i). Take $x \in S_o$; then $S_n \ni p_n(x) \to p_o(x) = x$. Therefore (8) is verified. Let now $S_{n_k} \ni x_{n_k} \rightharpoonup x_o$. Then, from (19),

$$\langle p_{n_k}(x_o) - x_{n_k}, \Phi(x_o - p_{n_k}(x_o))\rangle \geqslant 0, \quad \forall k.$$

But $x_o - p_{n_k}(x_o) \to x_o - p_o(x_o)$, so that $\Phi(x_o - p_{n_k}(x_o)) \to$ $\to \Phi(x_o - p_o(x_o))$ and $p_{n_k}(x_o) - x_{n_k} \rightharpoonup p_o(x_o) - x_o$. Therefore

$$\langle p_o(x_o) - x_o, \Phi(x_o - p_o(x_o))\rangle = -\|x_o - p_o(x_o)\|^2 \geqslant 0 ,$$

which shows that $x_o = p_o(x_o) \in S_o$ and (9) is verified. (Note we used only that X is strictly convex and X^* is an E-space).

(i) $\Rightarrow$ (iii). Let $x \in X$ and take $x_n \in S_n$, $\|x_n - x\| = \mathrm{dist}(x, S_n)$, $n \in N$. From (8) there exist $y_n \in S_n$, $n \in N$ such that $y_n \to x_o$. Therefore

$$\mathrm{dist}(x,S_o) = \lim\|y_n - x\| \geq \limsup\|x_n - x\| = \limsup \mathrm{dist}(x, S_n).$$

Take $\|x_{n_k} - x\| \to A$; we can suppose $x_{n_k} \rightharpoonup y \in S_o$. Then $x_{n_k} - x \rightharpoonup y - x$, so that

$$A = \lim \|x_{n_k} - x\| \geq \|y - x\| \geq \mathrm{dist}(x, S_o).$$

Therefore $\mathrm{dist}(x, S_n) \to \mathrm{dist}(x, S_o)$ $\forall x \in X$.

(Note we used only the reflexivity of X).

(iii) $\Rightarrow$ (ii). a) $x \in S_o$: then $S_n \ni p_n(x) \to x = p_o(x)$. Therefore $\forall x \in S_o$ $\forall n \in N^*$ $\exists x_n \in S_n : x_n \to x$.

b) Take $x \notin S_o$. For $p_o(x) \in S_o$ there exist $x_n \in S_n$, $n \in N^*$, such that $x_n \to p_o(x)$. Then, taking into account (19), we have

$$\langle x - x_n, \phi(x - p_n(x))\rangle \geq \|x - p_n(x)\|^2 = (\mathrm{dist}(x,S_n))^2, \ \forall n \in N^*.$$

Take $\phi(x - p_{n_k}(x)) \rightharpoonup x^*$. Then, taking the limit in the above inequalities, we get

$$\langle x - p_o(x), x^*\rangle \geq \|x - p_o(x)\|^2, \qquad (20)$$

so that $\|x^*\| \geq \|x - p_o(x)\|$. On the other hand $\|x^*\| \leq \liminf \|\phi(x - p_{n_k}(x))\| = \liminf \|x - p_{n_k}(x)\| = \|x - p_o(x)\|$, so that $\|x^*\| = \|x - p_o(x)\|$. Hence, in (2o) we have just equality. Therefore $x^* = \phi(x - p_o(x))$ and $\phi(x - p_n(x)) \rightharpoonup \phi(x - p_o(x))$. Since $\|\phi(x - p_n(x))\| \to \|\phi(x - p_o(x))\|$ we obtain that $\phi(x - p_n(x)) \to \phi(x - p_o(x))$. But ϕ^{-1} is continuous, so that $x - p_n(x) \to x - p_o(x)$, i.e., $p_n(x) \to p_o(x)$ $\forall x \in X$.

Remark 1. If X is a Hilbert space, Theorem 1 is just

[4 , Theorem 1] . For X and X^* uniformly convex the equivalence of (i) and (ii) is proved in [5] . R. Lucchetti communicated me that Zolezzi had shown the implication (iii) $\Rightarrow$ (ii) if X (and therefore X^*) is a strongly smooth E-space.

Let us study the various implications between (A) - (N). Firstly we have.

<u>Remark 2.</u> It is obvious that we always have : (M') $\Rightarrow$(N') $\Rightarrow$(J') ; (M') $\Rightarrow$ (H') $\Rightarrow$ (I') $\Rightarrow$ (J'); (F') $\Rightarrow$ (D') and (G') $\Rightarrow$ (D').

<u>Theorem 2.</u> Suppose X,Y are reflexive and X^* has property (h). Then (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C).

<u>Proof.</u> (A) $\Rightarrow$ (B). Let $x \in X$; $(x, L_o x) \in \text{gph } L_o$ so that there exists $(x_n) \subset X$ such that $(x_n, L_n x_n) \longrightarrow (x, L_o x)$. Therefore $x_n \rightharpoonup x$ and $L_n x_n \longrightarrow L_o x$, so that, taking into account (13), $L_n x \to L_o x$. Let now $x_n \rightharpoonup x$; then (x_n) is bounded so that, by (13), $(L_n x_n)$ is bounded. Take $L_{n_k}(x_{n_k}) \rightharpoonup y$; hence $(x_{n_k}, L_{n_k}(x_{n_k})) \rightharpoonup (x,y)$, so that $(x,y) \in \text{gph } L_o$, i.e., $y = L_o x$. Therefore $L_n x_n \rightharpoonup L_o x$.

(B) $\Rightarrow$ (C). We must show $L_n^* y^* \rightharpoonup L_o^* y^*$ for $y^* \in Y^*$. We have $\langle x, L_n^* y^* \rangle = \langle L_n x, y^* \rangle \to \langle L_o x, y^* \rangle = \langle x, L_o^* y^* \rangle$ for every $x \in X$. Therefore $L_n^* y^* \rightharpoonup L_o^* y^*$. Since X^* is reflexive there exist $x_n \in S_X$, $n \in N$ such that $\| L_n^* y^* \| = \langle x_n, L_n^* y^* \rangle =$ $= \langle L_n x_n, y^* \rangle$. Take $A = \limsup \| L_n^* y^* \|$ and (x_{n_k}) such that $\| L_{n_k}^* y^* \| \to A$; we can suppose $x_{n_k} \rightharpoonup x \in \bar{S}(o,1)$. By hypothesis $L_{n_k} x_{n_k} \rightharpoonup L_o x$, so that $\| L_{n_k}^* y^* \| = \langle x_{n_k}, L_{n_k}^* y^* \rangle =$ $= \langle L_{n_k} x_{n_k}, y^* \rangle \to A = \langle L_o x, y^* \rangle \leq \| L_o^* y^* \|$. Since the norm is w- l.s.c. we have $\liminf \| L_n^* y^* \| \geq \| L_o^* y^* \|$. Therefore $\| L_n y^* \| \to \| L_o^* y^* \|$. Since $L_n^* y^* \rightharpoonup L_o^* y^*$ and X^* has property (h) it follows that $L_n^* y^* \to L_o^* y^*$ $\forall y^* \in Y^*$.

(C) $\Longrightarrow$ (A). Let $(x, L_o x) \in \text{gph } L_o$; then $\text{gph } L_n \ni (x, L_n x) \to (x, L_o x)$. Let now $(x_{n_k}, L_{n_k} x_{n_k}) \to (x,y)$, i.e., $x_{n_k} \rightharpoonup x$, $L_{n_k} x_{n_k} \rightharpoonup y$. Then for $y^* \in Y^*$

$$\langle L_{n_k} x_{n_k}, y^* \rangle = \langle x_{n_k}, L^*_{n_k} y^* \rangle \to \langle x, L^*_o y^* \rangle = \langle L_o x, y^* \rangle.$$

Therefore $L_{n_k} x_{n_k} \rightharpoonup L_o x$, so that $y = L_o x$, i.e., $(x,y) \in \text{gph } L_o$. The proof is complete

<u>Theorem</u> 3. Suppose X is an E-space.

(i) (B) $\Longrightarrow$ (D), (E), (H) - (K), (M), (N),

(ii) (K') $\Longrightarrow$ (D').

(iii) Furthermore, if Y^* is an E-space and ψ_2 is continuous, then (B) $\Longrightarrow$ (L).

<u>Proof</u>. (i) (B) $\Longrightarrow$ (D). Fix $(\hat{x}, \hat{y})$. It is obvious that

$$\psi_1(\|\bar{x}_n - \hat{x}\|) + \psi_2(\|L_n \bar{x}_n - \hat{y}\|) \le \psi_1(\|x - \hat{x}\|) + \psi_2(\|L_n x - \hat{y}\|) \quad \forall x \in X, \tag{21}$$

so that

$$\psi_1(\|\bar{x}_n - \hat{x}\|) + \psi_2(\|L_n \bar{x}_n - \hat{y}\|) \le \psi_2(\|L_n \hat{x} - \hat{y}\|). \tag{22}$$

Therefore $(\bar{x}_n)$ is bounded. Let $\bar{x}_{n_k} \rightharpoonup x_o$. Then, from (B), $L_{n_k} \bar{x}_{n_k} \rightharpoonup L_o x_o$. Since f_1, f_2 are w- l.s.c., taking the lower limit in (21) for $n = n_k$, $k \to \infty$, we get

$$\psi_1(\|x_o - \hat{x}\|) + \psi_2(\|L_o x_o - \hat{y}\|) \le \psi_1(\|x - \hat{x}\|) + \psi_2(\|L_o x - \hat{y}\|) \ \forall x \in X,$$

so that $x_o = \bar{x}_o$. Therefore $\bar{x}_n \rightharpoonup \bar{x}_o$ and $L_n \bar{x}_n \rightharpoonup L_o \bar{x}_o$. Writing (21) for $x = \bar{x}_o$, we have

$$\psi_1(\|\bar{x}_n - \hat{x}\|) \le \psi_1(\|\bar{x}_o - \hat{x}\|) + \psi_2(\|L_n \bar{x}_o - \hat{y}\|) - \psi_2(\|L_n \bar{x}_n - \hat{y})\|,$$

so that, taking the upper limit, we get

$$\limsup \psi_1(\|\bar{x}_n - \hat{x}\|) \le \psi_1(\|\bar{x}_o - \hat{x}\|) + \psi_2(\|L_o \bar{x}_o - \hat{y}\|) - \psi_2(\|L_o \bar{x}_o - \hat{y}\|) = \psi_1(\|\bar{x}_o - \hat{x}\|).$$

Thus we obtained $\lim \psi_1(\|\bar{x}_n - \hat{x}\|) = \psi_1(\|\bar{x}_o - \hat{x}\|)$. But ψ_1 is increasing and continuous, so that $\|\bar{x}_n-\hat{x}\| \to \|\bar{x}_o-\hat{x}\|$, which together $\bar{x}_n-\hat{x} \rightharpoonup \bar{x}_o-\hat{x}$ imply $\bar{x}_n \to \bar{x}_o$.

(B) $\Rightarrow$ (N), (K). We just obtained $\bar{x}_n \to \bar{x}_o$, which implies $L_n\bar{x}_n \rightharpoonup L_o\bar{x}_o$, so that by the continuity of f_1 and f_2, we have $\min I_n \to \min I_o$.

In the same way, if we replace I_n by $I_n + \langle ., x^* \rangle$, $x^* \in X^*$, we obtain that $\min(I_n + \langle ., x^*\rangle) \to \min(I_o + \langle ., x^* \rangle)$, i.e. $I_n \xrightarrow{G} I_o$. So we have (B) $\Rightarrow$ (M). Thus, taking into account Remark 2, we must only show (B) $\Rightarrow$ (E). So, let $x \in X$; then $(x, L_nx) \rightharpoonup (x, L_ox)$, so that, by the continuity of f_2, $I_n(x) \to I_o(x)$. Let now $x_{n_k} \rightharpoonup x$; then $L_{n_k}x_{n_k} \rightharpoonup L_ox$, so that $\liminf I_{n_k}(x_{n_k}) \geq I_o(x)$. Therefore $I_n \xrightarrow{M} I_o$.

(ii) (K') $\Rightarrow$ (D'). Since $\bar{x}_n \rightharpoonup \bar{x}_o$, $L_n\bar{x}_n \rightharpoonup L_o\bar{x}_o$, we have

$$\liminf I_n(\bar{x}_n) = \liminf(\psi_1(\|\bar{x}_n-\hat{x}\|) + \psi_2(\|L_n\bar{x}_n-\hat{y}\|))$$
$$\geq \liminf \psi_1(\|\bar{x}_n-\hat{x}\|) + \liminf \psi_2(\|L_n\bar{x}_n-\hat{y}\|)$$
$$\geq \psi_1(\|\bar{x}_o-\hat{x}\|) + \psi_2(\|L_o\bar{x}_o-\hat{y}\|) = I_o(\bar{x}_o).$$

But, by hypotheses, $\limsup I_n(\bar{x}_n) \leq I_o(\bar{x}_o)$, so that $I_n(\bar{x}_n) \to I_o(\bar{x}_o)$. Therefore, $\liminf \psi_1(\|\bar{x}_n-\hat{x}\|) = \psi_1(\|\bar{x}_o-\hat{x}\|)$ and $\liminf \psi_2(\|L_n\bar{x}_n-\hat{y}\|) = \psi_2(\|L_o\bar{x}_o-\hat{y}\|)$. On the other hand

$$\limsup \psi_1(\|\bar{x}_n-\hat{x}\|) = \limsup [I_n(\bar{x}_n) - \psi_2(\|L_n\bar{x}_n-\hat{y}\|)] =$$
$$= I_o(\bar{x}_o) - \psi_2(\|L_o\bar{x}_o-\hat{y}\|) = \psi_1(\|\bar{x}_o-\hat{x}\|),$$

so that $\psi_1(\|\bar{x}_n-\hat{x}\|) \to \psi_1(\|\bar{x}_o-\hat{x}\|)$, and, as above, $\bar{x}_n \to \bar{x}_o$.

(iii) Suppose Y^* is an E-space and ψ_2 is continuous. Then, as noted at the beginning of the section, (D_n) has an unique solution $\bar{y}_n^* = -\Phi_2(L_n\bar{x}_n - \hat{y})$. But $\bar{x}_n \to \bar{x}_o$, so that $L_n\bar{x}_n \to L_o\bar{x}_o$. Since Y^* is an E-space, by Theorem A, Φ_2 is continuous. Therefore

$$\bar{y}_n^* = -\Phi_2(L_n\bar{x}_n - \hat{y}) \to -\Phi_2(L_o\bar{x}_o - \hat{y}) = \bar{y}_o^*.$$

Theorem 4. Suppose X is uniformly convex. Then (E') $\Rightarrow$ (D'); (J') $\Rightarrow$ (D'); (E') $\Rightarrow$ (F'). Furthermore, if Y^* is an E-space and φ_2 is continuous, then (B) $\Rightarrow$ (G).

Proof. (E') $\Rightarrow$ (D'). Let (E') be verified for $(\hat{x},\hat{y})$. Taking into account (22) and (13) we have that $\psi_1(\|\bar{x}_n-\hat{x}\|) \leq$ $\leq M_1$ for some $M_1 > 0$. Therefore, there exists $M > 0$ such that

$$\|\bar{x}_n-\hat{x}\| \leq M < a_1 \quad \forall n \in N. \qquad (23)$$

Since $I_n \xrightarrow{M} I_o$, there exists $(x_n) \subset X$, $x_n \to \bar{x}_o$ such that $I_n(x_n) \to I_o(\bar{x}_o)$, which implies $\limsup I_n(\bar{x}_n) \leq I_o(\bar{x}_o)$. We can suppose (x_n) also satisfies (23). Let now $\bar{x}_{n_k} \rightharpoonup x_o$; then $\liminf I_{n_k}(\bar{x}_{n_k}) \geqslant I_o(x_o) \geqslant I_o(\bar{x}_o)$. Therefore $x_o = \bar{x}_o$, $\bar{x}_n \rightharpoonup \bar{x}_o$ and $I_n(\bar{x}_n) \to I_o(\bar{x}_o)$. Since X is uniformly convex, f_1 is uniformly convex on $\bar{S}(o,M)$ so that I_n, $n \in N$ are equi-uniformly convex on $S(\hat{x},M)$. Therefore there exists $\delta \in \Delta$ such that

$$I_n(x_n) \geqslant I_n(\bar{x}_n) + \delta(\|x_n-\bar{x}_n\|), \forall n \in N. \qquad (24)$$

But $\lim I_n(x_n) = \lim I_n(\bar{x}_n) = I_o(\bar{x}_o)$, so that $\delta(\|x_n-\bar{x}_n\|) \to 0$ which implies $x_n-\bar{x}_n \to 0$. Since $x_n \to \bar{x}_o$ it follows $\bar{x}_n \to \bar{x}_o$.

(E') $\Rightarrow$ (F'). We saw above that $\bar{x}_n \rightharpoonup \bar{x}_o$ and $I_n(\bar{x}_n) \to I_o(\bar{x}_o)$. Since $I_n(x_n) - I_n(\bar{x}_n) \to 0$, it follows that $I_n(x_n) \to I_o(\bar{x}_o)$. There exists $M \in]o,a_1[$ such that

$$\|\bar{x}_n-\hat{x}\| \leq M, \|x_n-\hat{x}\| \leq M, \forall n \in N.$$

Thus, for some $\delta \in \Delta$ we have (24). Therefore $x_n-\bar{x}_n \to 0$ which together with $\bar{x}_n \rightharpoonup \bar{x}_o$ imply $x_n \rightharpoonup \bar{x}_o$.

(J') $\Rightarrow$ (D'). Since $I_n(\bar{x}_n) \leq I_n(\bar{x}_o)$, it follows, from the hypotheses, that $I_n(\bar{x}_n) \to I_o(\bar{x}_o)$ and $I_n(\bar{x}_o) \to I_o(\bar{x}_o)$. As above, for some $\delta \in \Delta$ we have

$$I_n(\bar{x}_o) \geqslant I_n(\bar{x}_n) + \delta(\|\bar{x}_o-\bar{x}_n\|), \forall n \in N,$$

which implies $\bar{x}_n \to \bar{x}_o$.

Suppose, furthermore, that Y^* is an E-space and φ_2 is continuous, and show (B) $\Rightarrow$ (G). Fix $(\hat{x},\hat{y})$ and take $\bar{x}_n$ the (unique) solution of (P_n) and $\bar{y}_n^*$ the (unique) solution of (D_n). From Theorem 3 we have $\bar{x}_n \to \bar{x}_o$ and $\bar{y}_n^* \to \bar{y}_o^*$. It is obvious that $(I_n(\bar{x}_n))$ is bounded which implies $(F(x_n,y_n))$ is so. But $0 \leq \varphi_1(\|x_n-\hat{x}\|) \leq F(x_n,y_n)$, so that there exists $M \in]o,a_1[$, such that

$$\|\bar{x}_n-\hat{x}\| \leq M, \ \|x_n-\hat{x}\| \leq M, \forall n \in N.$$

Then, taking into account Theorem A (xiii), there exists $\delta \in \Delta$ such that

$$f_1(x_n-\hat{x}) \geqslant f_1(\bar{x}_n-\hat{x}) + \langle x_n-\bar{x}_n, L_n^* \bar{y}_n^* \rangle + \delta(\|x_n-\bar{x}_n\|), \forall n \in N,$$

since, by Theorem B, $L_n^* \bar{y}_n^* \in \partial f_1(\bar{x}_n-\hat{x})$. On the other hand, since $-\bar{y}_n^* \in \partial f_2(L_n\bar{x}_n-\hat{y})$, we have

$$f_2(y_n-\hat{y}) \geqslant f_2(L_n\bar{x}_n-\hat{y}) + \langle y_n - L_n\bar{x}_n, -\bar{y}_n^* \rangle.$$

Adding the above inequalities, we get

$$F(x_n,y_n) \geqslant I_n(\bar{x}_n) + \langle L_nx_n-y_n, \bar{y}_n^* \rangle + \delta(\|x_n-\bar{x}_n\|), \forall n \in N.$$

Taking into account that $L_nx_n-y_n \to 0$, $\bar{y}_n^* \to \bar{y}_o^*$, $F(x_n,y_n) - I_n(\bar{x}_n) \to 0$ we obtain, as above, $x_n \to \bar{x}_o$. The proof is complete.

Theorem 5. (i) Suppose X is strictly convex, X^* is an E-space, Y has property (h), Y^* is uniformly convex, φ_1, φ_2 are continuous, $\lim_{t\to\infty} \varphi_2(t) = \infty$ and $\varphi_2(t) = 0 \Leftrightarrow t = 0$. Then (D) $\Rightarrow$ (C).

(ii) Suppose X is uniformly convex, X^* has property (h), Y is an E-space, Y^* is strictly convex, $\varphi_{1+}(o) = 0$, φ_2 is increasing and $\lim_{t\to\infty} \varphi_2(t) = \infty$. Then (L) $\Rightarrow$ (C).

Proof. (i) Note that in our conditions Φ_1, Φ_2 are single-valued mappings, Φ_1 is continuous on dom $\Phi_1 = S(o,a_1)$

and onto and ϕ_2 is onto and uniformly continuous on bounded subsets of Y (see Theorem A). So, by (17), for $\bar{x}_n = \bar{x}_n(\hat{x},\hat{y})$, we have

$$\phi_1(\bar{x}_n-\hat{x}) + L_n^* \phi_2(L_n\bar{x}_n-\hat{y}) = 0. \quad (25)$$

Let us show that

$$L_n^* \phi_2(L_n x+y) \to L_o^* \phi_2(L_o x+y) \quad \forall x \in X,\ y \in Y. \quad (26)$$

Since $\bar{x}_n \to \bar{x}_o$ and ϕ_1 is continuous, from (25), it follows that

$$L_n^* \phi_2(L_n\bar{x}_n-\hat{y}) = -\phi_1(\bar{x}_n-\hat{x}) \to -\phi_1(\bar{x}_o-\hat{x}) = = L_o^* \phi_2(L_o\bar{x}_o-\hat{y}). \quad (27)$$

But

$$\|L_n^* \phi_2(L_n\bar{x}_n-\hat{y}) - L_n^* \phi_2(L_n\bar{x}_o-\hat{y})\| \le C\|\phi_2(L_n\bar{x}_n-\hat{y}) - \phi_2(L_n\bar{x}_o-\hat{y})\|. \quad (28)$$

On the other hand $(L_n\bar{x}_n-\hat{y})_{n\in N}$ and $(L_n\bar{x}_o-\hat{y})_{n\in N}$ are bounded and

$$\|(L_n\bar{x}_n-\hat{y}) - (L_n\bar{x}_o-\hat{y})\| \le C\|\bar{x}_n-\bar{x}_o\|. \quad (29)$$

Since $\bar{x}_n \to \bar{x}_o$ and ϕ_2 is uniformly continuous on bounded sets, it follows from (28) and (29) that

$$L_n^* \phi_2(L_n\bar{x}_n-\hat{y}) - L_n^* \phi_2(L_n\bar{x}_o-\hat{y}) \to 0 \text{ as } n\to\infty. \quad (3o)$$

From (27) and (3o) we get

$$L_n^* \phi_2(L_n\bar{x}_o-\hat{y}) \to L_o^* \phi_2(L_o\bar{x}_o-\hat{y}).$$

But $\forall x \in X$, $\hat{y} \in Y$ $\exists \hat{x} \in X$ such that $x = \bar{x}_o(\hat{x},\hat{y})$. Indeed take $\hat{x}$ such that $\phi_1(x-\hat{x}) + L_o^* \phi_2(L_o x-\hat{y}) = 0$, which exists since ϕ_1 is onto. So we obtained that (26) is true. Taking now $x = 0$ in (26) we have $L_n^* \phi_2(y) \to L_o^* \phi_2(y)$ $\forall y \in Y$. But ϕ_2 is onto, so that

$$L_n^* y^* \to L_o^* y^* \quad \forall y^* \in Y^*. \quad (31)$$

Let now $x \in X$ and take $y = 0$ in (26). Then

$$L_n^* \Phi_2(L_n x) \longrightarrow L_o^* \Phi_2(L_o x),$$

so that

$$\langle x, L_n^* \Phi_2(L_n x)\rangle = \langle L_n x, \Phi_2(L_n x)\rangle = \|L_n x\| \varphi_2(\|L_n x\|) \rightarrow$$

$$\langle x, L_o^* \Phi_2(L_o x)\rangle = \langle L_o x, \Phi_2(L_o x)\rangle = \|L_o x\| \varphi_2(\|L_o x\|).$$

Since the function $0 \le t \longrightarrow t\varphi_2(t)$ is continuous and increasing, if follows that $\|L_n x\| \longrightarrow \|L_o x\|$. On the other hand

$$\langle L_n x, y^* \rangle = \langle x, L_n^* y^* \rangle \rightarrow \langle x, L_o^* y^* \rangle = \langle L_o x, y^* \rangle, \ \forall y^* \in Y^*,$$

so that $L_n x \rightharpoonup L_o x$. Since Y has property (h), we get $L_n x \rightarrow L_o x$, which together with (31) show that (C) is valid.

(ii) In our conditions, taking into account Theorem A, $\Phi_2^{-1} : Y^* \longrightarrow Y$ is a single-valued continuous mapping, $\Phi_1^{-1} : X^* \rightarrow X$ is a single-valued mapping, uniformly continuous on bounded sets and (D_n) has an unique solution $\bar{y}_n^* = \bar{y}_n^*(\hat{x},\hat{y})$ characterized by

$$L_n(\hat{x} + \Phi_1^{-1}(L_n^* \bar{y}_n^*)) + \Phi_2^{-1}(\bar{y}_n^*) - \hat{y} = 0. \qquad (32)$$

It follows easily that for $(\hat{x},\hat{y})$ fixed, $(\bar{y}_n^*)$ is bounded. As in (i) we obtain that

$$L_n(\hat{x} + \Phi_1^{-1}(L_n^* \bar{y}_o^*)) \longrightarrow L_o(\hat{x} + \Phi_1^{-1}(L_o^* \bar{y}_o^*)). \qquad (33)$$

It is obvious that $\forall \hat{x} \in X,\ y^* \in Y^*\ \exists \hat{y} \in Y$ such that $\bar{y}_o^*(\hat{x},\hat{y}) = y^*$. Thus, from (33), we get

$$L_n(x + \Phi_1^{-1}(L_n^* y^*)) \longrightarrow L_o(x + \Phi_1^{-1}(L_o^* y^*)) \ \forall x \in X,\ y^* \in Y^*. \ (34)$$

Taking $y^* = 0$ in (34) we obtain $L_n x \rightarrow L_o x \ \forall x \in X$, while, taking in (34) $x = 0$, we get

$$L_n \Phi_1^{-1}(L_n^* y^*) \longrightarrow L_o \Phi_1^{-1}(L_o^* y^*),\ \forall y^* \in Y^*.$$

As in the proof of part (i) we obtain that $L_n^* y^* \longrightarrow L_o^* y^* \ \forall y^* \in Y^*$, so that (C) is verified.

Summarizing the results of Theorems 2-5 we have

Theorem 6. Suppose the following conditions are verified : X, Y^* are uniformly convex, X^*, Y are E-spaces, φ_1, φ_2 are continuous and increasing, $\varphi_{1-}(a_1) = \infty$, $\lim_{t \to \infty} \varphi_2(t) = \infty$. Then the conditions (A) - (N) are pairwise equivalent. (By our conventions $\varphi_{1-}(o) = \varphi_{2-}(o) = 0$ so that $\varphi_{1+}(o) = \varphi_{2+}(o) = 0$.)

Remark 3. The equivalence of conditions (A) - (K) was obtained by Zolezzi [1] for X,Y Hilbert spaces and $\varphi_1(t) = \varphi_2(t) = t$.

Remark 4. We obtained that (C) $\Leftrightarrow$ (D) if X, X^* are E-spaces, Y has property (h) and Y^* is uniformly convex, φ_1 as in Theorem 6 and $\varphi_2 : R_+ \to R_+$ continuous, nondecreasing, $\varphi_2(t) = 0 \Leftrightarrow t = 0$ and $\lim_{t \to \infty} \varphi_2(t) = \infty$. Lucchetti and Mignanego [2] obtained the same equivalence when X,Y are strongly smooth E-spaces and $\varphi_1(t) = \varphi_2(t) = t^{p-1}$, $p > 1$. We are not convinced that condition Y^*- uniformly convex can be weakened, even for such φ_1, φ_2.

Let us observe that condition (D) can be written in the following form

(D) $\quad x_n(F) \longrightarrow \overline{x}_o(F) \ \forall F \in \mathcal{F}_1$,

where $\mathcal{F}_1 = \{F_{\hat{x},\hat{y}} : \hat{x} \in X, \hat{y} \in Y\}$. The conditions (E) - (N) may be written in a similar way. So as we pointed in Remark 3, Zolezzi proved the equivalences of (A) - (K) for X,Y Hilbert spaces and taking instead of $\mathcal{F}_1$ the class $\mathcal{F}_o$ obtained from $\mathcal{F}_1$ in the case $\varphi_1(t) = \varphi_2(t) = t^2$. Bennati [3] was interested in generalizing the results of Zolezzi taking a larger class of functions and keeping X,Y Hilbert spaces. More exactly, Bennati considered the class $\mathcal{F}$ of functions $F : X \times Y \to R$ which satisfy the following conditions :

F is convex and continuous, (35)

$$F\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \leq \frac{1}{2} F(x_1,y_1) + \frac{1}{2} F(x_2,y_2) - \frac{1}{4} \gamma \|x_1-x_2\|^2 \quad \forall x_1,x_2 \in X,\ y_1,y_2 \in Y, \tag{36}$$

$\gamma \in]0,1]$ being the same for all $F \in \mathcal{F}$, and

$$\begin{cases} F \text{ is Fréchet differentiable with respect to } y \text{ and} \\ X \times Y \ni (x,y) \longrightarrow \nabla_y F(x,y) \text{ is continuous.} \end{cases} \tag{37}$$

Remark 5. Bennati showed the equivalences (A) $\Leftrightarrow$ (C) $\Leftrightarrow$ (D) $\Leftrightarrow$ (E) $\Leftrightarrow$ (L) where the conditions (D), (E), (L) are taken for all $F \in \mathcal{F}$. Since $\mathcal{F}_0 \subset \mathcal{F}$ the essentially new implications are (C) $\Rightarrow$ (D), (E), (L).

In what follows we consider X a reflexive Banach space and Y a Banach space, $F : X \times Y \rightarrow R$ a continuous convex function, strictly convex with respect to x, i.e.

$$\begin{cases} F(\lambda(x_1,y_1) + (1-\lambda)(x_2,y_2)) < \lambda F(x_1,y_1) + (1-\lambda) F(x_2,y_2) \\ \qquad \forall x_1,x_2 \in X,\ x_1 \neq x_2,\ y_1,y_2 \in Y,\quad \lambda \in]0,1[. \end{cases} \tag{38}$$

For such a function we want to see what kind of assumptions we must impose such that (B) implies some of conditions (D') - (N'), where $(L_n) \subset L(X,Y)$ satisfies (13) and $I_n(x) = F(x, L_n x)$. Remark that (38) assures that I_n, $n \in N$ are strictly convex. We shall deal with one or many of the following conditions.

Firstly consider the following generalization of (36) :

$$\begin{cases} \exists\, \delta \in \Delta,\ \lim\limits_{t\to\infty} \dfrac{\delta(t)}{t} = \infty \text{ such that} \\ F\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right) \leq \dfrac{1}{2} F(x_1,y_1) + \dfrac{1}{2} F(x_2,y_2) - \\ \qquad - \delta(\|x_1-x_2\|) \quad \forall x_1,x_2 \in X,\ y_1,y_2 \in Y, \end{cases} \tag{39}$$

or, less restrictive,

$$\begin{cases} \forall M > 0 \ \exists \delta \in \Delta \quad \forall x_1, x_2 \in S(o,M),\ y_1, y_2 \in Y: \\ F\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \leq \frac{1}{2} F(x_1,y_1) + \frac{1}{2} F(x_2,y_2) - \delta(\|x_1-x_2\|). \end{cases} \tag{40}$$

We also consider the following coercivity conditions :

$$\lim_{\|x\| \to \infty} \inf_{y \in \bar{S}(o, c\|x\|)} F(x,y) = \infty \tag{41}$$

and

$$\lim_{\|x\| \to \infty} \frac{1}{\|x\|} \inf_{y \in \bar{S}(o,c\|x\|)} F(x,y) = \infty, \tag{42}$$

where $c > 0$ is the constant from (13).

As in [7] we can show that if F satisfies (39) then there exists $\delta \in \Delta$, $\lim_{t \to \infty} \frac{\delta(t)}{t} = \infty$ such that

$$\begin{cases} F(x,y) \geqslant F(\bar{x},\bar{y}) + \langle x-\bar{x}, x^* \rangle + \langle y-\bar{y}, y^* \rangle + \delta(\|x-\bar{x}\|) \\ \quad \forall x, \bar{x} \in X,\ y, \bar{y} \in Y,\ (x^*, y^*) \in \partial F(\bar{x},\bar{y}), \end{cases} \tag{43}$$

and, if F satisfies (4o) then

$$\begin{cases} \forall M > 0 \ \exists \delta \in \Delta \ \forall x, \bar{x} \in S(o,M),\ y, \bar{y} \in Y,\ (x^*, y^*) \in \partial F(\bar{x},\bar{y}): \\ F(x,y) \geqslant F(\bar{x},\bar{y}) + \langle x-\bar{x}, x^* \rangle + \langle y-\bar{y}, y^* \rangle + \delta(\|x-\bar{x}\|). \end{cases} \tag{44}$$

Taking in (43) $(\bar{x},\bar{y}) = (o,o)$ and $(x_o^*, y_o^*) \in \partial F(o,o)$ we get that F satisfies (42). Now, taking $L \in L(X,Y)$ and $I(x) = F(x, Lx)$, if F satisfies (39), then

$$I(x) \geqslant I(\bar{x}) + \langle x-\bar{x}, x^* \rangle + \delta(\|x-\bar{x}\|) \quad \forall x, \bar{x} \in X,\ x^* \in \partial I(\bar{x}), \tag{45}$$

and, if F satisfies (4o) then

$$\begin{cases} \forall M > 0 \ \exists \delta \in \Delta \ \forall x, \bar{x} \in S(O,M),\ x^* \in \partial I(\bar{x}): \\ I(x) \geqslant I(\bar{x}) + \langle x-\bar{x}, x^* \rangle + \delta(\|x-\bar{x}\|). \end{cases} \tag{46}$$

We can now formulate the following result.

<u>Theorem 7.</u> Let X be a reflexive Banach space, Y a Banach space, $F : X \times Y \to R$ a continuous convex function satisfying (38), $(L_n)_{n \in N} \subset L(X,Y)$, satisfying (13), $I_n(x) = F(x, L_n x)$.

(i) (B) $\Rightarrow$ (E')

(ii) if F satisfies (41) then (B) $\Rightarrow$ (J'), (K'), (N') ;

(iii) if F satisfies (42) then (B) $\Rightarrow$ (H'), (I'), (M') ;

(iv) if F satisfies (4o) and (41) then (E') $\Rightarrow$ (D'), (F') ;

(v) if F satisfies (37), (4o), (41) then (B) $\Rightarrow$ (L').

Furthermore, if (41) holds with c replaced by some $\alpha > c$ then (B) $\Rightarrow$ (G').

<u>Proof</u>. (i) By the continuity of F we have $\lim I_n(x) = \lim F(x, L_n x) = F(x, L_o x) = I_o(x)$ $\forall x \in X$. Take now $x_{n_k} \to x$; then, from (B), we have $L_{n_k} x_{n_k} \to L_o x$, which, by the w-lower semicontinuity of F implies $\liminf (I_{n_k}(x_{n_k}) = \liminf F(x_{n_k}, L_{n_k} x_{n_k}) \geq F(x, L_o x) = I_o(x)$. Therefore $I_n \xrightarrow{M} I_o$.

(ii) It is obvious that $F(\overline{x}_n, L_n \overline{x}_n) = I_n(\overline{x}_n) \leq I_n(o) = F(o,o)$, so that, from (41), we get that $(\overline{x}_n)$ is bounded. By (i) we have $I_n(\overline{x}_o) \to I_o(\overline{x}_o)$ so that $\limsup I_n(\overline{x}_n) \leq I_o(\overline{x}_o)$. Take $\overline{x}_{n_k} \rightharpoonup x_o$. Then $\liminf I_{n_k}(\overline{x}_{n_k}) \geq I_o(x_o) \geq I_o(\overline{x}_o)$. Therefore $x_o = \overline{x}_o$, $\overline{x}_n \rightharpoonup \overline{x}_o$, $\lim I_n(\overline{x}_n) = I_o(\overline{x}_o)$. Once again, by (B) it follows $L_n \overline{x}_n \rightharpoonup L_o \overline{x}_o$. Hence (B) $\Rightarrow$ (J'), (K'), (N').

(iv) (E') $\Rightarrow$ (D'). From (41) we have that $(\overline{x}_n)$ is bounded. Since $I_n \xrightarrow{M} I_o$, there exists $(x_n) \subset X$ such that $x_n \to \overline{x}_o$ and $I_n(x_n) \to I_o(\overline{x}_o)$. As in (ii) it follows that $I_n(\overline{x}_n) \to I_o(\overline{x}_o)$ and $\overline{x}_n \rightharpoonup \overline{x}_o$. Taking $M/2 \geq \|x_n\|, \|\overline{x}_n\|$ $\forall n \in N$, from (4o) it follows that (46) holds for very $n \in N$, i.e.

$$I_n(x_n) \geq I_n(\overline{x}_n) + \delta(\|x_n - \overline{x}_n\|), \forall n \in N. \qquad (47)$$

Taking the limit it follows that $x_n - \overline{x}_n \to 0$ which shows that $\overline{x}_n \to \overline{x}_o$.

(E') $\Rightarrow$ (F'). In our conditions $\overline{x}_n \rightharpoonup \overline{x}_o$. On the other hand, (47) is true, so that $x_n - \overline{x}_n \to 0$, which implies $x_n \to \overline{x}_o$.

(v) (B) $\Rightarrow$ (L'). We saw in part (iv) that $\bar{x}_n \to \bar{x}_o$. The solutions $\bar{y}_n^*$ of the dual problems (D_n)(which exists by Theorem B) verify

$$(L_n^* \bar{y}_n^*, -\bar{y}_n^*) \in \partial F(\bar{x}_n, L_n \bar{x}_n),$$

so that, from (37), we have that

$$\bar{y}_n^* = -\nabla_y F(\bar{x}_n, L_n \bar{x}_n).$$

This one shows that (D_n) has unique solution. From (B) it follows that $L_n \bar{x}_n \to L_o \bar{x}_o$, so that, by the continuity of $\nabla_y F$, we get

$$\bar{y}_n^* = -\nabla_y F(\bar{x}_n, L_n \bar{x}_n) \to -\nabla_y F(\bar{x}_o, L_o \bar{x}_o) = \bar{y}_o^*.$$

(B) $\Rightarrow$ (G'). Take $F(x_n, y_n) - I_n(\bar{x}_n) \to 0$ so that $y_n - L_n x_n \to 0$. Since $I_n(\bar{x}_n)$ is bounded, it follows that $(F(x_n, y_n))_{n \in N}$ is bounded. Suppose (x_n) is not bounded. Taking a subsequence, if necessary, we can suppose that $\|x_n\| \to \infty$. Since $y_n - L_n x_n \to 0$ there exists $M > 0$ such that $\|y_n - L_n x_n\| \le M$, so that $\|y_n\| \le M + c\|x_n\| \le \alpha \|x_n\|$ for n sufficiently large. Therefore $F(x_n, y_n) \geqslant \inf_{y \in S(o, \alpha\|x_n\|)} F(x_n, y)$ which implies that $\lim F(x_n, y_n) = \infty$, a contradiction. Therefore (x_n) is bounded. From (4o) there exists $\delta \in \Delta$ such that (44) holds. Since $(L_n^* \bar{y}_n^*, -\bar{y}_n^*) \in \partial F(\bar{x}_n, L_n \bar{x}_n)$, from (44), we get

$$F(x_n, y_n) \geqslant F(\bar{x}_n, L_n \bar{x}_n) + \langle x_n - \bar{x}_n, L_n \bar{y}_n^* \rangle + \langle y_n - L_n \bar{x}_n, -\bar{y}_n^* \rangle +$$
$$+ \delta(\|x_n - \bar{x}_n\|) = F(x_n, L_n x_n) + \langle L_n x_n - y_n, \bar{y}_n^* \rangle + \delta(\|x_n - \bar{x}_n\|).$$

Taking the limit and using the hypotheses and $\bar{y}_n^* \to \bar{y}_o^*$, proved above, we obtain $x_n - \bar{x}_n \to 0$. Since $\bar{x}_n \to \bar{x}_o$, we get $x_n \to \bar{x}_o$. The proof is complete.

(iii) For $x^* \in X^*$ take $\tilde{F}(x,y) = F(x,y) + \langle x, x^* \rangle$. If F satisfies (42) then $\tilde{F}$ also satisfies (42), and therefore (41). Applying (ii) for $\tilde{F}$ we obtain that (H'), (I'), (M') hold.

References

1. T.Zolezzi, Variational characterizations of some variational perturbations of the abstract linear quadratic problem. SIAM J.Control and Optimization 16, 1o6-121 (1978).

2. R.Lucchetti, F.Mignanego, Continuous dependence on the data in abstract control problems, Università di Genova, Preprint no.320 (1980).

3. M.L.Bennati, On the convergence of the dual variables in optimal control problems, Universita di Genova, Preprint no.319 (1980).

4. T.Zolezzi, Variational stability and well-posedness in the optimal control of ordinary differential equations (work presented at the Mathematical Theory of Optimal Control Semester of the Stefan Banach International Center, Warsaw, December, 1980).

5. N.Y.Sonntag, Interprétation géométrique de la convergence d'une suite de convexes, C.R.Acad.Sci.Paris, Sér.A, 282, 1o99-11oo (1976).

6. V.Barbu, Th.Precupanu, "Convexity and Optimization in Banach Spaces", Editura Academiei Bucureşti - Sijthoff & Noordhoff Int.Publishers, 1978.

7. R.T.Rockafellar, "Convex Analysis" Princeton Univ.Press, Princeton, N.J., 1969.

8. C.Zălinescu, On uniformly convex functions, to appear.

9. C.Zălinescu, The Fenchel-Rockafellar duality for mathematical programming in order complete vector latices and applications, INCREST-Bucureşti, Preprint no.451/ 1980.

10. J.Diestel, "Geometry of Banach spaces-Selected topics", Springer-Verlag Berlin-Heidelberg-New York, Lecture Notes in Mathematics, vol.485, 1975.