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HAMILTON-JACOBI EQUATIONS AND SYNTHESIS OF NONLINEAR CONTROL PROCESSES IN HILBERT SPACES

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Local existence for a class of Hamilton-Jacobi equations in Hilbert space is studied. The existence results are used to prove that certain control problems with nonlinear smooth state equations have synthesized controllers on sufficiently small intervals.

1. Introduction

This paper is concerned with existence for the Hamilton-Jacobi equation

$$(1.1)\quad \varphi_t(t,x) + \frac{1}{2}|\varphi_x(t,x)|^2 + (Ax + Fx, \varphi_x(t,x)) = g(t,x);\quad t \in [0,T],\ x \in D(A)$$

with initial value condition

$$(1.2)\quad \varphi(0,x) = \varphi_o(x) \qquad x \in H$$

and with its relevance in synthesis of optimal controllers for the constrained control process in a Hilbert space H,

$$(1.3)\quad \begin{aligned} &x' + Ax + Fx = u \quad ; \quad t \in [0,T] \\ &x(o) = x_o \ ; |x(t)| \le r \qquad \text{for } t \in [0,T] \end{aligned}$$

and with cost

$$(1.4)\quad \int_o^T (g_o(t,x(t)) + \frac{1}{2}|u(t)|^2)dt + \varphi_o(x(T))$$

Here $-A$ is the infinitesimal generator of semigroup of class C_o on H, F is a nonlinear smooth gradient mapping from H into itself and g, g_o, φ_o are real valued functions defined on $[0,T] \times H$ and H, respectively.

The main result, Theorem 1, gives existence of a local solution to

(1.1),(1.2) under the main assumption that F,g and φ_o are sufficiently regular (see also Theorem 2 below).This result is used in Theorem 3 to show that for F,g_o, φ_o sufficiently regular and T small enough,there exists an optimal feedback law for problem (1.3),(1.4) given by

$$(1.5)\quad u = -\varphi_x(T-t,x)\ ;\quad t\in[0,T]\ ,\ x\in H$$

where φ is a solution to (1.1),(1.2) where $g(t) = g_o(T-t)$; $t\in[0,T]$. This result completely solves the synthesis problem for the class of control problems considered here and the constructive approach used to prove Theorem 1 could lead to a numerical scheme to computing the synthesized optimal controller.

In the special case F = 0,Equation (1.1) has been studied in the class of convex functions in [3] .In [1] has been obtained by different methods a local existence result for a related operator equation (see also [2]).

2. Notation and preliminary results

Throughout this paper H will be a real Hilbert space with norm $|\cdot|$ and inner product $(\cdot,\cdot)$.

Given a Frechet differentiable mapping E from H to another Banach space X we shall denote either the symbol $E'(x)$ or $E_x(x)$ for the gradient of E at $x\in H$.

The following spaces will be frequently used in the sequel

1° C(H) is the space of all continuous and bounded operators $E: H\to H$ endowed with the norm

$$(2.1)\quad |E|_{oo} = \sup\{|Ex|\ ;\ x\in H\}.$$

Lip (H) is the space of all Lipschitzian operators from E into itself, endowed with the norm

$$(2.2)\quad \|E\| = \sup\{|Ex-Ey|/|x-y|\ ;\ x,y\in H\}.$$

2° We shall denote by $C^1(H)$ the space of all Fréchet differentiable functions $E \in C(H)$ such that E' is continuous and bounded as a function from H to the space $L(H,H)$ of all continuous operators on H. The notm i n $C^1(H)$ is defined by

(2.3) $$|E|_{1,\infty} = |E|_{\infty} + |E'|_{\infty}$$

where

(2.4) $$|E'|_{\infty} = \sup\{|E'(x)|_{L(H,H)} ; x \in H\} .$$

and $|\cdot|_{L(H,H)}$ is the operator norm of $L(H,H)$.

Let $C^1_{Lip}(H)$ be the space $\{E \in C^1(H);\ \|E'\| < \infty\}$ where

(2.5) $$\|E'\| = \sup\{|E'(x) - E'(y)|_{L(H,H)}/|x-y| ; x,y \in H\} .$$

3° $P(H)$ is the space of all continuous operators $E: H \to H$ which are of the form $E = \varphi'$ where φ is some Fréchet differentiable real valued function on H.

4° If X is one of the spaces $C(H)$, $Lip(H)$, $C^1(H)$ or $C^1_{Lip}(H)$ and $[0,T]$ a closed interval we shall denote by $C([0,T];X)$ the space of all continuous mappings $E: [0,T] \times H \to H$ such that $E(t,.) \in X$ for every $t \in [0,T]$ and $|E|_{T,X} = \sup\{|E(t)|_X ; t \in [0,T]\} < \infty$, where $|\cdot|_X$ is one of the above norms.

5° $C([0,T] \times H;R)$ is the space of all real valued continuous functions $\varphi : [0,T] \times H \to R$. If Σ_r is the closed ball of center 0 and radius r then $C([0,T] \times \Sigma_r; R)$ is the space of all real valued continuous functions on $[0,T] \times \Sigma_r$.

6° For $r > 0$, $C^1_{Lip}(\Sigma_r)$ is the space of all Fréchet differentiable mappings $E: \Sigma_r \to H$ such that

$$\|E'\|_r = \sup\{|E'(x) - E'(y)|_{L(H,H)}/|x-y| ; x,y \in \Sigma_r\} < \infty.$$

By $C^1_{Lip,loc}(H)$ we shall denote the space of all Fréchet differentiable mappings E on H such that $\|E'\|_r < \infty$ for all $r > 0$.

The spaces $C([0,T];\ C^1_{Lip}(\Sigma_r))$ and $C([0,T];\ C^1_{Lip,loc}(H))$ are defined as above.

For a given function $\varphi: [0,T]\times H \to R$ we shall denote by $\varphi_t(t,x)$ the partial derivative with respect to t and by $\varphi_x(t,x)$ the Fréchet derivative with respect to x.

Finally, we shall denote by $C([0,T];H)$ the space of all H-valued continuous on $[0,T]$. Let $E \in Lip(H)$ be fixed and let $J_\varepsilon: H \to H$ and $E_\varepsilon: H \to H$ be the mappings defined by

$$(2.7)\quad J_\varepsilon = (I+\varepsilon E)^{-1}\ ;\ E_\varepsilon = E J_\varepsilon = \varepsilon^{-1}(I - J_\varepsilon).$$

Clearly J_ε and E_ε are well defined for $0 < \varepsilon < \|E\|^{-1}$.

The next lemma gathers for later use some elementary properties of J_ε and E_ε.

LEMMA 1. Let $E \in C(H) \cap Lip(H)$ be given. Then for $0 < \varepsilon < \|E\|^{-1}$ the following estimates are satisfied

$$(2.8)\quad \|J_\varepsilon\| \leq (1 - \varepsilon\|E\|)^{-1}$$

$$(2.9)\quad \|E_\varepsilon\| \leq \|E\|(1-\varepsilon\|E\|)^{-1};\ |E_\varepsilon|_\infty \leq |E|_\infty .$$

$$(2.10)\quad |E_\varepsilon - \tilde{E}_\varepsilon|_\infty \leq |E - \tilde{E}|_\infty (1-\varepsilon\|E\|)^{-1}.$$

If $E \in C^1_{Lip}(H)$ then for all sufficiently small $\varepsilon > 0$ one has

$$(2.11)\quad E'_\varepsilon(x) = E'(J_\varepsilon x)(I + \varepsilon E'(J_\varepsilon x))^{-1},\quad x \in H$$

and the following estimates hold

$$(2.12)\quad |E'_\varepsilon|_\infty \leq |E'|_\infty (1 - \varepsilon|E'|_\infty)^{-1}$$

$$(2.13)\quad \|E'_\varepsilon\| \leq \|E'\|(1-\varepsilon\|E\|)^{-1}(1-\varepsilon|E'|_\infty)^{-1}(1 + \varepsilon|E'|_\infty(1-\varepsilon|E'|_\infty)^{-1}).$$

$$(2.14)\quad |E'_\varepsilon - \tilde{E}'_\varepsilon|_\infty \leq |E' - \tilde{E}'|_\infty (1-\varepsilon|E'|_\infty)^{-1} + \varepsilon|E'|_\infty(1-\varepsilon|E'|_\infty)^{-1}(|E'-\tilde{E}'|_\infty + |E-\tilde{E}|_\infty \|\tilde{E}'\|(1-\varepsilon|E'|_\infty)^{-1}(1-\varepsilon|E|_\infty)^{-1}).$$

Proof. Inequalities (2.8), (2.9) and (2.10) are immediate. As regard (2.12) and (2.13) they are implied by (2.11) and the following obvious inequalities

$$(2.15)\quad |(I+\varepsilon E'(J_\varepsilon x))^{-1}|_{L(H,H)} \leq (1-\varepsilon|E'|_\infty)^{-1}, \; x \in H$$

$$(2.16)\quad |(I+\varepsilon E'(J_\varepsilon x))^{-1}-(I+\varepsilon E'(J_\varepsilon y))^{-1}|_{L(H,H)} \leq \varepsilon(1-\varepsilon|E'|_\infty)^{-2}\|E'\|(1-\varepsilon\|E\|)^{-1}.$$

Inequality (2.14) follows from (2.11) by an elementary calculation.

3. Local existence for the Hamilton-Jacobi equation

Consider the Cauchy problem (1.1),(1.2) where $g \in C([0,T]\times H;R)$ and $\varphi_0 : H \to R$ are given functions satisfying

$$(3.1)\quad \varphi_0' \in C^1_{Lip}(H)\;;\; g_x \in C([0,T]\;;\; C^1_{Lip}(H))$$

and the mapping $F: H \to H$ satisfies the conditions

$$(3.2)\quad F, F' \in C^1_{Lip}(H)\;;\; F \in P(H).$$

As regards the linear operator A, as is explicitly stated in Introduction, we shall assume that $-A$ is the infinitesimal generator of a strongly continuous semigroup of linear bounded operators on H denoted e^{-At}.

By a solution to problem (1.1),(1.2) we mean a function $\varphi : [0,T] \times H \to R$ which is differentiable in t for every $x \in D(A)$ (the domain of A) is Fréchet differentiable in x for every $t \in [0,T]$ and satisfies equation (1.1) along with initial condition (1.2).

THEOREM 1 <u>Let</u> g, φ_0, F, A <u>satisfy the above assumptions. Then for</u> $T > 0$ <u>sufficiently small, problem</u> (1.1),(1.2) <u>has at least one solution</u> $\varphi \in C([0,T]\times H;\, R)$ <u>satisfying</u>

$$(3.3)\quad \varphi_x \in C([0,T];C(H) \cap Lip(H)).$$

<u>Moreover, there exists</u> $C > 0$ <u>depending only on</u> $|\varphi_0|_\infty, |F|_{1,\infty}$ <u>and</u> $|g_x|_{T,C(H)}$ <u>such that</u>

$$(3.4)\quad |\varphi_x(t)|_\infty \leq C \qquad \text{for } t \in [0,T].$$

Now we shall derive a variant of Theorem 1 which seems to be more appropriate for the applications we have in mind.

THEOREM 2. Let $\varphi_o, g \in C([0,T] \times H;R)$ and $F: H \to H$ satisfy the following conditions

(3.5) $\varphi_o \in C^1_{Lip,loc}(H)$; $g_x \in C([0,T] \,;\, C^1_{Lip,loc}(H))$

(3.6) $F, F' \in C^1_{Lip,loc}(H)$; $F \in P(H)$.

Then for every $r > 0$ there exists $T_r \in]0,T]$ and $\varphi_r \in C([0,T_r] \times H;R)$ which is a solution to (1.1)~(1.2) on $[0,T_r] \times \Sigma_r$ and satisfies the condition

(3.7) $(\varphi_r)_x \in C([0,T_r] \,;\, C(H) \cap Lip(H))$.

Proof of Theorem 2 Let $r > 0$ be arbitrary but fixed. Then we may choose an infinitely differentiable real function α_r such that

$\alpha_r(u) = 1$ for $|u| \leq r$ and $\alpha_r(u) = 0$ for $|u| > r + 1$.

Since $F \in P(H)$ there exists a Frechet differentiable function $\chi: H \to R$ such that $F = \chi'$.

Now we apply Theorem 1 where φ_o, g and F are taken as

(3.8) $\varphi_{o,r}(x) = \varphi_o(x)\alpha_r(|x|)$; $g_r(t,x) = g(t,x)\alpha_r(|x|)$

and

(3.9) $F_r(x) = (\chi(x)\alpha_r(|x|))'$; $x \in H$,

respectively.

It is clear that the corresponding solution φ_r to (1.1),(1.2) (which exists on some interval $[0,T_r]$) satisfies equation (1.1) on $[0,T_r] \times \Sigma_r$, thereby completing the proof.

Remark 1 It is tempting to hope that under the assumptions of Theorem 2 the solution φ_r to problem (1.1),(1.2) is unique, but we failed to prove this. However, we shall see later that by variational arguments one can derive the following partial uniqueness result: for every $r > 0$ there exists $T_r > 0$ and $\gamma(r) < r$ such that problem (1.1),(1.2) has a solution φ_r on $[0,T_r] \times \Sigma_r$ which is uniquely defined on $[0,T_r] \times \Sigma_{\gamma(r)}$.

On the other hand, there exists an approximating convergent process (see equation (4.1) below) which uniquely defines a solution φ to

(1.1), (1.2) and is locally Lipschitzian as a function on φ_0 and g(see Remark 2).

4. Proof of Theorem 1.

We shall prove Theorem 1 in several steps. To begin with, we consider for $\varepsilon > 0$ the Cauchy problem

$$(4.1)\quad \begin{cases} E_t(t,x) + A^* E(t,x) + E_x(t,x)Ax + \varepsilon^{-1}(D(t,x) - D_\varepsilon(t,x)) = \\ \qquad = B(t,x); \quad t \in [0,T], \ x \in H \\ E(0,x) = E_0(x) \qquad x \in H \end{cases}$$

where A^* is the adjoint of A, $D_\varepsilon = D(I + \varepsilon D)^{-1}$ and

$$(4.2)\quad E_0 = \varphi_0'; \ B = g_x + F'F$$

$$(4.3)\quad D = E + F .$$

We shall study the following "mild" form of Eq (4.1)

$$(4.4)\quad E(t,x) = e^{-(\varepsilon^{-1}+A^*)t} E_0(e^{-At}x) + \int_0^t e^{-(\varepsilon^{-1}+A^*)(t-s)} B(s, e^{-A(t-s)}x)ds +$$
$$+ \varepsilon^{-1}\int_0^t e^{-(\varepsilon^{-1}+A^*)(t-s)}[D_\varepsilon(s, e^{-A(t-s)}x) - F(e^{-A(t-s)}x)]ds;$$
$$t \in [0,T], \ x \in H.$$

LEMMA 2 <u>Let</u> φ_0, g, A <u>and F satisfy assumptions of Theorem</u> 1. <u>Then there exists</u> T <u>independent of</u> ε <u>such that</u> Eq.(4.4) <u>has on</u> $[0,T]$ <u>a unique solution</u> $E^\varepsilon \in C([0,T]; C^1_{Lip}(H))$ <u>satisfying</u>

$$(4.5)\quad E^\varepsilon(t) \in P(H) \qquad \forall\ t \in [0,T]$$

$$(4.6)\quad |E^\varepsilon(t)|_{1,\infty} + \|E^\varepsilon(t)\| + \|E^\varepsilon_x(t)\| \le M \qquad \forall\ t \in [0,T]$$

<u>where</u> M <u>is independent of</u> ε .

<u>Proof</u>. Let K be the closed subset of $C([0,T]; C^1_{Lip}(H))$ defined by

$$K = \{E;\ |E(t)|_{1,\infty} + \|E(t)\| + \|E_x(t)\| \le M;\ E(t) \in P(H),\ t \in [0,T]\}.$$

Here M is a positive number such that

$$2(|E_0|_{1,\infty} + \|E_0\| + \|E_0'\|) \le M .$$

Let $\Gamma : K \to C([0,T]; C^1_{Lip}(H))$ be the mapping defined by the right hand side of Eq. (4.4). For the sake of simplicity and without no loss of generality we shall assume from now on that

(4.7) $\left| e^{-At} \right|_{L(H,H)} \leq 1 \quad , \quad t \in [0,T]$.

For any $E \in K$ we have (by direct calculation)

(4.8) $$\left| \Gamma_\varepsilon E(t) \right|_\infty \leq e^{-t/\varepsilon} \left| E_0 \right|_\infty + \int_0^t e^{-(t-s)/\varepsilon} \left| B(s) \right|_\infty ds + \varepsilon^{-1} \int_0^t e^{-(t-s)/\varepsilon} \left| D_\varepsilon(s) - F \right|_\infty ds \, , \quad t \in [0,T]$$

(4.9) $$\left\| \Gamma_\varepsilon E(t) \right\| \leq e^{-t/\varepsilon} \left\| E_0 \right\| + \int_0^t e^{-\varepsilon^{-1}(t-s)} \left\| B(s) \right\| ds + \varepsilon^{-1} \int_0^t e^{-\varepsilon^{-1}(t-s)} \left\| D_\varepsilon(s) - F \right\| ds ; \quad t \in [0,T] .$$

Recalling that

(4.10) $$D_\varepsilon - F = E(I + \varepsilon D)^{-1} + F((I + \varepsilon D)^{-1}) - F$$

and

(4.11) $$F((I + \varepsilon D)^{-1} x) - Fx = -\varepsilon \int_0^1 F' \left(\lambda (I + \varepsilon D)^{-1} x + (1 - \lambda) x \right) D_\varepsilon x \, d\lambda$$

we see by Lemma 1 that

(4.12) $$\left| D_\varepsilon - F \right|_\infty \leq \left| E \right|_\infty + \varepsilon \left\| F \right\| \left| D \right|_\infty .$$

(4.13) $$\left\| D_\varepsilon - F \right\| \leq \left\| E \right\| (1 - \varepsilon \left\| D \right\|)^{-1} + \varepsilon \left(\left| F' \right|_\infty \left\| D \right\| (1 - \varepsilon \left\| D \right\|)^{-1} + \left| D \right|_\infty \left\| F' \right\| \left((1 - \varepsilon \left\| D \right\|)^{-1} + 1 \right) \right) .$$

Om the other hand, we have

(4.14) $$(\Gamma_\varepsilon E)_x(t,x) = e^{-(\varepsilon^{-1} + A^*)t} E_0' (e^{-At} x) e^{-At} + \int_0^t e^{-(\varepsilon^{-1} + A^*)(t-s)} B_x(s, e^{-A(t-s)} x) e^{-A(t-s)} ds + \varepsilon^{-1} \int_0^t e^{-(\varepsilon^{-1} + A^*)(t-s)} ((D_\varepsilon)_x(s) - F') (e^{-A(t-s)} x) e^{-A(t-s)} x ds$$

while by (2.12)

(4.15) $$D_\varepsilon'(x) - F'(x) = D'((I + \varepsilon D)^{-1} x)(I + \varepsilon D'((I + \varepsilon D)^{-1} x))^{-1} - F'(x) .$$

This yields

$$(4.16)\ |D'_\varepsilon - F'|_\infty \leq |E'|_\infty (1-\varepsilon|D'|_\infty)^{-1} + \varepsilon(1-\varepsilon|D'|_\infty)^{-1} (\|F'\| \, |D|_\infty + |F'|_\infty |D'|_\infty).$$

Now using (4.11) and (4.15) we get after some calculation that

$$(4.17)\ \|D'_\varepsilon - F'\| \leq (1-r\varepsilon)^{-2}\|E'\| + \mu(r)\varepsilon$$

for all $E \in C^1_{Lip}(H)$ satisfying $|E|_{1,\infty} + \|E\| + \|E'\| \leq r$.
Here μ is a certain real continuous and positive function whose expression is to complicated to be written here.

Inserting (4.12), (4.13), (4.16) and (4.17) into (4.8), (4.9) and (4.14) respectively, we conclude that there exists $t_\varepsilon \in]0,T]$ such that

$$(4.18)\ |(\Gamma_\varepsilon E)(t)|_{1,\infty} + \|(\Gamma_\varepsilon E)(t)\| + \|(\Gamma_\varepsilon E)_x(t)\| \leq M; t \in [0,t_\varepsilon]$$

for all $E \in K$.
Now we recall that a mapping $E \in C^1(H)$ belongs to $P(H)$ if and only if E' is a self-adjoint linear operator on H. Inasmuch as E_o, F and $B(s)$ belong to $P(H)$ while by (4.15) $E_\varepsilon \in P(H)$ for any $E \in P(H)$, it follows from (4.14) that

$$(\Gamma_\varepsilon E)(t) \in P(H) \quad \text{for all } \varepsilon > 0, \quad t \in [0,T], \ E \in K.$$

Thus for $T = t_\varepsilon$, Γ_ε maps K into itself. On the other hand, for any pair $E_1, E_2 \in K$, by (2.14) and (4.14) we have

$$|(\Gamma_\varepsilon E_1)(t) - (\Gamma_\varepsilon E_2)(t)|_{1,\infty} \leq \varepsilon^{-1}\int_0^t e^{-\varepsilon^{-1}(t-s)} \left(|D^1_\varepsilon(s) - D^2_\varepsilon(s)|_\infty + |(D^1_\varepsilon)_x(s) - (D^2_\varepsilon)_x(s)|_\infty\right) ds \leq C(1-e^{-\varepsilon^{-1}t}) \sup |E_1(s) - E_2(s)|_{1,\infty};\ s \in [0,t_\varepsilon].$$

Hence for $T = t_\varepsilon$ sufficiently small Γ_ε is a contraction in $C([0,T]; C^1(H))$ and therefore Eq. (4.4) has a unique solution $E^\varepsilon \in K$. Moreover, by a standard extension argument we may infer that exists a maximal interval $[0,T_\varepsilon]$ such that Eq. (4.4) has a unique solution $E^\varepsilon \in K$.

Using Eq. (4.4) where $E = E^\varepsilon$ along with inequalities (4.12), (4.13), (4.16) and (4.17) one obtains for ε sufficiently small the following estimates:

$$(4.18)\quad |E^{\varepsilon}(t)|_{\infty} \leq e^{-\varepsilon^{-1}t}|E_o|_{\infty} + \varepsilon^{-1}\int_0^t e^{-\varepsilon^{-1}(t-s)}|E^{\varepsilon}(s)|_{\infty}ds + C\varepsilon(1-e^{-\varepsilon^{-1}t})\ ;\quad t\in[0,T]$$

$$(4.19)\quad \|E^{\varepsilon}(t)\| \leq e^{-\varepsilon^{-1}t}\|E_o\| + \varepsilon^{-1}(1-\tilde{M}\varepsilon)^{-1}\int_0^t e^{-\varepsilon^{-1}(t-s)}\|E^{\varepsilon}(s)\|ds + C\varepsilon(1-e^{-\varepsilon^{-1}t});\ t\in[0,T]$$

$$(4.20)\quad |E_x^{\varepsilon}(t)|_{\infty} \leq e^{-\varepsilon^{-1}t}|E_o'|_{\infty} + \varepsilon^{-1}(1-\tilde{M}\varepsilon)^{-1}\int_0^t e^{-\varepsilon^{-1}(t-s)}|E_x^{\varepsilon}(s)|_{\infty}ds + C\varepsilon(1-e^{-\varepsilon^{-1}t})\ ;\ t\in[0,T]$$

and

$$(4.21)\quad \|E_x^{\varepsilon}(t)\| \leq e^{-\varepsilon^{-1}t}\|E_o'\| + \varepsilon^{-1}(1-\tilde{M}\varepsilon)^{-2}\int_0^t e^{-\varepsilon^{-1}(t-s)}\|E_x^{\varepsilon}(s)\|ds + C\varepsilon(1-e^{-\varepsilon^{-1}t})\ ;\ t\in[0,T]$$

where $\tilde{M} = M + |F|_{1,\infty} + \|F\| + \|F'\|$ and C is independent of ε. Setting $\rho_{\varepsilon}(t) = |E^{\varepsilon}(t)|_{1,\infty} + \|E^{\varepsilon}(t)\| + \|E_x^{\varepsilon}(t)\|$ we have by (4.18), (4.19), (4.20) and (4.21)

$$\rho_{\varepsilon}(t) \leq 2^{-1}Me^{-\varepsilon^{-1}t} + \varepsilon^{-1}(1-\tilde{M}\varepsilon)^{-2}\int_0^t e^{-\varepsilon^{-1}(t-s)}\rho_{\varepsilon}(s)ds + C\varepsilon(1-e^{-\varepsilon^{-1}t})\ ;\ t\in[0,T],$$

and by Gronwall's lemma

$$\rho_{\varepsilon}(t) \leq C(1-e^{\delta_{\varepsilon}t}) + 2^{-1}Me^{\delta_{\varepsilon}t}\ ;\ t\in[0,T]$$

where $\delta_{\varepsilon} = (2\tilde{M} + \varepsilon\tilde{M}^2)(1-\tilde{M}\varepsilon)^2$.

Thus there is $T > 0$ independent of ε such that

$$\rho_{\varepsilon}(t) \leq 3M/4 \quad \text{for } 0\leq t\leq \inf\ (T,T_{\varepsilon}).$$

The latter clearly implies that $T_{\varepsilon} > T$ for all ε thereby completing the proof of Lemma 2.

LEMMA 3. <u>Let E_1 and E_2 be two solutions to Eq. (4.4) corresponding to $E_{o,1}, g_1$ and $E_{o,2}, g_2$ respectively. Then on the common interval of existence $[0,T]$ one has</u>

$$(4.22)\quad |E_1(t) - E_2(t)|_{\infty} \leq C(|E_{o,1} - E_{o,2}|_{\infty} + \sup\{|(g_1)_x(s) - (g_2)_x(s)|_{\infty}\ ;\ s\in[0,T]\})\ ;\ t\in[0,T]$$

where C is independent of ε.

Proof. We have

$$(4.23)\quad |E_1^{\varepsilon}(t) - E_2^{\varepsilon}(t)|_{\infty} \le e^{-\varepsilon^{-1}t}|E_{o,1} - E_{o,2}|_{\infty} +$$

$$+ \varepsilon^{-1}\int_0^t e^{-\varepsilon^{-1}(t-s)}|D_{\varepsilon}^1(s) - D_{\varepsilon}^2(s)|_{\infty}\, ds +$$

$$+ \int_0^t e^{-\varepsilon^{-1}(t-s)}|(g_1)_x(s) - (g_2)_x(s)|_{\infty}\, ds;\ t \in [0,T].$$

where $D_{\varepsilon}^i = D_i^{\varepsilon}(I + \varepsilon D_i^{\varepsilon})^{-1}$, $D_i^{\varepsilon} = E_i^{\varepsilon} + F$; $i = 1,2$.

Using once again inequality (2.10) and estimate (4.6) we get (4.22) by a simple calculation involving Gronwall's lemma.

Proof of Theorem 1 (continued) Let E^{ε} be the solution to Eq. (4.4) provided by Lemma 2. For any $E \in C^1_{Lip}(H)$ and $\varepsilon > 0$ define the mapping $R_{\varepsilon}(E)$: $H \to H$ by

$$(4.24)\quad R_{\varepsilon}(E)x = \varepsilon^{-1}(Ex - E_{\varepsilon}x) - E'(x)Ex = \int_0^1 (E'(\lambda x + (1-\lambda)(I + \varepsilon E)^{-1}x)\, E_{\varepsilon}x - E'(x)Ex)\, d\lambda .$$

We have,

$$(4.25)\quad |R_{\varepsilon}(E)|_{\infty} \le \varepsilon |E|_{\infty}(\|E\|\,|E'|_{\infty} + \|E'\|\,|E|_{\infty}).$$

Then by (4.6) it follows that for all $\varepsilon, \lambda > 0$ one has

$$(4.26)\quad |R_{\varepsilon}(E^{\lambda}(t))|_{\infty} \le C\varepsilon \qquad \forall\, t \in [0,T]$$

and by (4.16), (4.17) we have

$$(4.27)\quad |R_{\varepsilon}(D^{\varepsilon}(t))|_{\infty} \le C\varepsilon \qquad \forall\, t \in [0,T]$$

where C is some positive constant independent of ε and λ; $D^{\varepsilon} = E^{\varepsilon} + F$.

On the other hand, as is easily verified,

$$(4.28)\quad \frac{d}{dt}(E^{\varepsilon}(t,x),y) + (E^{\varepsilon}_x(t,x)y, Ax) + (E^{\varepsilon}(t,x), Ay) + $$

$$+ \varepsilon^{-1}(D(t,x) - D_{\varepsilon}^{\varepsilon}(t,x), y) = (B(t,x), y)$$

for all $x, y \in D(A)$ and $t \in [0,T]$.

To prove (4.28) we have used Eq. (4.4) along with the obvious relatio

(4.29) $\frac{d}{ds}(E(t,e^{-As}x),e^{-As}y) = -(E_x(t,e^{-As}x)Ae^{-As}x,e^{-As}y) - (E(t,e^{-As}y);x,y \in D(A),\ s \in [0,T]$

Let $\eta_\varepsilon : [0,T] \times [0,T] \to R$ be the function

$$\eta_\varepsilon(t,s) = (E^\varepsilon(t,e^{-As}x),e^{-As}y).$$

By (4.28) and (4.29) it follows that

(4.31) $\frac{\partial \eta_\varepsilon}{\partial t} - \frac{\partial \eta_\varepsilon}{\partial s} = (B(t,e^{-As}x) + \varepsilon^{-1}(D^\varepsilon_\varepsilon(t,e^{-As}x) - D^\varepsilon(t,e^{-As}x),e^{-As}y);$

$s,t \in [0,T]$.

Integrating equation (4.31) we obtain since $\eta_\varepsilon(0,s) = (E_0(e^{-As}x),e^{-As}y)$ and y,x are arbitrary,

(4.32) $E^\varepsilon(t,x) = e^{-A^*t}E_0(e^{-At}x) + \int_0^t e^{-A^*(t-s)}B(s,e^{-A(t-s)}x)ds + \varepsilon^{-1}\int_0^t e^{-A^*(t-s)}(D^\varepsilon_\varepsilon(s,e^{-A(t-s)}x) - D^\varepsilon(s,e^{-A(t-s)}x))ds$

for all $x \in H$ and $t \in [0,T]$.

Conversely, by the same argument we deduce that every solution to Eq.(4.32) is a solution to Eq.(4.4). This shows that for every $\lambda > 0$, E^λ is a solution to (4.4) where B has been replaced by $B + R_\varepsilon(D^\lambda) - R_\lambda(D^\lambda)$.

Then Lemma 3 along with estimate (4.27) yields

(4.33) $|E^\varepsilon(t) - E^\lambda(t)|_\infty \leq C \sup\{|R_\varepsilon(D^\lambda(s))|_\infty + |R_\lambda(D^\lambda(s))|_\infty ;\ s \in [0,T]\} \leq C(\varepsilon+\lambda)$ for $t \in [0,T]$.

Hence there exists $E = \lim_{\varepsilon\to 0} E^\varepsilon$ in $C([0,T]; C(H))$ and by Lemma 2 it follows that $E(t) \in P(H) \cap Lip(H)$ for all $t \in [0,T]$ and

$$\sup\{\|E(t)\| ;\ t \in [0,T]\} < \infty .$$

Let φ_ε and φ be the real valued functions defined by

$$\varphi_\varepsilon(t,x) = \int_0^1 (E^\varepsilon(t,\lambda x),x)d\lambda ;\ t \in [0,T] ,\ x \in H$$

and

$$\varphi(t,x) = \int_0^1 (E(t,\lambda x),x)d\lambda ;\ t \in [0,T] ,\ x \in H.$$

Since $(\varphi_\varepsilon)_x = E^\varepsilon$ and $\varphi_x = E$ we conclude that

(4.34) $\lim_{\varepsilon\to 0} \varphi_\varepsilon(t,x) = \varphi(t,x)$ uniformly on $[0,T] \times \Sigma_r$.

for any $r > 0$, and

$$(4.35) \quad \lim_{\varepsilon \to 0} (\varphi_\varepsilon)_x = \varphi_x \text{ in } C([0,T];C(H)).$$

Next in (4.28) we replace x by λx, y by x and integrate in λ over $[0,1]$. We get

$$(4.36) \quad (\varphi_\varepsilon)_t(t,x) + ((\varphi_\varepsilon)_x(t,x), Ax) + \varepsilon^{-1}\psi_\varepsilon(t,x) = g(t,x) + \frac{1}{2}|Fx|^2$$

for all $t \in [0,T]$ and $x \in D(A)$.

Here ψ_ε is the function defined by

$$(4.37) \quad \psi_\varepsilon(t,x) = \int_0^1 (D^\varepsilon - D^\varepsilon_\varepsilon)(t, \lambda x)x d\lambda .$$

On the other hand, it follows from (4.27) and (4.37) that

$$|\varepsilon^{-1}\psi_\varepsilon(t,x) - \int_0^1 D^\varepsilon_x(t,\lambda x)D^\varepsilon(t,\lambda x)d\lambda| \le \delta(\varepsilon)|x| \; ; \; t \in [0,T], x \in H$$

where $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$. Keeping in mind that

$$D^\varepsilon_x D^\varepsilon = \frac{1}{2}|D^\varepsilon|^2_x$$

we conclude that

$$|\varepsilon^{-1}\psi_\varepsilon(t,x) - \frac{1}{2}|E^\varepsilon(t,x) + Fx|^2| \le \delta(\varepsilon)|x|, \quad t \in [0,T], \; x \in H.$$

The latter inequality, when added to (4.36) and to (4.34), (4.36) shows that φ is a solution to (1.1), (1.2). As regards estimate (3.4) it follows by (4.18) via Gronwall's lemma thereby completing the proof of Theorem 1.

<u>Remark</u> 2 Let $\varphi(t,E_0,G)$; $E_0 = \varphi_0'$, $G = g_x$ be the solution to (1.1) (1.2) obtained by the above convergent process. By Lemma 3 it follows that the map $(E_0,G) \to (t,E_0,G)$ is locally Lipschitzian from $C(H) \times C([0,T];C(H))$ to $C([0,T] \times H;R)$.

Moreover, arguing as in the proof of Lemma 3 we see that also the map $F \to \varphi$ is locally Lipschitzian from $C^1_{Lip}(H)$ to $C([0,T] \times H;R)$. These facts could be used to prove that Theorem 1 remains true if instead of (3.1) and (3.2) one merely assumes that $\varphi_0' \in C(H) \cap Lip(H)$, $g_x \in C([0,T];C(H) \cap Lip(H))$ and $F \in C^1_{Lip}(H)$. However, in order to avoid a tedious and lengthy argument we did not put Theorem 1 into this general form.

5. Synthesis of optimal control

Consider here the control problem (1.3),(1.4): minimize

(5.1) $\int_0^T (g_o(t,x(t)) + \frac{1}{2}|u(t)|^2)dt + \varphi_o(x(T))$

in $u \in L^2(0,T;H)$ and $x \in C([0,T];H)$, subject to the constraints

(5.2) $x' + Ax + Fx = u \quad ; \quad t \in [0,T]$

(5.3) $x(o) = x_o$

(5.4) $|x(t)| \leq r \qquad$ for $t \in [0,T]$.

Here $x_o \in H$, and $r > |x_o|$ are arbitrary but fixed while φ_o, g_o are continuous real valued functions on H and $[0,T] \times H$, respectively, satisfying

(5.5) $\varphi_o' \in C^1_{Lip,loc}(H); \ (g_o)_x \in C([0,T] ; C^1_{Lip,loc}(H))$.

As regard the operators A and F we shall assume that they satisfy assumptions of Theorem 2, i.e., -A is the infinitesimal generator of a semigroup of class C_o and

(5.6) $F \in P(H) \ ; \ F, F' \in C^1_{Lip,loc}(H)$.

Also we shall assume that there exists some real ω such that

(5.7) $((A + F)x,x) \geq -\omega|x|^2$ for all $x \in D(A)$.

(Condition (5.7) is not absolutely necessary but it simplifies problem avoiding some tedious arguments.)

It is well known that under these assumptions, the Cauchy problem (5.2),(5.3) has a unique "mild" solution $x \in C([0,T];H)$, i.e.,

(5.8) $x(t) = e^{-At}x_o + \int_0^t e^{-A(t-s)}(u(s) - F(x(s)))ds \ ; t \in [0,T]$

Moreover, one has the following estimate

(5.9) $|x(t)| \leq e^{\omega t}|x_o| + \int_0^t e^{\omega(t-s)}|u(s)|ds \ ; \ 0 \leq t \leq T$.

We associate with control problem (5.1) the equation

$$(5.10)\quad \psi_t(t,x) - \tfrac{1}{2}|\psi_x(t,x)|^2 - (\psi_x(t,x), Ax + Fx) + g_o(t,x) = 0$$

$$\text{for } t \in [0,T], \quad x \in D(A); |x| < 2r$$

with final value condition

$$(5.11)\quad \psi(T,x) = \varphi_o(x) \ ; \quad |x| \leq 2r.$$

By substitution $\varphi(t,x) = \psi(T-t,x)$, problem (5.10),(5.11) reduces to (1.1),(1.2) where $g(t,x) = g_o(T-t,x)$. Then by Theorem 2 we may infer that there exists $T_r > 0$ such that for any $0 < T \leq T_r$ problem (5.10),(5.11) has at least one solution denoted ψ_T which satisfies

$$(\psi_T)_x \in C([0,T] \ ; C(H) \cap \mathrm{Lip}\,(H))$$

and

$$(5.12)\quad |(\psi_T)_x(t)|_{oo} \leq C_r \qquad \text{for } t \in [0,T]$$

where C_r is a positive constant which depends only on $|\varphi'_{o,2r}|_{oo}$, $|F'_{2r}|_{oo}$ and $|g'_{2r}|_{oo}$ (see (3.8) and (3.9)).

Now consider the closed loop system

$$(5.13)\quad \begin{aligned} & x' + Ax + (\psi_T)_x(t,x) = 0 \qquad t \in [0,T] \\ & x(o) = x_o . \end{aligned}$$

Since $(\psi_T)_x \in C([0,T] \ ; \mathrm{Lip}(H))$ it follows by (5.7) and (5.9) that (5.13) has a unique "mild" solution $x = x^*(t)$ which satisfies the estimate

$$(5.14)\quad |x^*(t)| \leq e^{\omega t}|x_o| + C_r\,\omega^{-1}(e^{\omega t} - 1) \ ; \ t \in [0,T].$$

We may therefore conclude that for every T satisfying

$$0 < T \leq \inf(T_r, \ \omega^{-1} \ln(C_r + \omega r)(C_r + \omega|x_o|)^{-1}$$

the solution x^* to (5.13) remains in the closed ball Σ_r, i.e.,

$$|x^*(t)| \leq r \quad \text{for } t \in [0,T].$$

The main result of this section, Theorem 3 below, simply says that under the above assumptions the feedback law

$$(5.16)\qquad u(t) = -(\psi_T)_x\,(t,x)$$

is optimal in problem (5.1).

THEOREM 3. Let Ψ_T be a solution to (5.10), (5.11) where T satisfies condition (5.15) and let x^* be the solution to (5.13). Then x^* is an optimal arc to problem (5.1) corresponding to optimal control $u = u^*$ given by

$$u^*(t) = -(\Psi_T)_x(t,x^*(t)) \qquad t \in [0,T]$$

Moreover, there exists $0 < \gamma(r) \leq r$ such that for all $t \in [0,T]$ and $h \in \Sigma_{\gamma(r)}$ one has

$$(5.17) \quad \Psi_T(t,h) = \inf \left\{ \int_t^T (g_0(s,x(s)) + \frac{1}{2}|u(s)|^2)ds + \varphi_0(x(T)); \right.$$

$$\left. x(t) = h; x' + Ax + Fx = u \text{ and } |x(s)| \leq r \text{ for } s \in [t,T] \right\}.$$

Proof. Let $u \in L^2(t,T;H)$ be arbitrary and let x be the corresponding "mild" solution to (5.13) on the interval $[t,T]$ with initial value condition $x(t) = h$ and such that $|x(s)| \leq r$ for $s \in [t,T]$. Let $\{u_\varepsilon\} \subset C^1([t,T];H)$ and $\{h_\varepsilon\} \subset D(A)$ be two sequences strongly convergent for $\varepsilon \to 0$ to u and h in $L^2(t,T;H)$ and H, respectively. Then the corresponding solutions x_ε are differentiable on $[t,T]$ and $x_\varepsilon \to x$ uniformly on $[t,T]$. In particular, it follows that $|x_\varepsilon(s)| \leq 2r$ for $s \in [0,T]$ and all ε sufficiently small. On the other hand, we have by (5.10) and (5.13)

$$\frac{d}{ds}\Psi_T(s,x_\varepsilon(s)) = (\Psi_T)_s(s,x_\varepsilon(s)) + ((\Psi_T)_x(s,x_\varepsilon(s)),x_\varepsilon'(s)) =$$

$$= -g_0(s,x_\varepsilon(s)) + \frac{1}{2}|(\Psi_T)_x(s,x_\varepsilon(s))|^2 - (u_\varepsilon(s),$$

$$(\Psi_T)_x(s,x_\varepsilon(s))) \geq -g_0(s,x_\varepsilon(s)) - \frac{1}{2}|u_\varepsilon(s)|^2.$$

Integrating on $[t,T]$ we get

$$\Psi_T(t,h) \leq \int_t^T (g_0(s,x_\varepsilon(s)) + \frac{1}{2}|u_\varepsilon(s)|^2)ds + \varphi_0(x_\varepsilon(T))$$

and letting ε tend to zero,

$$(5.18) \quad \Psi_T(t,h) \leq \int_t^T (g_0(s,x(s)) + \frac{1}{2}|u(s)|^2)ds + \varphi_0(x(T)).$$

Let us denote by $S(t,h)$ the optimal value of problem (5.1) on interval $[t,T]$, i.e, the right hand side of relation (5.17). By (5.18) it follows that

$$(5.19)\quad \psi_T(t,h) \le S(t,h) \quad \text{for all} \quad t \in [0,T],\ h \in \Sigma_r .$$

Now we fix h in $\Sigma_{\gamma(r)}$ where

$$(5.20)\qquad \gamma(r) = e^{-\omega T}(r + C_r\,\omega^{-1}) - C_r\,\omega^{-1}.$$

Then by (5.14) we see that the solution $x = y(s)$ to the Cauchy problem

$$y' + Ay + Fy + (\psi_T)_y(s,y) = 0\ ,\ t \le s \le T$$

$$y(t) = h$$

remains in Σ_r. Proceeding as above we may assume that y is differentiable on $[t,T]$ so that using once again equation (5.10) we have

$$\frac{d}{ds}\psi_T(s,y(s)) = (\psi_T)_s(s,y(s)) + ((\psi_T)_x(s,y(s))\ ,\ y'(s)) =$$

$$= g_o(s,y(s)) - \frac{1}{2}|(\psi_T)_y\ (s,y(s))|^2 \ \text{for}\ s \in [t,T].$$

Then integrating on $[t,T]$ we get

$$\psi_T(t,h) = \int_t^T (g_o(s,y(s)) + \frac{1}{2}|(\psi_T)_y(s,y(s))|^2)ds + \varphi_o(y(T)).$$

Along with (5.19) the latter yields (5.17). In particular for $t = 0$, $h = x_o$ and $y = x^*$ it follows by (5.17) that the pair (x^*, u^*) is optimal in problem (5.1). This completes the proof.

<u>Remark 3</u>. By (5.17) it follows that any solution ψ_T to the Hamilton Jacobi equation (5.10) is uniquely determined on $[0,T] \times \Sigma_{\gamma(r)}$ where $\gamma(r)$ is given by formula (5.20). As mentioned eqrlier this fact could be viewed as a uniqueness result for the Hamilton-Jacobi equation (1.1).

<u>Remark 4</u>. In particular, it follows by Theorem 3 that for a sufficiently small T the optimal control problem (5.1) has at least one optimal pair (x^*, u^*).

Theorem 3 admith a dual formulation in the sense that if T is given, then for F, g and φ_o "sufficiently small" there exists an optimal feedback law of the form (5.16) for the problem (5.1). To be more specific we consider the following optimal control problem: minimize.

$$(5.21) \quad \int_0^T \left(\lambda^2 g_o(t,x(t)) + \frac{1}{2} |u(t)|^2 \right) dt + \lambda \varphi_o(x(T))$$

in $u \in L^2(0,T;H)$ and $x \in C([0,T];H)$ subject to the constraints

$$(5.22) \quad \begin{aligned} & x' + Ax + \lambda Fx = u \qquad t \in [0,T] \\ & x(o) = x_o \\ & |x(t)| \leq r \qquad\qquad t \in [0,T] \end{aligned}$$

where $r > |x_o|$ is arbitrary but fixed, λ is a positive parameter and F, φ_o, g satisfy conditions (5.5), (5.6). In addition we shall strengthen condition (5.7) to

$$(5.23) \qquad (Ax + Fx, x) \geq 0 \qquad \forall \; x \in D(A).$$

COROLLARY 1. <u>Let</u> $[0,T]$ <u>be an arbitrary bounded interval. Then there exists</u> $\lambda_o > 0$ <u>such that for every</u> $0 < \lambda < \lambda_o$ <u>the conclusions of Theorem 3 remain valid for problem</u> (5.21), (5.22).

<u>Proof</u>. By (5.14) and (5.23) we see that Theorem 3 is applicable to problem (5.1) for all $T = T_o$ satisfying the condition

$$0 < T_o \leq \inf (T_r, (r - |x_o|) C_r^{-1}).$$

Since by substitution $t \to \lambda t$ problem (5.1) reduces to (5.21), (5.22) where $T = \lambda^{-1} T_o$, we may infer that all the conclusions of Theorem 3 are valid for the latter problem if

$$0 < \lambda \leq \lambda_o = T^{-1} \inf (T_r, (r - |x_o|) C_r^{-1}).$$

Thus the proof of Corollary 1 is complete.

Theorem 3 and Corollary 1 are in particular applicable in the case of distributed control problems governed by semilinear parabolic equations with smooth nonlinearities. A typical example is the following problem: minimize

$$(5.24) \quad \int_Q (h(t,x)\,|y(t,x) - y_d(t,x)|^2 + |u(t,x)|^2)\,dxdt + \\ + \int_\Omega \ell(x)\,|y(T,x) - y_1(x)|^2 dx$$

in $u \in L^2(Q)$ and $y \in L^2(0,T;H_0^1(\Omega) \cap H^2(\Omega))$, $y_t \in L^2(Q)$ subject to the constraints

$$(5.25) \quad \begin{aligned} & y_t - \Delta y + \beta(y) = u && \text{in } Q =]0,T[\times \Omega \\ & y = 0 && \text{in } \Sigma =]0,T[\times \Gamma \end{aligned}$$

$$(5.26) \quad \begin{aligned} & y(0,x) = y_o(x) && \text{a.e. } x \in \Omega \\ & \int_\Omega |y(t,x)|^2 \; dx \le r^2 && \text{for } t \in [0,T]\,. \end{aligned}$$

Here Ω is a bounded and open subset of the Euclidean space R^N, Γ is the boundary of Ω; $H_o^1(\Omega)$, $H^2(\Omega)$ are usual Sobolev spaces on Ω and β is a monotone function on real axis. The functions $y_o, y_1 \in L^2(\Omega)$ and $h \in L^\infty(Q)$, $\ell \in L^\infty(\Omega)$ are given.

In this case $H = L^2(\Omega)$, $A = -\Delta$; $D(A) = H_o^1(\Omega) \cap H^2(\Omega)$ and $(Fy)(x) = \beta(y(x))$ a.e. $x \in \Omega$, $y \in L^2(\Omega)$. Conditions (5.6) are satisfied if β is of class C^2 and $\beta^{(i)}$, $i = 0,1,2$ are of linear growth.

More general control problems of the form (5.24), (5.25) with nondifferentiable or even multivalued maximal monotone graphs can be treated as follows (For a direct treatment see [4], [5] [6]). Consider the control problem with performance criterion (5.24), constraint (5.26) and state equation

$$(5.27) \quad \begin{aligned} & y_t - \Delta y + \beta^\varepsilon(y) = u && \text{in } Q \\ & y = 0 && \text{in} \\ & y(0,x) = y_o(x) && x \in \Omega \end{aligned}$$

where

$$\beta^\varepsilon(r) = \int_{-\infty}^{\infty} \beta_\varepsilon(r - \varepsilon\theta)\,\rho(\theta)\,d\theta\,, \qquad \varepsilon > 0$$

and $\beta_\varepsilon = \varepsilon^{-1}(1 - (1 + \varepsilon\beta)^{-1})$. Here ρ is a C_o^∞ function on R such that $\rho > 0$ on $]-1,1[$, $\rho = 0$ for $|r| > 1$, $\rho(r) = \rho(-r)$ for all $r \in R$ and $\int_{-\infty}^{\infty} \rho(r)\,dr = 1$.

It is easily verified that control problem (5.24),(5.26),(5.27) has at least one optimal pair $(y_\varepsilon, u_\varepsilon)$. Moreover, by Lemma 5 in [4] we have for $\varepsilon \to 0$

(5.28) $u_\varepsilon \to u^*$ strongly in $L^2(Q)$

and

(5.29) $y_\varepsilon \to y^*$ strongly in $C([0,T];L^2(\Omega))$ and weakly in $L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))$

where (y^*,u^*) is an optimal pair of problem (5.24),(5.25),(5.26).

Let $S_\varepsilon :[0,T]\times H \to R$ and $S: [0,T]\times H \to R$ be the optimal value functions of problem (5.24)∼(5.26) and (5.24),(5.25),(5.27), respectively. By (5.28) and (5.29) it follows that

(5.20) $\lim_{\varepsilon \to 0} S_\varepsilon(t,h) = S(t,h)$

for all $t \in [0,T]$ and all h in the closed ball Σ_r of $L^2(\Omega)$.

Thus if ψ_ε is the solution to the corresponding Hamilton-Jacobi equation associated with problem (5.24),(5.27) we may regarde

$$u = -(\psi_\varepsilon)_y(t,y)$$

as an approximating optimal feedback law for problem (5.24)∼(5.26).

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