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by

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HERMITIAN GEOMETRY AND INVOLUTIVE ALGEBRAS

Mircea MARTIN

Let M be a connected complex manifold and let $\mathcal{H}(M)$ denote the category of all Hermitian holomorphic vector bundles over M, with real-analytic metrics.

A classical problem in Hermitian geometry, the so called equivalence problem, is to determine when two objects in $\mathcal{H}(M)$ are locally or globally equivalent.

Certain recent results obtained by M.J.Cowen and R.G.Douglas [4], pointed out relevant connections between the equivalence problem, and some important problems in the theory of holomorphic curves and in operator theory. Cowen and Douglas related the study of a class of bounded linear operators on a separable infinite dimensional Hilbert space, which possess an open connected set Ω of eigenvalues, and the study of holomorphic curves defined on an open connected subset Ω of the complex line, to the equivalence problem in $\mathcal{H}(\Omega)$, and they also formulated and proved a notable answer to the equivalence problem in $\mathcal{H}(\Omega)$. More precisely, if Ω is an open connected subset of the complex line, Cowen and Douglas defined the relation of pointwise equivalence in $\mathcal{H}(\Omega)$ and showed that two objects in $\mathcal{H}(\Omega)$ are locally equivalent if and only if they are pointwise equivalent.

The purpose of this paper is to prove that, with an appropiate definition of pointwise equivalence, an analogous result holds in general.

Many of the ideas in our paper are derived from [1].

In Section 1 we state the main technical result of the paper, Theorem A, and in Section 2 we prove Theorem A using some elementary facts of the theory of finite dimensional C^*-algebras. The proof is strongly influenced by [1].

In Section 3 we deal with linear connections on Hermitian vector bundles.

In Section 4 we give the definition of pointwise equivalence and we show, in Theorem B, that two pointwise equivalent C^∞-differentiable Hermitian vector bundle over a complex manifold M are locally equivalent on an open dense subset M_o of M.

Finally, a discussion of some applications of Theorem B is carried out in Section 5.

I am very grateful to Professor C. Apostol for numerous discussions on the subject of this paper, in particular for suggesting that the results of Cowen and Douglas might generalize to arbitrary complex manifolds.

1. INVOLUTIVE ALGEBRAS AND HERMITIAN VECTOR BUNDLES

Throughout this section, M will denote a finite-dimensional C^{∞}-differentiable connected real manifold, without boundary.

For each C^{∞}-differentiable Hermitian vector bundle E over M, of finite rank, we define an unital involutive algebra, denoted by $C^{\infty}(M,L(E))$. The main body of this section is a discussion of involutive subalgebras of a such algebra $C^{\infty}(M,L(E))$.

In order to state the main result of this section, Theorem A, we need some remarks regarding involutive unital algebras and Hermitian vector bundles.

1.1. Let A be an unital complex algebra.

A complex linear map ∂ from A to A is called a derivation on A if

$$\partial(aa') = (\partial a)a' + a(\partial a') \qquad (a, a' \in A). \tag{1.1}$$

The space $\mathfrak{X}(A)$ of all derivations on A is a complex Lie algebra, with the following bracket operation

$$[\partial, \partial'] = \partial \circ \partial' - \partial' \circ \partial \qquad (\partial, \partial' \in \mathfrak{X}(A)) \tag{1.2}$$

For each a in A we will denote by $\theta(a)$ the inner derivation defined by a as follows

$$\theta(a)a' = aa' - a'a \qquad (a' \in A) \tag{1.3}$$

For all a in A and ∂ in $\mathfrak{X}(A)$ we have

$$[\partial, \theta(a)] = \theta(\partial a) \tag{1.4}$$

An element a in A is called central, if $\theta(a)=0$. The set $\mathcal{C}$ of all central elements in A is a commutative subalgebra of A. If ∂ is in $\mathfrak{X}(A)$ and a is a central element, then from

(1.4) we infer that ∂a is also a central element, therefore we can define the complex-linear map

$$(1.5) \qquad \mathrm{res}: \mathfrak{X}(A) \longrightarrow \mathfrak{X}(\mathcal{C}), \quad (\mathrm{res}\,\partial)a = \partial a \qquad (\partial \in \mathfrak{X}(A),\ a \in \mathcal{C})$$

We remark that

$$(1.6) \qquad \mathrm{res}[\partial, \partial'] = [\mathrm{res}\,\partial, \mathrm{res}\,\partial'] \qquad (\partial, \partial' \in \mathfrak{X}(A))$$

It is immediate that $\mathfrak{X}(A)$ and A are modules over $\mathcal{C}$ and the map res is $\mathcal{C}$-linear. For a derivation ∂ in $\mathfrak{X}(A)$ we have res $\partial = 0$ if and only if ∂ is $\mathcal{C}$-linear. Note that all inner derivations are $\mathcal{C}$-linear.

1.2. Suppose that A is an unital involutive algebra. For each derivation ∂ on A we define a map $\partial^{\#}$ from A to A by the equations

$$(1.7) \qquad \partial^{\#} a = (\partial a^{*})^{*} \qquad (a \in A)$$

It is plain that $\partial^{\#}$ is a derivation on A, the map $\partial \to \partial^{\#}$ is conjugate-linear and, moreover, for all ∂, ∂' in $\mathfrak{X}(A)$ and a in A we have

$$(1.8) \qquad [\partial, \partial']^{\#} = [\partial^{\#}, \partial'^{\#}]$$

$$(1.9) \qquad (\partial^{\#})^{\#} = \partial$$

$$(1.10) \qquad \theta(a)^{\#} = -\theta(a^{*})$$

$$(1.11) \qquad \mathrm{res}(\partial^{\#}) = (\mathrm{res}\,\partial)^{\#}$$

1.3. Let M be a C^{∞}-differentiable real manifold. The set $C^{\infty}(M)$ of all complex-valued C^{∞}-differentiable functions on M is an unital commutative involutive algebra, using complex conjugation of functions as involution.

The space of derivations $\mathfrak{X}(C^{\infty}(M))$ is exactly the complex Lie algebra $\mathfrak{X}(M)$ of all complex vector fields on M.

1.4. Suppose that M is as above and let E be a C^{∞}-differentiable complex vector bundle over M, with the projection $\pi_E : E \to M$.

As usual, for each point p in M we denote by E_p the fiber $\pi_E^{-1}(p)$ over p.

A section of the bundle E over M is a C^{∞}-differentiable map σ from M to E, such that, $\sigma(p) \in Ep$, for all p in M. The space $C^{\infty}(M,E)$ of all sections of the bundle E over M has a natural structure of $C^{\infty}(M)$-module.

Let L(E) denote the C^{∞}-differentiable complex vector bundle over M, constructed such that

$$(1.12) \qquad L(E)_p = L(E_p)$$

for all p in M, where by $L(E_p)$ we denote the complex algebra of all linear operators from Ep to Ep.

If $T:M \longrightarrow L(E)$ is a section of the bundle L(E), there $T(p):Ep \to Ep$ is a linear operator, for all p in M.

The section T in $C^{\infty}(M,L(E))$ induces a $C^{\infty}(M)$-linear map from $C^{\infty}(M,E)$, which we also denote by T, defined by

$$(1.13) \qquad T:C^{\infty}(M,E) \to C^{\infty}(M,E), \ (T\sigma)(p) = T(p)(\sigma(p)) \quad (\sigma \in C^{\infty}(M,E), p \in M)$$

It is a standard fact that the space of all sections $C^{\infty}(M,L(E))$ is isomorphic as a $C^{\infty}(M)$-module to the space of all $C^{\infty}(M)$-linear maps from $C^{\infty}(M,E)$ to $C^{\infty}(M,E)$, therefore $C^{\infty}(M,L(E))$ is in a natural manner an unital complex algebra, with the identity denoted by I.

Identifying each λ in $C^{\infty}(M)$ with λI in $C^{\infty}(M,L(E))$ we obtain that $C^{\infty}(M)$ is a subalgebra of $C^{\infty}(M,L(E))$. We clearly have that $C^{\infty}(M)$ coincide with the center of $C^{\infty}(M,L(E))$.

In the following, we denote the space of derivations $\mathfrak{X}(C^{\infty}(M,L(E)))$ simply by $\mathfrak{X}(M,E)$. From 1.1 we have that $\mathfrak{X}(M,E)$ is a $C^{\infty}(M)$-module and we obtain the $C^{\infty}(M)$-linear map

$$\text{(1.14)} \qquad \text{res}: \mathfrak{X}(M,E) \longrightarrow \mathfrak{X}(M), \quad (\text{res}\,\partial)\lambda = \partial(\lambda I) \qquad (\lambda \in C^{\infty}(M)).$$

1.5. In order that $C^{\infty}(M,L(E))$ becomes an involutive algebra in a canonical manner, we assume that the bundle E is a Hermitian vector bundle, with a specified Hermitian structure

$$\text{(1.15)} \qquad \mu : C^{\infty}(M,E) \times C^{\infty}(M,E) \rightarrow C^{\infty}(M,E), \quad \mu(\sigma,\tau) = \langle \sigma, \tau \rangle .$$

Each fiber Ep is an inner product space, and the inner products vary in a C^{∞} fashion for p in M.

For each T in $C^{\infty}(M,L(E))$ we define T^{*} in $C^{\infty}(M,L(E))$ by the implicit equations

$$\text{(1.16)} \qquad \langle T\sigma, \tau \rangle = \langle \sigma, T^{*}\tau \rangle \qquad (\sigma, \tau \in C^{\infty}(M,E)$$

The map $T \longrightarrow T^{*}$ is an involution of the complex algebra $C^{\infty}(M,L(E))$, called the involution associated with the Hermitian structure .

Note that for each ∂ in $\mathfrak{X}(M,E)$ and T in $C^{\infty}(M,L(E))$ we have

$$\text{(1.17)} \qquad \langle (\partial^{\#}T)\sigma, \tau \rangle = \langle \sigma, (\partial T^{*})\tau \rangle \qquad (\sigma, \tau \in C^{\infty}(M,E)$$

1.6. In what follows we are interested in subalgebras of $C^{\infty}(M,L(E))$, where E is, as above, a Hermitian vector bundle over the manifold M.

Let $\mathcal{S}$ be a fixed subset of $C^{\infty}(M,L(E))$ and let $\mathcal{X}$ be a fixed subset of $\mathcal{X}(M,E)$.

Using the notations

(1.18) $$\mathcal{X}\mathcal{S} = \{\partial s : s \in \mathcal{S}, \partial \in \mathcal{X}\}$$

(1.19) $$\mathcal{X}^{\#} = \{\partial^{\#} : \partial \in \mathcal{X}\}$$

we define two collections of subsets of $C^{\infty}(M,L(E))$, $(\mathcal{S}_k)_k$ and $(\mathcal{T}_k)_k$, where $0 \leq k$ is an integer

$$\mathcal{S}_o = \mathcal{S}$$

$$\mathcal{T}_k = \mathcal{S}_k \cup \mathcal{X}\mathcal{S}_k \cup \mathcal{X}^{\#}\mathcal{S}_k \qquad (0 \leq k)$$

$$\mathcal{S}_{k+1} = \mathcal{T}_k \cup \mathcal{X}\mathcal{X}^{\#}\mathcal{S}_k \cup \mathcal{X}^{\#}\mathcal{X}\mathcal{S}_k \qquad (0 \leq k)$$

Let A_k and B_k denote the involutive subalgebras of $C^{\infty}(M;L(E))$ generated in $C^{\infty}(M,L(E))$ by the sets $\mathcal{S}_k \cup C^{\infty}(M)$ and, respectively, $\mathcal{T}_k \cup C^{\infty}(M)$, $k=0,1,2,\ldots$. The union $A_{\infty} = \bigcup_{0 \leq k} A_k$ is obviously an involutive algebra and a $C^{\infty}(M)$-module. We remark that A_{∞} is the smallest involutive sublagebra of $C^{\infty}(M,L(E))$ such that

(1.21)
$$\mathcal{S} \cup C^{\infty}(M) \subseteq A$$
$$\mathcal{X} A_{\infty} \subseteq A$$

We have

(1.22)
$$A_k \cup \mathcal{X}A_k \cup \mathcal{X}^{\#}A_k \subseteq B_k$$
$$B_k \cup \mathcal{X}\mathcal{X}^{\#}A_k \cup \mathcal{X}^{\#}\mathcal{X}A_k \subseteq A_{k+1}$$

If M_o is an open subset of M, we shall denote by E/M_o the restriction of the bundle E to M_o. For all T in $C^{\infty}(M,L(E))$ and ∂ in $\mathfrak{X}(M,E)$, there exist the well-defined restriction T/M_o in $C^{\infty}(M_o,L(E|M_o))$ and $\partial|M_o$ in $\mathfrak{X}(M_o,E|M_o)$. We shall use the notations

$$\mathfrak{X}|M_o=\{\partial|M_o:\ \partial\in\mathfrak{X}\}$$

$$A_k|M_o=\{T|M_o:T\in A_k\}$$

$$B_k|M_o=\{T|M_o:T\in B_k\}$$

$$A_{\infty}|M_o=\{T|M_o:T\in A_{\infty}\}$$

Now, we can state the main result of this section.

1.7. THEOREM A. Suppose that E is a C^{∞}-differentiable Hermitian vector bundle over M of rank n, and let $\mathcal{S}$ and $\mathfrak{X}$ be as above. Then there exist an open nonempty subset M_o of M and an integer $1\leq k\leq \max(1,n-1)$, with the properties

(i) $A_k|M_o=A_{\infty}|M_o$

(ii) if A is an unital involutive algebra and

$$\psi:A_k|M_o\to A,\quad \varphi:\mathfrak{X}\cup\mathfrak{X}^{\#}|M_o\to\mathfrak{X}(A)$$

are, respectively, a morphism of unital complex algebras and a map, such that

$$\psi(\partial T|M_o)=\varphi(\partial|M_o)\,\psi(T|M_o),\quad \psi(\partial^{\#}T|M_o)=\varphi(\partial^{\#}|M_o)\,\psi(T|M_o)$$

$$\psi(\partial\partial'^{\#}T(M_o)=\varphi(\partial|M_o)\varphi(\partial'^{\#}|M_o)\,\psi(T|M_o)$$

$$\psi(\partial^{\#}\partial'T|M_o)=\varphi(\partial^{\#}|M_o)\,\varphi(\partial'|M_o)\,\psi(T|M_o)$$

for all ∂, ∂' in $\mathfrak{X}$ and T in A_{k-1}, then we have

$$\psi(\partial T|M_o) = \varphi(\partial|M_o)\psi(T|M_o), \quad \psi(\partial^{\#}T|M_o) = \varphi(\partial^{\#}|M_o)\psi(T|M_o)$$

for all ∂ in $\mathfrak{X}$ and T in A_{∞}.

2. THE PROOF OF THEOREM A

Throughout this section we continue with the notations of the previous section.

For each p in M let us consider the finite dimensional C^*-algebras defined by

$$A_\ell(p)=\{T(p):\ T\in A_\ell\} \qquad (0\leq\ell)$$

$$B_\ell(p)=\{T(p):\ T\in B_\ell\} \qquad (0\leq\ell)$$

$$A_\infty(p)=\{T(p):\ T\in A_\infty\}$$

2.1. We first review some elementary facts about the structure of finite dimensional C^*-algebras. For more details the reader is advised to consult [9 , Ch. I-11].

Let A be a finite dimensional C^*-algebra. Then there exist an uniquely determined sequence of positive integers,

$$\{n_1,n_2,\ldots,n_m\}$$

and a system of orthogonal selfadjoint central projections in A,

$$\{q_1,q_2,\ldots,q_m\}$$

such that A is decomposed into the direct sum

$$A=Aq_1\oplus Aq_2\oplus\ldots\oplus Aq_m \qquad (2.1)$$

and each Aq_i is isomorphic to the algebra of all $n_i \times n_i$ matrices, $1\leq i\leq m$.

For each i, $1\leq i\leq m$, we can find a system of orthogonal selfadjoint minimal projections in Aq_i ,

$$\{p^i_\alpha:\ 1\leq\alpha\leq n_i\}$$

and a system of elements of Aq_i

$$\{u^i_{\alpha\beta} : 1 \leq \alpha, \beta \leq n_i\}$$

satisfying the conditions

$$\sum_\alpha p^i_\alpha = q_i \tag{2.2}$$

$$u^i_{\alpha\alpha} = p^i_\alpha \tag{2.3}$$

$$u^{i\,*}_{\alpha\beta} = u^i_{\beta\alpha} \tag{2.4}$$

$$u^i_{\alpha\beta} u^i_{\gamma\delta} = \Delta_{\beta\gamma} u^i_{\alpha\delta} \tag{2.5}$$

for all $1 \leq \alpha, \beta, \gamma, \delta \leq n_i$, where $\Delta_{\beta\gamma}$ means the Kronecker symbol.

If a is an element of A, then there exists an uniquely determined collection of complex numbers

$$\{\lambda^i_{\alpha\beta}(a) : 1 \leq i \leq m, \quad 1 \leq \alpha, \beta \leq n_i\}$$

such that

$$p^i_\alpha a p^i_\beta = \lambda^i_{\alpha\beta}(a) u^i_{\alpha\beta} \qquad (1 \leq \alpha, \beta \leq n_i) \tag{2.6}$$

We remark that

$$a = \sum_{i=1}^{m} \sum_{\alpha,\beta=1}^{n_i} \lambda^i_{\alpha\beta}(a) u^i_{\alpha\beta} \tag{2.7}$$

2.2. The next result is a restatement of [4] , Lemma 3.4 .

LEMMA. Let T be a selfadjoint element of $C^\infty(M, L(E))$. Then there exist an open nonempty subset M_o of M, a positive integer m and two collections

$$\{P_\alpha : 1 \le \alpha \le m\} \subset C^\infty(M_o, L(E|M_o))$$

$$\{\lambda_\alpha : 1 \le \alpha \le m\} \subset C^\infty(M_o)$$

with the properties

(i) $P_\alpha = P_\alpha^* = P_\alpha P_\alpha$ $\qquad (1 \le \alpha \le m)$

(ii) $I|M_o = \sum_\alpha P_\alpha$

where I is the identity of $C^\infty(M, L(E))$.

(iii) $T|M_o = \sum_\alpha \lambda_\alpha P_\alpha$

(iv) each P_α, $1 \le \alpha \le m$, is an element of the subalgebra generated in $C^\infty(M_o, L(E|M_o))$ by $C^\infty(M_o) \cup \{T|M_o\}$.

2.3. LEMMA. Let $(A_k)_k$, A_∞ and $\mathfrak{X}$ be as in 1.6. If M_o is an open subset of M and k is an integer such that

$$\mathfrak{X} A_k|M_o \subset A_k|M_o \tag{2.8}$$

then we have

$$A_k|M_o = A_\infty|M_o$$

PROOF. From (2.8) we obtain

$$A_k \cup \mathfrak{X} A_k \cup \mathfrak{X}^{\#} A_k \cup \mathfrak{X}\mathfrak{X}^{\#} A_k \cup \mathfrak{X}^{\#}\mathfrak{X} A_k \,|\, M_o \subseteq A_k|M_o$$

hence

$$A_{k+1}|M_o = A_k|M_o \tag{2.9}$$

and similarly we get

$$A_{k+\ell}|M_o = A_k|M_o \qquad (0 \le \ell) \tag{2.10}$$

Therefore, it suffices to remark that

$$A_\infty|M_o = \bigcup_{0 \le \ell} A_{k+\ell}|M_o \; .$$

2.4. LEMMA. Let A be an involutive algebra and let v, w in A be given such that vwv=v. Then for each derivation δ on A we have

$$\delta v = v(\delta f) + (\delta e)v - v(\delta w)v \tag{2.11}$$

where f=wv and e=vw.

PROOF. Since vf=ev=v we obtain

$$\begin{aligned} v(\delta f)+(\delta e)v &= v(\delta w)v+vw(\delta v)+(\delta e)v= \\ &= v(\delta w)v+e(\delta v)+(\delta e)v= \\ &= v(\delta w)v+\delta v. \end{aligned}$$

2.5. Our proof of Theorem A divides naturally into four parts.

STEP 1. For each p in M we clearly have

$$A_{\ell}(p) \subseteq B_{\ell}(p) \subseteq A_{\ell+1}(p) \subseteq A_{\infty}(p) \subseteq L(E_p) \qquad (0 \leq \ell) \tag{2.12}$$

From (2.12), applying 2.1 and by a repeated use of Lemma 2.2, we can find:

(i) an open nonempty subset M_o of M

(ii) an integer k with $1 \leq k \leq \max(1, n-1)$

(iii) a sequence of positive integers

$$\{n_1, n_2, \ldots, n_m\}$$

and a system of orthogonal selfadjoint central projections in $B_{k-1}|M_o$

$$\{Q_1, Q_2, \ldots, Q_m\}$$

such that, for each p in M_o, the algebra $B_{k-1}(p)$ is decomposed into the direct sum

(2.13) $B_{k-1}(p)=B_{k-1}(p)Q_1(p) \oplus B_{k-1}(p)Q_2(p) \oplus \ldots \oplus B_{k-1}(p)Q_m(p)$

and each $B_{k-1}(p)Q_i(p)$ is isomorphic to the algebra of all $n_i \times n_i$ matrices, $1 \leq i \leq m$.

(iv) moreover, for each i, $1 \leq i \leq m$, there exist a system of orthogonal selfadjoint projections in $A_{k-1}|M_o Q_i$,

$$\{P^i_\alpha : 1 \leq \alpha \leq n_i\}$$

and a system of elements of $B_{k-1}|M_o Q_i$,

$$\{U^i_{\alpha\beta} : 1 \leq \alpha, \beta \leq n_i\}$$

such that, all $P^i_\alpha(p)$ are minimal projections in $A_k(p)Q_i(p)$ for all p in M_o, $1 \leq \alpha \leq n_i$, and the following conditions are satisfied

(2.14) $$\sum_\alpha P^i_\alpha = Q_i$$

(2.15) $$U^i_{\alpha\alpha} = P^i_\alpha$$

(2.16) $$U^{i\,*}_{\alpha\beta} = U^i_{\beta\alpha}$$

(2.17) $$U^i_{\alpha\beta}\, U^i_{\gamma\delta} = \Delta_{\beta\gamma} U^i_{\alpha\delta} .$$

STEP 2. We clearly have, from (2.14), that all projections Q_i, $1 \leq i \leq m$ are elements of $A_{k-1}|M_o$.

Let ∂ be a derivation in $\mathfrak{X} \cup \mathfrak{X}^\#$. Since Q_i is central in $B_{k-1}|M_o$, we obtain

(2.18) $$\partial Q_i = \partial(Q_i Q_i) = (\partial Q_i)Q_i + Q_i(\partial Q_i) = 2Q_i(\partial Q_i)Q_i = 0$$

From the last equality it is easy to check that

$$Q_i T = T Q_i$$

for all T in the set $B_{k-1} \cup \mathfrak{X} B_{k-1} \cup \mathfrak{X}^\# B_{k-1}|M_o$.

In particular, it follows that the projections Q_i, $1 \leq i \leq m$, are central projections in $A_k|M_o$.

Since the projections $P^i_\alpha(p)$ are minimal in $A_k(p)Q_i(p)$ for all p in M_o and $1\leq\alpha\leq n_i$, we obtain that for each T in $A_k|M_o$ there exists an uniquely determined collection of functions

$$\{\lambda^i_{\alpha\beta}(T): 1\leq i\leq m,\ 1\leq\alpha,\beta\leq n_i\}\subset C^\infty(M_o)$$

such that

$$P^i_\alpha T P^i_\beta = \lambda^i_{\alpha\beta}(T)U^i_{\alpha\beta} \tag{2.19}$$

$$T=\sum_{i=1}^{m}\ \sum_{\alpha,\beta=1}^{n_i}\lambda^i_{\alpha\beta}(T)U^i_{\alpha\beta} \tag{2.20}$$

STEP 3. To conclude the proof of Theorem A it sufficies to prove that we have

$$\partial U^i_{\alpha\beta}\in A_k|M_o \tag{2.21}$$

$$\psi(\partial U^i_{\alpha\beta})=\varphi(\partial)\psi(U^i_{\alpha\beta}) \tag{2.22}$$

$$\psi(\partial^\# U^i_{\alpha\beta})=\varphi(\partial^\#)\psi(U^i_{\alpha\beta})$$

for all $1\leq i\leq m$, $1\leq\alpha,\beta\leq n_i$ and ∂ in $\mathfrak{X}|M_o$.

Indeed, (i) of our theorem will follow by Lemma 2.3 from (2.21) and (2.20), whence (ii) will be a consequence of (2.22) and (2.20).

Let us consider the subsets of $B_{k-1}|M_o$ defined by

$$\mathcal{G}_i=\{P^i_\alpha R(\partial S)TP^i_\beta : 1\leq\alpha,\beta\leq n_i,\ R,S,T\in A_{k-1}|M_o,\ \partial\in\mathfrak{X}\cup\mathfrak{X}^\#\}$$

where $1\leq i\leq m$.

If p is a point in M_o, then each $U^i_{\alpha\beta}(p)$ is a finite product of elements belonging to $\mathcal{G}_i$, evaluated at p. Therefore, eventually decreasing M_o, we may suppose that any $U^i_{\alpha\beta}$ is a finite product of $U^i_{\gamma\delta}$'s belonging to $\mathcal{G}_i$. Thus we are allowed to prove (2.21) and (2.22) assuming that $U^i_{\alpha\beta}$ is an element of $\mathcal{G}_i$.

STEP 4. We first assume that

$$U^i_{\alpha\beta} = P^i_\alpha R(\partial_o S) T P^i_\beta \tag{2.23}$$

where R, S and T are in $A_{k-1}|M_o$ and ∂_o is in $\mathfrak{X}$.

Let ∂ be a derivation in $\mathfrak{X}$. We derive easily that

$$\partial^\# U^i_{\alpha\beta}\ ,\ \partial U^{i*}_{\alpha\beta} \in A_k|M_o \tag{2.24}$$

$$\psi(\partial^\# U^i_{\alpha\beta}) = \varphi(\partial^\#)\psi(U^i_{\alpha\beta}) \tag{2.25}$$

$$\psi(\partial U^{i*}_{\alpha\beta}) = \varphi(\partial)\psi(U^{i*}_{\alpha\beta})$$

Putting in Lemma 2.4

$$\delta = \partial\ ,\ v = U^i_{\alpha\beta}\ ,\ w = U^{i*}_{\alpha\beta}\ ,\ e = P^i_\alpha\ ,\ f = P^i_\beta$$

we have

$$\partial U^i_{\alpha\beta} = U^i_{\alpha\beta}(\partial P^i_\beta) + (\partial P^i_\alpha)U^i_{\alpha\beta} - U^i_{\alpha\beta}(\partial U^{i*}_{\alpha\beta})U^i_{\alpha\beta} \tag{2.26}$$

If we put in Lemma 2.4

$$\delta = \varphi(\partial)\ ,\ v = \psi(U^i_{\alpha\beta})\ ,\ w = \psi(U^{i*}_{\alpha\beta})\ ,\ e = \psi(P^i_\alpha)\ ,\ f = \psi(P^i_\beta)$$

then, since $\varphi(\partial) \in \mathfrak{X}(A)$, we obtain

$$\begin{aligned}\varphi(\partial)\psi(U^i_{\alpha\beta}) = {} & \psi(U^i_{\alpha\beta})(\varphi(\partial)\psi(P^i_\beta)) + (\varphi(\partial)\psi(P^i_\alpha))\psi(U^i_{\alpha\beta}) - \\ & -\psi(U^i_{\alpha\beta})(\varphi(\partial)\psi(U^{i*}_{\alpha\beta}))\psi(U^i_{\alpha\beta})\end{aligned} \tag{2.27}$$

From (2.26) and (2.24) it follows that

$$\partial U^i_{\alpha\beta} \in A_k|M_o \tag{2.28}$$

On the other hand, from (2.27), (2.25) and (2.26), under our assumptions we have

$$\varphi(\partial)\psi(U^{i}_{\alpha\beta})=\psi(\partial U^{i}_{\alpha\beta}) \tag{2.29}$$

The equalities (2.21) and (2.22) are contained in (2.28), (2.29) and the first part of (2.25).

For the second case, when

$$U^{i}_{\alpha\beta}=P^{i}_{\alpha}R(\partial^{\#}_{o}S)TP^{i}_{\beta} \tag{2.30}$$

we proceed analogously.

The proof of Theorem A is complete.

3. LINEAR CONNECTIONS

In this section we shall be concerned with linear connections on C^{∞}-differentiable vector bundles.

3.1. Let E be a C^{∞}-differentiable Hermitian vector bundle over the C^{∞}-differentiable manifold M, with the Hermitian structure μ .

By a linnear connection on the bundle E we mean a complex bilinear map

$$\Gamma : \mathfrak{X}(M) \times C^{\infty}(M,E) \longrightarrow C^{\infty}(M,E)$$

such that

$$\Gamma(\lambda X,\sigma)=\lambda\Gamma(X,\sigma) \tag{3.1}$$

$$\Gamma(X,\lambda\sigma)=X(\lambda)\sigma+\lambda\Gamma(X,\sigma) \tag{3.2}$$

for all X in $\mathfrak{X}(M)$, σ in $C^{\infty}(M,E)$ and λ in $C^{\infty}(M)$.

We say that the linear connection Γ preserves the Hermitian structure μ on E, if we have

$$X\langle\sigma,\tau\rangle=\langle\Gamma(X,\sigma),\tau\rangle+\langle\sigma,\Gamma(X^{\#},\tau)\rangle \tag{3.3}$$

for all X in $\mathfrak{X}(M)$ and σ, τ in $C^{\infty}(M,E)$.

In general, there exist many linear connections on the Hermitian bundle E which preserve the metric μ .

In what follows we fixe a such linear connection Γ and for each X in $\mathfrak{X}(M)$ let $\Gamma(X)$ be the complex linear map defined by

$$\Gamma(X):C^{\infty}(M,E)\to C^{\infty}(M,E), \quad \Gamma(X)\sigma=\Gamma(X,\sigma) \tag{3.4}$$

Let us denote by K the map

$$(3.5)\qquad K:\mathfrak{X}(M)\times\mathfrak{X}(M) \longrightarrow C^{\infty}(M,L(E))$$

$$(3.6)\qquad K(X,Y)=\Gamma(X)\circ\Gamma(Y)-\Gamma(Y)\circ\Gamma(X)-\Gamma([X,Y]) \qquad (X,Y\in\mathfrak{X}(M))$$

It is plain that K is a well-defined $C^{\infty}(M)$-bilinear map and, moreover, we have

$$(3.7)\qquad K(X,Y)=-K(Y,X)$$

$$(3.8)\qquad K(X,Y)^{*}=K(Y^{\#},X^{\#})$$

for all X,Y in $\mathfrak{X}(M)$.

The map K will be called the curvature of the linear connection Γ.

3.2. Let Γ be as above. The connection Γ can be used to construct a linear connection on the bundle L(E),

$$\nabla : \mathfrak{X}(M)\times C^{\infty}(M,L(E)) \longrightarrow C^{\infty}(M,L(E))$$

as follows

$$(3.9)\qquad \nabla(X,T)=\Gamma(X)\circ T-T\circ\Gamma(X) \qquad (X\in\mathfrak{X}(M),\ T\in C^{\infty}(M,L(E))).$$

It is easy to check that ∇ is a linear connection on L(E).

More important is the fact that for each X in $\mathfrak{X}(M)$, the map $\nabla(X)$ defined by

$$(3.10)\qquad \nabla(X):C^{\infty}(M,L(E)) \longrightarrow C^{\infty}(M,L(E)), \qquad \nabla(X)T=\nabla(X,T)$$

is a derivation on the involutive algebra $C^{\infty}(M,L(E))$ and, therefore, ∇ induces a $C^{\infty}(M)$-linear map from $\mathfrak{X}(M)$ to $\mathfrak{X}(M,E)$.

Note that for all X in $\mathfrak{X}(M)$ we have

$$(3.11)\qquad \operatorname{res}\ \nabla(X)=X$$

(3.12) $\nabla(X)^{\#} = \nabla(X^{\#})$

We also obtain

(3.13) $[\nabla(X), \nabla(Y)] - \nabla([X,Y]) = \theta(K(X,Y))$

for all X,Y in $\mathfrak{X}(M)$, where by $\theta(K(X,Y))$ we denote, as in 1.1, the inner derivation on $C^{\infty}(M,L(E))$ defined by $K(X,Y)$.

3.3. Let E and Γ be as above and let M_o be an open subset of M. The linear connection Γ induces a metric-preserving linear connection $\Gamma|M_o$ on the bundle $E|M_o$.

Assume that $\{\sigma_\alpha : 1 \leq \alpha \leq n\}$ is an orthonormal frame of E on M_o, that is

(i) $\{\sigma_\alpha : 1 \leq \alpha \leq n\}$ is a subset of $C^{\infty}(M_o, E|M_o)$

(ii) for each point p in M_o the values $\{\sigma_\alpha(p) : 1 \leq \alpha \leq n\}$ form an orthonormal basis for the fiber E_p.

Then there exists a collection of differential one-forms on M_o

$$\{\eta_{\alpha\beta} : 1 \leq \alpha, \beta \leq n\}$$

such that

(3.14) $$\Gamma|M_o(X, \sigma_\beta) = \sum_\alpha \eta_{\alpha\beta}(X)\sigma_\alpha \qquad (1 \leq \beta \leq n)$$

for all X in $\mathfrak{X}(M_o)$.

From (3.14) we obtain that there exists a collection of differential two-forms on M_o

$$\{\omega_{\alpha\beta} : 1 \leq \alpha, \beta \leq n\}$$

such that

$$K|M_o(X,Y)\sigma_\beta = \sum_\alpha \omega_{\alpha\beta}(X,Y)\sigma_\alpha \qquad (1 \le \beta \le n) \tag{3.15}$$

for all X,Y in $\mathfrak{X}(M_o)$, where by $K|M_o$ we denote the curvature of the linear connection $\Gamma|M_o$.

Each two form $\omega_{\alpha\beta}$ can be obtained from the one-forms $\{\eta_{\alpha\beta} : 1 \le \alpha, \beta \le n\}$ using the exterior derivative d and exterior products. For example, if we have

$$\eta_{\alpha\beta} = \Delta_{\alpha\beta}\eta \qquad (1 \le \alpha, \beta \le n) \tag{3.16}$$

where η is an one-form on M_o and $\Delta_{\alpha\beta}$ are the Kronecker symbols, then we find

$$\omega_{\alpha\beta} = \Delta_{\alpha\beta}\, d\eta \tag{3.17}$$

3.4. The next lemma is a converse of this last remark. The result is a simple extension of [4], Lemma 3.2.

LEMMA. If Γ is a metric-preserving linear connection on E and the curvature K is of the form

$$K(X,Y) = \omega(X,Y)I \qquad (X,Y \in \mathfrak{X}(M)) \tag{3.18}$$

where ω is a two-form on M, then for each point p in M there exist

(i) an open neighborhood M_o of p

(ii) a differential one-form η on M_o

(iii) an orthonormal frame $\{\sigma_\alpha : 1 \le \alpha \le n\}$ of E on M_o such that

$$\Gamma|M_o(X,\sigma_\alpha) = \eta(X)\sigma_\alpha \qquad (1 \le \alpha \le n) \tag{3.19}$$

for all X in $\mathfrak{X}(M_o)$.

Moreover, the choice of η depends only on ω.

PROOF. Let ∇ be the linear connection on L(E) defined by the equations (3.9).

Using (3.9) and (3.6) we have in $\mathfrak{X}(M,E)$

(3.20) $\quad \mathfrak{S}\,[\nabla(X),\theta(K(Y,Z))] = \mathfrak{S}\,\theta(K([X,Y],Z))$

for all X,Y,Z in $\mathfrak{X}(M)$, where $\mathfrak{S}$ denotes the cyclic sum with respect to X,Y and Z.

From (3.20) and (3.18) we obtain that ω is a closed two-form.

Since $\omega(X,Y)^* = -\omega(X^{\#},Y^{\#})$ for all X,Y in $\mathfrak{X}(M)$, we have that locally there exists an one-form η such that

(3.21) $\quad d\eta = \omega$

(3.22) $\quad \eta(X)^* = -\eta(X^{\#})$

The rest of the proof is a consequence of the Frobenius theorem and is similar to the proof of [4], Lemma 3.2.

3.5. Let n be a positive integer and let $\mathcal{HC}_n(M)$ denote the collection of all pairs (E,Γ) where E is a C^∞-differentiable Hermitian vector bundle over M of rank n and Γ is a metric-preserving linear connection on E.

DEFINITION. (i) Two objects (E,Γ) and $(\tilde{E},\tilde{\Gamma})$ in $\mathcal{HC}_n(M)$ are called equivalent in $\mathcal{HC}_n(M)$ if and only if there exists a $C^\infty(M)$-linear map

$$U: C^\infty(M,E) \longrightarrow C^\infty(M,E)$$

such that

$$\langle \sigma, \tau \rangle = \langle U\sigma, U\tau \rangle^{\sim} \tag{3.23}$$

$$U \circ \Gamma(X) = \tilde{\Gamma}(X) \circ U \tag{3.24}$$

for all σ, τ in $C^{\infty}(M,E)$ and X in $\mathfrak{X}(M)$.

Each map U with the properties (3.23) and (3.24) will be called an equivalence between (E,Γ) and $(\tilde{E},\tilde{\Gamma})$.

(ii) Two objects (E,Γ) and $(\tilde{E},\tilde{\Gamma})$ in $\mathcal{HC}_n(M)$ are called locally equivalent in $\mathcal{HC}_n(M)$ if and only if there exists an open covering (M_i) of M such that $(E|M_i, \Gamma|M_i)$ and $(\tilde{E}|M_i, \tilde{\Gamma}|M_i)$ are equivalent in $\mathcal{HC}_n(M_i)$ for each open subset M_i.

3.6. The following lemma is a direct consequence of Lemma 3.4.

LEMMA. Let (E,Γ) and $(\tilde{E},\tilde{\Gamma})$ two elements of $\mathcal{HC}_n(M)$ and let ω be a differential two-form on M such that

$$K(X,Y)=\omega(X,Y)I, \quad \tilde{K}(X,Y)=\omega(X,Y)\tilde{I} \tag{3.25}$$

for all X,Y in $\mathfrak{X}(M)$.

Then (E,Γ) and $(\tilde{E},\tilde{\Gamma})$ are locally equivalent in $\mathcal{HC}_n(M)$.

PROOF. From Lemma 3.4 we have that for each point p there exist:

(i) an open neighborhood M_o of p

(ii) a differential one-form η on M_o

(iii) two orthonormal frames

$$\{ \sigma_\alpha : 1 \leq \alpha \leq n \} \quad \text{of } E \text{ on } M_o$$

$$\{ \tilde{\sigma}_\alpha : 1 \leq \alpha \leq n \} \quad \text{of } \tilde{E} \text{ on } M_o$$

such that

$$\Gamma|M_o(X,\sigma_\alpha)=\eta(X)\sigma_\alpha \tag{3.26}$$

$$\tilde{\Gamma}|M_o(X,\tilde{\sigma}_\alpha)=\eta(X)\tilde{\sigma}_\alpha \tag{3.27}$$

for all $1\leq\alpha\leq n$ and X in $\mathfrak{X}(M_o)$.

It is now easy to check that the $C^\infty(M_o)$-linear map

$$U:C^\infty(M_o,E|M_o) \longrightarrow C^\infty(M_o,\tilde{E}|M_o)$$

defined by

$$U\sigma_\alpha=\tilde{\sigma}_\alpha \tag{3.28}$$

is an equivalence between $(E|M_o,\Gamma|M_o)$ and $(\tilde{E}|M_o,\tilde{\Gamma}|M_o)$.

3.7. Let U be an equivalence between (E,Γ) and $(\tilde{E},\tilde{\Gamma})$. We denote by U_* the $C^\infty(M)$-linear map

$$U_*:C^\infty(M,L(E)) \longrightarrow C^\infty(M,L(\tilde{E}))$$

defined by

$$U_*(T)=UTU^{-1} \qquad (T\in C^\infty(M,L(E))) \tag{3.29}$$

We clearly have that U_* is an isomorphism of involutive algebras. From (3.24) and (3.6) we have

$$U_*(K(X,Y))=\tilde{K}(X,Y) \qquad (X,Y\ X(M)) \tag{3.30}$$

where K and $\tilde{K}$ are the curvatures of Γ and $\tilde{\Gamma}$, and also, from (3.24) and (3.9) we find

$$U_*\circ\nabla(X)=\tilde{\nabla}(X)\circ U_* \qquad (X\in\mathfrak{X}(M)) \tag{3.31}$$

where ∇ and $\widetilde{\nabla}$ are the linear connections on $L(E)$ and $L(\widetilde{E})$ associated with Γ and $\widetilde{\Gamma}$, respectively.

By a repeated use of (3.30) and (3.31) it follows that

$$(3.32)\qquad U_*(\nabla(Z_1)\dots\nabla(Z_k)K(X,Y))=\widetilde{\nabla}(Z_1)\dots\widetilde{\nabla}(Z_k)\widetilde{K}(X,Y)$$

for all X,Y and $Z_1,\dots,Z_k$ in $\mathfrak{X}(M)$.

3.8. Assume now that thepaires (E,Γ) and $(E,\widetilde{\Gamma})$ are locally equivalent. Arguing as above we obtain that for each point p in M there exists an isometry

$$(3.33)\qquad U_p : E_p \longrightarrow \widetilde{E}_p$$

with the properties

$$(3.34)\qquad U_p K(X,Y)(p)=\widetilde{K}(X,Y)(p)U_p$$

$$(3.35)\qquad U_p(\nabla(Z_1)\dots\nabla(Z_k)K(X,Y))(p)=(\widetilde{\nabla}(Z_1)\dots\widetilde{\nabla}(Z_k)\widetilde{K}(X,Y))(p)U_p$$

for all X, Y and $Z_1,\dots,Z_k$ in $\mathfrak{X}(M)$.

We use this last remark in the next section.

4. POINTWISE EQUIVALENCE

Throughout this section we will suppose that M is a connected complex manifold with its natural C^∞ structure.

As in the previous section, let $\mathcal{HE}_n(M)$ denote the collection of all C^∞-differentiable Hermitian vector bundles over M of rank n, endowed with metric-preserving linear connections.

We know what means that two objects in $\mathcal{HE}_n(M)$ are locally equivlaent.

In this section we define the pointwise equivalence between the elements of $\mathcal{HE}_n(M)$, using a slightly restrictive version of the definition proposed by Cowen and Douglas [4], in the case when M is an open subset of the complex line.

We need a strengthened definition because we consider a complex manifold of an arbitrary dimension and the main result of this section, Theorem B, is proved using Theorem A.

4.1. Let M be a complex manifold and let O(M) denote the complex algebra of all complex-valued holomorphic functions on M.

The space $\mathfrak{X}(M)$ is decomposed into the direct sum

$$\mathfrak{X}(M)=\mathfrak{X}^{1,0}(M)\oplus\mathfrak{X}^{0,1}(M)$$

where by definition

$$\mathfrak{X}^{1,0}(M)=\{X\in\mathfrak{X}(M):X(\lambda^*)=0,\ \lambda\in O(M)\} \tag{4.1}$$

$$\mathfrak{X}^{0,1}(M)=\{X\in\mathfrak{X}(M):X(\lambda)=0,\lambda\in O(M)\} \tag{4.2}$$

We clearly have $\mathfrak{X}^{1,0}(M)^{\#}=\mathfrak{X}^{0,1}(M)$

4.2. DEFINITION. Let (E,Γ) and $(\tilde{E},\tilde{\Gamma})$ be two objects in $\mathcal{HC}_n(M)$. Let K and $\tilde{K}$ be the curvatures of Γ and $\tilde{\Gamma}$.

We say that (E,Γ) and $(\tilde{E},\tilde{\Gamma})$ are pointwise equivalent if and only if for each point p in M there exists an isometry

$$U_p : E_p \longrightarrow \tilde{E}_p$$

such that

$$U_p K(X,Y)(p) = \tilde{K}(X,Y)(p) U_p \tag{4.3}$$

$$U_p(\nabla(Z_1)\dots\nabla(Z_t)\nabla(Z^{\#}_{t+1})\dots\nabla(Z^{\#}_{t+s})K(X,Y))(p) = \tag{4.4}$$

$$=(\tilde{\nabla}(Z_1)\dots\tilde{\nabla}(Z_t)\tilde{\nabla}(Z^{\#}_{z+1})\dots\tilde{\nabla}(Z^{\#}_{t+s})\tilde{K}(X,Y))(p)U_p$$

for all X,Y in $\mathfrak{X}(M)$, $0\le t,s\le n-1$, $1\le t+s$ and $Z_1,\dots,Z_{t+s}$ in $\mathfrak{X}^{1,0}(M)$.

4.3. From 3.8 we clearly obtain that two locally equivalent objects in $\mathcal{HC}_n(M)$ are pointwise equivalent.

As a partial converse we have the following

PROPOSITION. Let (E,Γ) and $(\tilde{E},\tilde{\Gamma})$ be two pointwise equivalent elements of $\mathcal{HC}_n(M)$. Then there exists an open non-empty subset M_o of M such that $(E|M_o, \Gamma|M_o)$ and $(\tilde{E}|M_o, \tilde{\Gamma}|M_o)$ are equivalent in $\mathcal{HC}_n(M_o)$.

PROOF. We consider separately the cases $n=1$ and n 2.

(i) If $n=1$, then there exist the two-forms ω and $\tilde{\omega}$ on M such that

$$K(X,Y)=\omega(X,Y)I$$

$$\tilde{K}(X,Y)=\tilde{\omega}(X,Y)\tilde{I}$$

for all X,Y in $\mathfrak{X}(M)$.

From (4.3) we obtain $\omega=\tilde{\omega}$ and from Lemma 3.6 we conclude that (E,Γ) and $(\tilde{E},\tilde{\Gamma})$ are locally equivalent in $\mathcal{H}\mathcal{C}_n(M)$.

(ii) Consider now the case $n\geq 2$. Let us put

$$\mathcal{Y}=\left\{K(X,Y):X,Y\in \mathfrak{X}(M)\right\}\subseteq C^{\infty}(M,L(E))$$

$$\tilde{\mathcal{Y}}=\left\{\tilde{K}(X,Y):X,Y\in \mathfrak{X}(M)\right\}\subseteq C^{\infty}(M,L(\tilde{E}))$$

$$\mathcal{X}=\left\{\nabla(Z):Z\in\mathfrak{X}^{1,0}(M)\right\}\subseteq \mathfrak{X}(M,E)$$

$$\tilde{\mathcal{X}}=\left\{\ (Z):Z\in\mathfrak{X}^{1,0}(M)\right\}\subseteq \mathfrak{X}(M,E).$$

Using $\mathcal{Y}$ and $\mathcal{X}$ we construct, as in 1.6, the involutive subalgebras A_{ℓ} and B_{ℓ} of $C^{\infty}(M,L(E))$, for each positive integer ℓ. Let $A_{\infty}=\bigcup_{0\leq \ell} A_{\ell}$.

Let M_o and $1\leq k\leq n-1$ be produced by Theorem A.

Arguing as in 2.5 we may assume that in the involutive algebra $A_{\infty}|M_o$ there exist:

(i) a system of orthogonal selfadjont central projections

$$\left\{Q_1,Q_2,\ldots,Q_m\right\}$$

(ii) for each i, $1\leq i\leq m$, a system of orthogonal selfadjoint minimal projections

$$\left\{P^i_{\alpha}:1\leq\alpha\leq n_i\right\}$$

(iii) for each i, $1\leq i\leq m$, a system of elements of $A_{\infty}|M_o$

$$\left\{U^i_{\alpha\beta}:1\leq\alpha,\beta\leq n_i\right\}$$

such that

(4.5) $$\sum_{\alpha} P^{i}_{\alpha} = Q_i$$

(4.6) $$U^{i}_{\alpha\alpha} = P^{i}_{\alpha}$$

(4.7) $$U^{i*}_{\alpha\beta} = U^{i}_{\beta\alpha}$$

(4.8) $$U^{i}_{\alpha\beta} U^{i}_{\gamma\delta} = \Delta_{\beta\gamma} U^{i}_{\alpha\delta}$$

We define

$$\psi : A_k|M_o \longrightarrow C^{\infty}(M_o, L(\tilde{E}|M_o))$$

$$\varphi : \mathfrak{X} \cup \mathfrak{X}^{\#}|M_o \longrightarrow \mathfrak{X}(M_o, \tilde{E}|M_o)$$

by the equations

(4.9) $$\psi(T)(p) = U_p T(p) U_p^{-1} \qquad (p \in M_o \ , \ T \in A_k|M_o)$$

(4.10) $$\varphi(\nabla(Z)|M_o) = \tilde{\nabla}(Z)|M_o \qquad (Z \in \mathfrak{X}^{1,0}(M) \cup \mathfrak{X}^{0,1}(M))$$

Using (4.3) and (4.4) we obtain that

(4.11) $$\psi(K(X,Y)|M_o) = \tilde{K}(X,Y)|M_o$$

and moreover

(4.12) $$\psi(\nabla(Z_1)\ldots\nabla(Z_t)\nabla(Z^{\#}_{t+1})\ldots\nabla(Z^{\#}_{t+s})K(X,Y)|M_o) =$$

$$= \tilde{\nabla}(Z_1)\ldots\tilde{\nabla}(Z_t)\,\tilde{\nabla}(Z^{\#}_{t+1})\ldots\tilde{\nabla}(Z^{\#}_{t+s})\tilde{K}(X,Y)|M_o$$

for all X,Y in $\mathfrak{X}(M)$, $0\leq t,s\leq n-1$, $1\leq t+s$ and $Z_1,\ldots,Z_{t+s}$ in $\mathfrak{X}^{1,0}(M)$.

Since $A_k|M_o=A_\infty|M_o$, from (4.11), (4.12) and by a repeated use of (3.13) we obtain that the map ψ is a well defined morphism of unital complex algebras and applying Theorem A we have

$$\psi(\nabla(X)T)=\tilde{\nabla}(X)\psi(T) \tag{4.13}$$

for all T in $A_\infty|M_o$ and X in $\mathfrak{X}(M_o)$.

Let $\tilde{A}_\ell$, $\tilde{B}_\ell$ and $\tilde{A}_\infty$ be the subalgebras of $C^\infty(M,L(\tilde{E}))$ produced by $\tilde{\mathcal{T}}$ and $\tilde{\mathfrak{X}}$, where $0\leq\ell$ is an integer.

From (4.10) and (4.11) we derive that ψ induces an isomorphism of unital involutive algebras from $A_\infty|M_o$ onto $\tilde{A}_\infty|M_o$.

Let us put

$$\tilde{\Omega}_i=\psi(\Omega_i) \qquad (1\leq i\leq m) \tag{4.14}$$

$$\tilde{P}^i_\alpha=\psi(P^i_\alpha) \qquad (1\leq\alpha\leq n_i) \tag{4.15}$$

$$\tilde{U}^i_{\alpha\beta}=\psi(U^i_{\alpha\beta}) \qquad (1\leq\alpha,\beta\leq n_i) \tag{4.16}$$

For each i, $1\leq i\leq m$ and each α, $1\leq\alpha\leq n_i$, let E^i_α denote the subbundle of $E|M_o$ defined by

$$(E^i_\alpha)_p=P^i_\alpha(p)(E_p) \qquad (p\in M_o). \tag{4.17}$$

and let Γ^i_α denote the linear connection on E^i_α induced by Γ

as follows

$$\Gamma_\alpha^i(X)=P_\alpha^i\,\Gamma(X)P_\alpha^i \qquad (X\in\mathfrak{X}(M_o)) \tag{4.18}$$

It is easy to check that Γ_α^i is a metric-preserving linear connection on E_α^i and from (4.18), (3.6) we obtain that the curvature K_α^i of the connection Γ_α^i is of the form:

$$K_\alpha^i(X,Y)=P_\alpha^i K(X,Y)P_\alpha^i+P_\alpha^i(\nabla(X)P_\alpha^i)(\nabla(Y)P_\alpha^i)P_\alpha^i- \tag{4.19}$$
$$-P_\alpha^i(\nabla(Y)P_\alpha^i)(\nabla(X)P_\alpha^i)P_\alpha^i$$

for all X,Y in $\mathfrak{X}(M_o)$, where by K we denote the curvature of Γ.

Since P_α^i is a minimal projection in $A_\infty|M_o$, from (4.19) we have that

$$K_\alpha^i(X,Y)=\omega_\alpha^i(X,Y)P_\alpha^i \qquad (X,Y\in\mathfrak{X}(M_o)) \tag{4.20}$$

where ω_α^i is a two-form on M_o.

In a similar fashion we construct the subbundle $\tilde{E}_\alpha^i$ of $\tilde{E}|M_o$ and the linear connection $\tilde{\Gamma}_\alpha^i$ on $\tilde{E}_\alpha^i$ whith the curvature $\tilde{K}_\alpha^i$, such that

$$\tilde{K}_\alpha^i(X,Y)=\omega_\alpha^i(X,Y)\tilde{P}_\alpha^i\ . \tag{4.21}$$

From (4.20) and (4.21), applying Lemma 3.6 and eventually decreasing M_o, we may suppose that each $(E_\alpha^i, \Gamma_\alpha^i)$ is equivalent to $(\tilde{E}_\alpha^i, \tilde{\Gamma}_\alpha^i)$ as Hermitian vector bundles over M_o endowed with metric-preserving linear connections.

For each $1\leq i\leq m$, let V_1^i denote the isometric bundle map from E_1^i into $\tilde{E}_1^i$ such that

$$(4.22) \qquad \tilde{\Gamma}^i_1(X)V^i_1 = V^i_1 \Gamma^i_1(X)$$

for all X in $\mathfrak{X}(M_o)$.

We extend V^i_1 to a bundle map U^i_1 from $E|M_o$ to $\tilde{E}|M_o$ such that

$$(4.23) \qquad U^i_1 = \tilde{P}^i_1 V^i_1 P^i_1 .$$

From (4.22) we obtain

$$(4.24) \qquad \tilde{P}^i_1 \tilde{\Gamma}(X) U^i_1 = U^i_1 \Gamma(X) P^i_1$$

For each i, $1 \leq i \leq m$ and each α, $1 \leq \alpha \leq n_i$ let U^i_α denote the bundle map from $E|M_o$ to $\tilde{E}|M_o$ defined by

$$(4.25) \qquad U^i_\alpha = \tilde{U}^i_{\alpha 1} U^i_1 U^i_{1\alpha}$$

We remark that U^i_α maps isometrically E^i_α into $\tilde{E}^i_\alpha$.

Moreover we can prove the following

LEMMA. We have:

$$(4.26) \qquad \tilde{P}^i_\alpha \tilde{\Gamma}(X) U^i_\alpha = U^i_\alpha \Gamma(X) P^i_\alpha$$

$$(4.27) \qquad \tilde{P}^i_\beta \tilde{\Gamma}(X) U^i_\alpha = U^i_\alpha \Gamma(X) P^i_\alpha$$

for all X in $\mathfrak{X}(M_o)$ and $1 \leq i \leq m$, $1 \leq \alpha, \beta \leq n_i$, $\alpha \neq \beta$.

PROOF OF LEMMA. For all i, $1 \leq i \leq m$ and $1 \leq \alpha, \beta \leq n_i$ there exists an one-form on M_o denoted by $\lambda^i_{\alpha\beta}$ such that

$$P^i_\alpha(\nabla(X)U^i_{\alpha\beta})P^i_\beta=\lambda^i_{\alpha\beta}(X)U^i_{\alpha\beta}$$

$$\tilde{P}^i_\alpha(\tilde{\nabla}(X)\tilde{U}^i_{\alpha\beta}\ )\tilde{P}^i_\alpha=\lambda^i_{\alpha\beta}(X)\tilde{U}^i_{\alpha\beta}$$

for all X in $\mathfrak{X}(M_o)$.

Since $U^i_{\alpha\beta}U^i_{\beta\alpha}=P^i_\alpha$ we obtain

$$P^i_\alpha(\nabla(X)U^i_{\alpha\beta})U^i_{\beta\alpha}+U^i_{\alpha\beta}(\nabla(X)U^i_{\beta\alpha})P^i_\alpha=P^i_\alpha(\nabla(X)P^i_\alpha)P^i_\alpha=0$$

hence

$$\lambda^i_{\alpha\beta}+\lambda^i_{\beta\alpha}=0\ . \tag{4.28}$$

From (4.25) and (4.24) we have

$$\tilde{P}^i_\alpha\tilde{\Gamma}(X)U^i_\alpha=\tilde{P}^i_\alpha(\tilde{\nabla}(X)\tilde{U}^i_{\alpha 1}+\tilde{U}^i_{\alpha 1}\tilde{\Gamma}(X))U^i_1U^i_{1\alpha}=$$

$$=\lambda^i_{\alpha 1}(X)U^i_\alpha+\tilde{U}^i_{\alpha 1}U^i_1\Gamma(X)U^i_{1\alpha}$$

and also

$$U^i_\alpha\Gamma(X)P^i_\alpha=\tilde{U}^i_{\alpha 1}U^1_iU^i_{1\alpha}\Gamma(X)P^i_\alpha=\tilde{U}^i_{\alpha 1}U^i_1(-\nabla(X)U^i_{1\alpha}+\Gamma(X)U^i_{1\alpha})P^i_\alpha=$$

$$=-\lambda^i_{1\alpha}(X)U^i_\alpha+\tilde{U}^i_{\alpha 1}U^i_1\Gamma(X)U^i_{1\alpha}$$

But from (4.28) we have $\lambda^i_{\alpha 1}=-\lambda^i_{1\alpha}$ hence (4.23) is proved.

Suppose now that $1\leq\alpha,\beta\leq n_i$ and $\alpha\neq\beta$. Under our assumptions we have

$$P^i_\beta\Gamma(X)P^i_\alpha=P^i_\beta(\nabla(X)P^i_\alpha+P^i_\alpha\Gamma(X))P^i_\alpha=\mu^i_{\beta\alpha}(X)U^i_{\beta\alpha}$$

where $\mu^i_{\beta\alpha}$ is an one-form on M_o , and also we find

$$\tilde{P}^i_\beta \tilde{\Gamma}(X)\tilde{P}^i_\alpha = \mu^i_{\beta\alpha}(X)\tilde{U}^i_{\beta\alpha}$$

for all X in $\mathfrak{X}(M_o)$.

The relation (4.27) is a consequence of the relations

$$\tilde{P}^i_\beta \tilde{\Gamma}(X)U^i_\alpha = \mu^i_{\beta\alpha}(X)\tilde{U}^i_{\beta\alpha}U^i_\alpha = \mu^i_{\beta\alpha}(X)\tilde{U}^i_{\beta 1}U^i_1 U^i_{1\alpha}$$

$$U^i_\beta \Gamma(X)P^i_\alpha = \mu^i_{\beta\alpha}(X)U^i_\beta U^i_{\beta\alpha} = \mu^i_{\beta\alpha}(X)\tilde{U}^i_{\beta 1}U^i_1 U^i_{1\alpha}$$

The proof of Lemma is complete.

Our aim is to show that the pairs $(E|M_o, \Gamma|M_o)$ and $(\tilde{E}|M_o, \tilde{\Gamma}|M_o)$ are equivalent in $\mathcal{H}\mathcal{C}_n(M_o)$.

Let us consider for each i, $1\le i\le m$,

$$U^i = \sum_\alpha U^i_\alpha \tag{4.29}$$

and

$$U = \sum_i U^i \tag{4.30}$$

Since each U^i_α induces an isometry from E^i_α into $\tilde{E}^i_\alpha$, it is straightforward to check that U is an isometry of $E|M_o$ into $\tilde{E}|M_o$.

We claim that

$$\tilde{\Gamma}(X)U = U\Gamma(X) \tag{4.31}$$

for all X in $\mathfrak{X}(M_o)$.

It is sufficient to prove that:

$$\tilde{\Gamma}(X)U^i = U^i\Gamma(X) \tag{4.32}$$

for all i, $1\le i\le m$.

Since each projection Q_i is a central projection in $A_\infty|M_o$, and each $\tilde{Q}_i$ is a central projection in $\tilde{A}_\infty|M_o$, arguing as in 2.5 we have

$$(4.33)\qquad \nabla(X)Q_i=0 \qquad\qquad \tilde{\nabla}(X)\tilde{Q}_i=0$$

hence we obtain, using (4.26) and (4.27)

$$\tilde{\Gamma}(X)U^i=\tilde{\Gamma}(X)\tilde{Q}_iU^i=\tilde{Q}_i\tilde{\Gamma}(X)U^i=(\sum_\beta \tilde{P}^i_\beta)\,\tilde{\Gamma}(X)(\sum_\alpha U^i_\alpha)=$$

$$=\sum_\alpha \tilde{P}^i_\alpha\tilde{\Gamma}(X)U^i_\alpha+\sum_{\alpha\neq\beta}\tilde{P}^i_\beta\tilde{\Gamma}(X)U^i_\alpha=\sum_\alpha U^i_\alpha\Gamma(X)P^i_\alpha+\sum_{\alpha\neq\beta}U^i_\beta\Gamma(X)P^i_\alpha=$$

$$=(\sum_\beta U^i_\beta)\Gamma(X)(\sum_\alpha P^i_\alpha)=U^i\Gamma(X)Q_i=U^iQ_i\Gamma(X)=U^i\Gamma(X).$$

The proof of Proposition is complete.

4.4. We are now in position to state and prove the main result of this section.

THEOREM B. Let (E,Γ) and $(\tilde{E},\tilde{\Gamma})$ be two pointwise equivalent elements of $\mathcal{H}\mathcal{C}_n(M)$. Then there exists an open dense subset M_o of M such that $(E|M_o,\Gamma|M_o)$ and $(\tilde{E}|M_o,\tilde{\Gamma}|M_o)$ are locally equivalent in $\mathcal{H}\mathcal{C}_n(M_o)$.

PROOF. Let M_o be a maximal open subset of M such that $(E|M_o,\Gamma|M_o)$ and $(\tilde{E}|M_o,\tilde{\Gamma}|M_o)$ are locally equivalent in $\mathcal{H}\mathcal{C}_n(M_o)$. By Proposition 4.2 the subset M_o is nonempty.

If M_o is not dense in M, then there exists an open nonempty subset M_1 of M such that M_o and M_1 are disjoint.

But $(E|M_1,\Gamma|M_1)$ and $(\tilde{E}|M_1,\tilde{\Gamma}|M_1)$ are pointwise equivalent in $\mathcal{H}\mathcal{C}_n(M_1)$, hence by Proposition 4.2, there exists an open nonempty subset M_{10} of M_1 such that $(E|M_{10}, \Gamma|M_{10})$ and $(\tilde{E}|M_{10}, \tilde{\Gamma}|M_{10})$ are equivalent in $\mathcal{H}\mathcal{C}_\eta(M_{10})$. It follows that $(E|M_o \cup M_{10}, \Gamma|M_o \cup M_{10})$ and $(\tilde{E}|M_o \cup M_{10}, \tilde{\Gamma}|M_0 \cup M_{10})$ are locally equivalent in $\mathcal{H}\mathcal{C}_\eta(M_0 \cup M_{10})$, a contradiction.

Hence M_o is a dense subset of M.

5. SOME REMARKS

Let M be a connected complex manifold and let $\mathcal{H}_n(M)$ denote the category of all Hermitian holomorphic vector bundles over M of rank n, with real-analytic metrics.

Let E be an object in $\mathcal{H}_n(M)$. Then there exists on E a canonical connection Γ_E with preserves both the Hermitian and holomorphic structures.

In Theorem C below we show that two elements E and $\tilde{E}$ of $\mathcal{H}_n(M)$ are locally equivalent as Hermitian holomorphic vector bundles if and only if (E, Γ_E) and $(\tilde{E}, \Gamma_{\tilde{E}})$ are pointwise equivalent in $\mathcal{HC}_n(M)$.

We end this section with a brief discussion of holomorphic maps into a Grassmann manifold.

5.1. Let E be a Hermitian holomorphic vector bundle over the connected complex manifold M of rank n.

We denote by O(M) the complex algebra of all complex-valued holomorphic functions on M and by O(M,E) the O(M)-module of all holomorphic sections of the bundle E.

As in the previous section, we consider the decomposition of $\mathfrak{X}(M)$ into the direct sum

$$\mathfrak{X}(M)=\mathfrak{X}^{1,0}(M)\oplus\mathfrak{X}^{0,1}(M)$$

It is a standard fact that on E there exists a unique metric-preserving linear-connection Γ_E such that

$$\Gamma_E(X^{\#},\sigma)=0 \tag{5.1}$$

for all X in $\mathfrak{X}^{1,0}(M)$ and σ in O(M,E).

Using this canonical connection Γ_E we obtain the element (E, Γ_E) of $\mathcal{HC}_n(M)$.

The Hermitian metric μ on E is called real-analytic, if for each σ in O(M,E) the real-valued map $\langle\sigma,\sigma\rangle$ is a real-analytic map on M.

5.2. DEFINITION. (i). Two objects E and $\tilde{E}$ in $\mathcal{H}_n(M)$ are called equivalent in $\mathcal{H}_n(M)$ if and only if there exists an isometric holomorphic bundle map from E onto $\tilde{E}$.

(ii) Two objects E and $\tilde{E}$ in $\mathcal{H}_n(M)$ are called locally equivalent in $\mathcal{H}_n(M)$ if and only if there exist an open covering (M_i) of M such that $E|M_i$ are equivalent in $\mathcal{H}_n(M_i)$ for each open subset M_i.

The next result is well-known (see [4], Lemma 2.13).

LEMMA. Let E and $\tilde{F}$ be Hermitian holomorphic vector bundles over M with the canonical connections Γ_E and $\Gamma_{\tilde{E}}$. Then E and $\tilde{E}$ are equivalent (respectively locally equivalent) in $\mathcal{H}_n(M)$ if and only if the pairs (E, Γ_E) and $(\tilde{E}, \Gamma_{\tilde{E}})$ are equivalent (respectively locally equivalent) in $\mathcal{HC}_n(M)$.

<u>PROOF</u>

Let U be an isometric holomorphic bundle map from E onto $\tilde{E}$. For each σ in O(M,E) and $\tilde{\sigma}$ in $O(M,\tilde{E})$ we have

$$(5.2) \qquad \langle U\sigma, \tilde{\sigma}\rangle^{\sim} = \langle \sigma, U^{-1}\tilde{\sigma}\rangle$$

Let X in $\mathcal{X}^{1,0}(M)$. Since $U\sigma$ and $U^{-1}\tilde{\sigma}$ are holomorphic sections, from (5.2), from the fact that Γ_E and $\Gamma_{\tilde{E}}$ are metric-preserving and from (5.1) we obtain

$$\langle \tilde{\Gamma}(X)U\sigma, \tilde{\sigma}\rangle = \langle \Gamma(X)\sigma, U^{-1}\tilde{\sigma}\rangle$$

hence

$$\tilde{\Gamma}(X)U = U\Gamma(X)$$

Since obviously we have

$$\tilde{\Gamma}(X^{\#})U = U\Gamma(X^{\#})$$

we conclude that U is an equivalence between (E, Γ_E) and $(\tilde{E}, \Gamma_{\tilde{E}})$ in $\mathcal{HC}_n(M)$.

The converse follows similarly.

5.3. The following lemma is a restatement of [4], Lemma 3.25.

LEMMA. Let E and $\tilde{E}$ be two objects in $\mathcal{H}_n(M)$. If there exists an open nonempty subset M_o of M such that $E|M_o$ and $\tilde{E}|M_o$ are equivalent in $\mathcal{H}_n(M_o)$, then E and $\tilde{E}$ are locally equivalent in $\mathcal{H}_n(M)$.

The proof can be obtain arguing as in [4], therefore we omit it.

5.4. Using Theorem B, Lemma 5.2 and Lemma 5.3 we obtain

THEOREM C. Two elements E and $\tilde{E}$ of $\mathcal{H}_n(M)$ are locally equivalent in $\mathcal{H}_n(M)$ if and only if (E, Γ_E) and $(\tilde{E}, \Gamma_{\tilde{E}})$ are pointwise equivalent in $\mathcal{HC}_n(M)$.

5.5. We conclude this section with one geometrical application of Theorem C.

We need some preliminaries. For more details the reader can consult [4], § 2.

Let H be a separable complex Hilbert space and $1\leq n$ a positive integer.

Let $Gr(n,H)$ denote the Grassmann manifold, that is, the set of all n-dimensional subspaces of H.

As before, let M be a connected complex manifold. A map $f:M \to Gr(n,H)$ is called holomorphic if for each point p_o in M there exist an open neighbourhood M_o of p_o and a collection $\{\sigma_\alpha : 1 \le \alpha \le n\}$ of holomorphic H-valued functions on M_o, such that

$$f(p)=V\{\sigma_\alpha(p):1\le\alpha\le n\},$$

that is $\{\sigma_\alpha(p):1\le\alpha\le n\}$ span $f(p)$, for all p in M_o.

For $f:M\to Gr(n,H)$ we can construct a natural Hermitian holomorphic vector bundle $E(f)$ over M of rank n, defined by

(5.3) $$E(f)_p=f(p) \qquad (p\in M) .$$

DEFINITION. Two holomorphic maps $f:M\to Gr(n,H)$ and $\tilde{f}:M\to Gr(n,H)$ are called congruent if and only if there exists a unitary operator U on H such that

(5.4) $$U(f(p))=\tilde{f}(p) \qquad (p\in M) .$$

We note the following important result

THEOREM (cf. [4], Theorem 2.2). Let $f:M\to Gr(n,H)$ and $\tilde{f}:M\to Gr(n,H)$ be two holomorphic maps such that

$$H=V\{f(p):p\in M\}=V\{\tilde{f}(p):p\in M\} .$$

Then f and $\tilde{f}$ are congruent if and only if $E(f)$ and $E(\tilde{f})$ are equivalent in $\mathcal{H}_n(M)$.

5.6. Let $\{e_i\}$ be an orthonormal basis of H. For each C^∞-differentiable map σ from M to H we have a decomposition of the form

$$(5.5) \qquad \sigma = \sum_i \sigma^i e_i \ , \qquad \sigma^i \in C^\infty(M) \ .$$

For each X in $\mathcal{X}(M)$ we put

$$(5.6) \qquad X\sigma = \sum_i X(\sigma^i)e_i \ .$$

We recall a definition (cf. [7], [4])

DEFINITION. Let $0 \leq k$ be an integer and $f:M \to \mathcal{G}r(n,H)$, $\tilde{f}:M \to \mathcal{G}r(n,H)$ be two holomorphic maps.

We say that f and $\tilde{f}$ have order of contact k if for each point p in M there exist

(i) a unitary U on H

(ii) two collections

$$\{\sigma_\alpha : 1 \leq \alpha \leq n\} \subset O(M,E(f))$$

$$\{\tilde{\sigma}_\alpha : 1 \leq \alpha \leq n\} \subset O(M,E(\tilde{f}))$$

such that

$$(5.7) \qquad f(p) = V\{\sigma_\alpha(p) : 1 \leq \alpha \leq n\}$$

$$(5.8) \qquad \tilde{f}(p) = V\{\tilde{\sigma}_\alpha(p) : 1 \leq \alpha \leq n\}$$

$$(5.9) \qquad U\sigma_\alpha(p) = \tilde{\sigma}_\alpha(p)$$

$$(5.10) \qquad U(X_1 \ldots X_j \sigma_\alpha)(p) = (X_1 \ldots X_j \tilde{\sigma}_\alpha)(p)$$

for all $1 \leq j \leq k$, $X_1, \ldots, X_j$ in $\mathcal{X}(M)$ and $1 \leq \alpha \leq n$.

The next result can be proved arguing as in [4], Proposition 2.18.

PROPOSITION. The holomorphic maps f and $\tilde{f}$ have order of contact n if and only if the pairs $(E(f), \Gamma_{E(f)})$ and $(E(\tilde{f}), \Gamma_{E(\tilde{f})})$

are pointwise equivalent in $\mathcal{HC}_n(M)$.

5.7. Now, using Theorem C, from 5.5 and 5.6 we have

PROPOSITION. Let $f:M \to \mathcal{G}r(n,H)$ and $\tilde{f}:M \to \mathcal{G}r(n,H)$ be two holomorphic maps such that

$$H = V\{f(p):p \in M\} = V\{\tilde{f}(p):p \in M\}.$$

Then f and $\tilde{f}$ are congruent if and only if f and $\tilde{f}$ have order of contact n.

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