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(Preliminary version)

The aim of this note is to give a labelling of all solutions to the following problem:

(*) Let $H=H_1 \oplus H_2$, $K=K_1 \oplus K_2$ be Hilbert spaces and $A\in L(H_1,K_1)$, $B\in L(H_2,K_1)$, $C\in L(H_1,K_2)$ be such that $A_\ell=(A,B)$ and $A_c=\begin{pmatrix} A \\ C \end{pmatrix}$ are contractions. How many operators $X\in L(H_2,K_2)$ do exist such that $A=\begin{pmatrix} A & B \\ C & X \end{pmatrix}$ be a contraction?

Some applications of this labelling to dual pairs of subspaces in a Kreĭn space and to dual pairs of accretive operators are given.

1. MAIN THEOREM

We consider complex Hilbert spaces and we denote by $L(H,K)$ the set of all (linear bounded) operators from the Hilbert space H to the Hilbert space K. For $T\in L_1(H,K)$ (i.e. T is a contraction, that is $\|T\|\leq 1$) let $D_T=(I-T^*T)^{1/2}$ and $\mathcal{D}_T=\overline{D_T(H)}$ be the defect operator, respectively the defect space of T. We shall use the following result, which is proved in this form in [3], Lemma 1.1.2.

LEMMA 1.1. *Let* H *and* K *be Hilbert spaces,* $H_o \subset H$ *a closed subspace of* H, *and* $T_o \in L_1(H_o, K)$. *The formula*

$$T = (T_o, D_{T_o^*}\Gamma) \tag{1.1}$$

establishes a one-to-one correspondence between all $T \in L_1(H,K)$ *such that* $T|H_o = T_o$ *and all* $\Gamma \in L_1(H \ominus H_o, \mathcal{D}_{T_o^*})$. *Moreover the operators*

$$Z(T_o;T) = Z : \mathcal{D}_{T_o} \oplus \mathcal{D}_\Gamma \longrightarrow \mathcal{D}_T,\quad Z(D_{T_o} \oplus D_\Gamma) = (D_T|H_o) \oplus (P^{\mathcal{D}_T}_{\mathcal{D}_T \ominus \overline{D_T(H_o)}} D_T|H \ominus H_o), \tag{1.2}$$

$$Z_*(T_o;T) = Z_* : \mathcal{D}_{\Gamma^*} \longrightarrow \mathcal{D}_{T^*},\quad Z_* D_{\Gamma^*} D_{T_o^*} = D_{T^*}, \tag{1.3}$$

are unitaries.

(For $H_o \subset H$, $P^H_{H_o}$ denotes the orthogonal projection of H on H_o).

The proof of (1.1) uses a well known result on the factorization (see for example [4],   ); the fact that Z and Z_* are unitaries is proved by direct computations. Lemma 1.1 has a natural variant for the "column" T^* instead of "line" T.

Coming back to our problem, we have from Lemma 1.1 that

$$B = D_{A^*}\Gamma_1, \quad \text{where} \quad \Gamma_1 \in L_1(H_2, \mathcal{D}_{A^*}), \tag{1.4}$$

and

$$C = \Gamma_2 D_A, \quad \text{where} \quad \Gamma_2 \in L_1(\mathcal{D}_A, K_2). \tag{1.5}$$

Moreover the unitaries $Z_\ell = Z(A;A_\ell)$, $Z_{*\ell} = Z_*(A;A_\ell)$, $Z_c = Z(A^*;A_c^*)$, $Z_{*c} = Z_*(A^*;A_c^*)$ give the possibility of identifying the spaces

$\mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1}$ and $\mathcal{D}_{A_\ell}$, $\mathcal{D}_{\Gamma_1^*}$ and $\mathcal{D}_{A_\ell^*}$, $\mathcal{D}_{A^*} \oplus \mathcal{D}_{\Gamma_2^*}$ and $\mathcal{D}_{A_c^*}$, $\mathcal{D}_{\Gamma_2}$ and $\mathcal{D}_{A_c}$, respectively.

In this situation we have the following

LEMMA 1.2. *The operator* $D_{A_c^*}Z_c : \mathcal{D}_{A^*} \oplus \mathcal{D}_{\Gamma_2^*} \longrightarrow K_1 \oplus K_2$ *has the matrix*

$$\begin{pmatrix} D_{A^*} & 0 \\ -\Gamma_2 A^* & D_{\Gamma_2^*} \end{pmatrix} .$$

Proof. First we have

$$D_{A_c^*}Z_c D_{A^*} = D_{A_c^*}^2 | K_1 = \begin{pmatrix} D_{A^*}^2 \\ -\Gamma_2 D_A A^* \end{pmatrix} = \begin{pmatrix} D_{A^*}^2 \\ -\Gamma_2 A^* D_{A^*} \end{pmatrix} = \begin{pmatrix} D_{A^*} \\ -\Gamma_2 A^* \end{pmatrix} D_{A^*} ,$$

which gives the first column of the desired matrix. (We use here the well-known relation $D_A A^* = A^* D_{A^*}$; see [7], Ch. I, (3.4*).) Denote now by $Q = P_{\mathcal{D}_{A_c^*} \ominus \overline{D_{A_c^*}(K_1)}}^{\mathcal{D}_{A_c^*}}$; then

$$\left(D_{A_c^*}Z_c D_{\Gamma_2^*}k_2 \, , \, k_1' \oplus k_2'\right) = \left(Z_c D_{\Gamma_2^*}k_2 \, , \right.$$

$$\left. Q D_{A_c^*}(k_1' \oplus k_2')\right) = \left(D_{\Gamma_2^*}k_2 \, , \, Z_c^* Q D_{A_c^*}k_2'\right) =$$

$$= \left(D_{\Gamma_2^*}k_2, \, D_{\Gamma_2^*}k_2'\right) = \left(D_{\Gamma_2^*}^2 k_2, \, k_1' \oplus k_2'\right) ,$$

for any $k_1' \in K_1$ and k_2, $k_2' \in K_2$. This gives that $D_{A_c^*}Z_c D_{\Gamma_2^*} = D_{\Gamma_2^*}^2$, which finishes the proof of the lemma.

Our main result is the following

THEOREM 1.3. *The answer to the problem* (*) *is that the formula*

(1.6) $$X=-\Gamma_2 A^*\Gamma_1+D_{\Gamma_2^*}\Gamma D_{\Gamma_1}$$

establishes a one-to-one correspondence between all the operators $X\in L(H_2,K_2)$ *such that* $\tilde{A}=\begin{pmatrix} A & D_{A^*}\Gamma_1 \\ \Gamma_2 D_A & X\end{pmatrix}$ *is a contraction, and all* $\Gamma\in L_1(\mathcal{D}_{\Gamma_1},\mathcal{D}_{\Gamma_2^*})$. *Moreover,* $\mathcal{D}_{\tilde{A}}$ *can be identified with* $\mathcal{D}_{\Gamma_2}\oplus\mathcal{D}_{\Gamma}$ *and* $\mathcal{D}_{\tilde{A}^*}$ *with* $\mathcal{D}_{\Gamma_1^*}\oplus\mathcal{D}_{\Gamma^*}$.

Proof. We can apply Lemma 1.1 for $H_o=H_1$, $H=H_1\oplus H_2$, $K=K_1\oplus K_2$, and $T_o=\begin{pmatrix} A \\ \Gamma_2 D_A\end{pmatrix}=A_c$. Thus there exists a one-to-one correspondence between the contractions $\tilde{A}\in L(H_1\oplus H_2, K_1\oplus K_2)$ with $\tilde{A}|H_1=A_c$ and the contractions $\Gamma_c\in L(H_2,\mathcal{D}_{A_c^*})$ given by

(1.7) $$\tilde{A}=(A_c, D_{A_c^*}\Gamma_c).$$

Moreover $Z^*(A_c;\tilde{A})$ and $Z_*^*(A_c;\tilde{A})$ are unitaries between $\mathcal{D}_{\tilde{A}}$ and $\mathcal{D}_{A_c}\oplus\mathcal{D}_{\Gamma_c}$, respectively between $\mathcal{D}_{\tilde{A}^*}$ and $\mathcal{D}_{\Gamma_2^*}$. We have the supplementary condition

(1.8) $$D_{A^*}\Gamma_1=P_{K_1}^{K_1\oplus K_2}D_{A_c^*}\Gamma_c .$$

Using the definition of Z_c we have that

$$(Z_c|\mathcal{D}_{A^*})D_{A^*}=D_{A_c^*}|K_1 .$$

Denoting by $\Gamma_c'=Z_c^*\Gamma_c$, the relation (1.8) becomes then

$$\Gamma_1^* D_{A^*}=\Gamma_c'^{\,*}Z_c^*D_{A_c^*}|K_1=(\Gamma_c'^{\,*}|\mathcal{D}_{A^*})D_{A^*}$$

which means that there exists a one-to-one correspondence between the solutions $\tilde{A}$ for the problem (*) and the contractions $\Gamma_c' \in L(H_2, \mathcal{D}_{A^*} \oplus \mathcal{D}_{\Gamma_2^*})$ which verifies that $\Gamma_c'^* | \mathcal{D}_{A^*} = \Gamma_1^*$. In this correspondence the spaces $\mathcal{D}_{\tilde{A}}$ and $\mathcal{D}_{\Gamma_2} \oplus \mathcal{D}_{\Gamma_c'}$ (resp. $\mathcal{D}_{\tilde{A}^*}$ and $\mathcal{D}_{\Gamma_c'^*}$) are identified by the unitary $(Z_*^*(A^*;A_c^*) \oplus I) Z^*(A_c;\tilde{A})$ (resp. by $Z_c^* Z_*^*(A_c;\tilde{A})$. The above correspondence is given by

$$(1.7)' \qquad \tilde{A} = (A_c, D_{A_c^*} Z_c \Gamma_c') \ .$$

It remains to describe all the contractions $\Gamma_c' \in L(H_2, \mathcal{D}_{A^*} \oplus \mathcal{D}_{\Gamma_2^*})$ which verifies that $\Gamma_c'^* | \mathcal{D}_{A^*} = \Gamma_1^*$; this can be done again by Lemma 1.1. We obtain then a one-to-one correspondence between such contractions Γ_c' and the contractions $\Gamma^* \in L(\mathcal{D}_{\Gamma_2^*}, \mathcal{D}_{\Gamma_1})$, given by

$$(1.9) \qquad \Gamma_c'^* = (\Gamma_1^*, D_{\Gamma_1} \Gamma^*) \ .$$

The operator $Z(\Gamma_1^*;\Gamma_c'^*)$ (resp. $Z_*(\Gamma_1^*, \Gamma_c'^*)$) is unitary between $\mathcal{D}_{\Gamma_c'^*}$ and $\mathcal{D}_{\Gamma_1^*} \oplus \mathcal{D}_{\Gamma^*}$ (resp. between $\mathcal{D}_{\Gamma_c'}$ and $\mathcal{D}_{\Gamma}$).

Combining (1.7)' and (1.9), it results a one-to-one correspondence between the solutions $\tilde{A}$ of (*) and the contractions Γ in $L(\mathcal{D}_{\Gamma_1}, \mathcal{D}_{\Gamma_2^*})$ given by

$$\tilde{A} = \left(A_c \ , \ D_{A_c^*} Z_c \begin{pmatrix} \Gamma_1 \\ \Gamma D_{\Gamma_1} \end{pmatrix}\right) .$$

Using Lemma 1.2 we obtain that

$$(1.10) \qquad \tilde{A} = \begin{pmatrix} A & D_{A^*} \Gamma_1 \\ \Gamma_2 D_A & -\Gamma_2 A^* \Gamma_1 + D_{\Gamma_2^*} \Gamma D_{\Gamma_1} \end{pmatrix} ,$$

Finally note that the operators

$$(1.11)\qquad (Z^*_{*c} \oplus Z_*(\Gamma_1^*;\Gamma_c'^*))Z^*(A_c;\tilde{A}):\mathcal{D}_{\tilde{A}} \longrightarrow \mathcal{D}_{\Gamma_2} \oplus \mathcal{D}_{\Gamma}$$

and

$$(1.12)\qquad Z(\Gamma_1^*;\Gamma_c'^*)Z_c^* Z_*^*(A_c;\tilde{A}):\mathcal{D}_{\tilde{A}^*} \longrightarrow \mathcal{D}_{\Gamma_1^*} \oplus \mathcal{D}_{\Gamma^*}$$

are unitaries, which finishes the proof.

REMARKS.

(1) From the identifications made above results that the solutions of problem (*) can be indexed by the contractions between $\mathcal{D}_{A_\ell} \ominus \overline{D_{A_\ell}(H_1)}$ and $\mathcal{D}_{A_c^*} \ominus \overline{D_{A_c^*}(K_1)}$.

(2) In the above theorem exactly the same correspondence results if one start the completion of A from A_ℓ (and not from A_c). This implies some relations between the unitaries (1.11), (1.12) and the analogue ones for A_ℓ.

(3) For $A \in L_1(H_1,K_1)$, take $H_2=\mathcal{D}_{A^*}$, $K_2=\mathcal{D}_A$, $\Gamma_1=I_{\mathcal{D}_{A^*}}$ and $\Gamma_2=I_{\mathcal{D}_A}$. Theorem 1.3 implies then the known fact that the operator

$$J(A)=\begin{pmatrix} A & D_{A^*} \\ D_A & -A^* \end{pmatrix}:H_1 \oplus \mathcal{D}_{A^*} \longrightarrow K_1 \oplus \mathcal{D}_A$$

is unitary.

(4) With the notations of Theorem 1.3, denote by

$$\hat{\Gamma}_1=\begin{pmatrix} I & 0 \\ 0 & \Gamma_1 \end{pmatrix}:H_1 \oplus H_2 \longrightarrow H_1 \oplus \mathcal{D}_{A^*}$$

$$\hat{\Gamma}_2=\begin{pmatrix} I & 0 \\ 0 & \Gamma_2 \end{pmatrix}:K_1 \oplus \mathcal{D}_A \longrightarrow K_1 \oplus K_2$$

$$\hat{\Gamma} = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma \end{pmatrix} : H_1 \oplus \mathcal{D}_{\Gamma_1} \longrightarrow K_1 \oplus \mathcal{D}_{\Gamma_2^*} .$$

Then (1.10) shows that the solutions of the problem (*) are

$$(1.10)' \qquad \tilde{A} = \hat{\Gamma}_2 J(A) \hat{\Gamma}_1 + D_{\hat{\Gamma}_2^*} \hat{\Gamma} D_{\hat{\Gamma}_1} .$$

2. MAXIMAL DUAL PAIRS OF SUBSPACES

Let K be a Kreĭn space with the fundamental symmetry J, that is K is a Hilbert space and J is a selfadjoint unitary operator on K. Denote by $(\cdot,\cdot)$ the Hilbert space scalar product, and by $\langle\cdot,\cdot\rangle$ the (generally indefinite) inner product given by $\langle x,y\rangle = (Jx,y)$ for $x,y \in K$. The notation and the terminology concerning Kreĭn spaces is that of [2]. In particular if $J=J^+-J^-$ is the Jordan decomposition of J, then $K^+=J^+K$ and $K^-=J^-K$ are the positive (resp. negative) part of K. A set $A \subset K$ is positive (negative, neutral) if $\langle x,x\rangle \geqslant 0$ (resp. $\leqslant 0$, $=0$) for every $x \in A$.

The J-orthogonal of A is $A^{\langle\perp\rangle} = \{x \in K;\ \langle x,y\rangle = 0 \ \ \forall\, y \in A\}$.

A dual pair of subspaces of K is a pair $\{M,N\}$ of closed subspaces of K such that M is positive, N is negative and $M \langle\perp\rangle N$ (i.e. $\langle x,y\rangle = 0$ for $x \in M$ and $y \in N$). Consider the following problem:

(**) For a given dual pair $\{M,N\}$ find all the maximal dual pairs $\{\tilde{M},\tilde{N}\}$ which contain $\{M,N\}$.

(A dual pair $\{M,N\}$ is maximal if M is maximal positive and N is maximal negative). The existence part of problem (**) was proved in [5] (see also [1]). We will give here a labelling of all solutions to this problem. The proof of the existence uses the notion of angular operator introduced by R.S.Phillips. For a positive subspace M, the operator $T:M^{+}=J^{+}M \longrightarrow K^{-}$ defined by $T(J^{+}x)=J^{-}x$, $x \in M$ is a contraction; T is called the angular operator of M and we have that M is exactly the graph of T. Conversely if T is a contraction from $M^{+} \subset K^{+}$ into K^{-}, then the graph of T is a positive subspace with the angular operator T. The angular operator for a negative subspace is defined analogously. In this correspondence the objects associated to T can be also described "geometrically". For example if M is a positive subspace with the angular operator T, then the negative subspace associated to T^{*} is the only maximal negative subspace N, J-orthogonal to M, and with $J^{+}N \subset M^{+}$. Moreover $\ker D_{T}=J^{+}(M^{O})$, where $M^{O}=M \cap M^{\langle\perp\rangle}$ and $D_{T}=J^{+}(M \ominus M^{O})$.

Coming back to problem (**), let $\{M,N\}$ be a dual pair with angular operators T, resp.S. The condition that $M \langle\perp\rangle N$ means $\langle x \oplus Tx, Sy \oplus y\rangle=0$ for every $x \in M^{+}$ and $y \in N^{-}$. This gives

$$\langle Tx,y\rangle = \langle x,Sy\rangle \quad \text{for } x \in M^{+},\ y \in N^{-}. \tag{2.1}$$

Write $T=\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ and $S=\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$ with respect to the decompositions $K^{-}=N^{-} \oplus (K^{-} \ominus N^{-})$, respectively $K^{+}=M^{+} \oplus (K^{+} \ominus M^{+})$. The relation (2.1) means that

$$T_1=S_1^{*}. \tag{2.2}$$

From these considerations follows that in order to describe all the maximal positive subspaces $\tilde{M}$ which contain M and are

J-orthogonal on N one has to find all the contraction completions of the matrix

$$\begin{pmatrix} T_1 & S_2^* \\ T_2 & ? \end{pmatrix} .$$

This is also the solution to the problem (**), because $\tilde{N}$ is exactly $\tilde{M}^{\langle\perp\rangle}$ (from the maximality). Using Theorem 1.3 we obtain

COROLLARY 2.1. *There exists a one-to-one correspondence between all the solutions of problem (**) and all the contractions from* $\mathcal{D}_{S^*} \ominus \overline{D_{S^*}(M^+)}$ *into* $\mathcal{D}_{T^*} \ominus \overline{D_{T^*}(N^-)}$.

A "geometrical" variant of this corollary can be given. Similar results are obtained for maximal uniform (resp. maximal strict uniform) dual pairs.

3. MAXIMAL DUAL PAIRS OF ACCRETIVE OPERATORS

A densely defined closed operator $A:\mathcal{D}(A)\longrightarrow H$ is called <u>accretive</u> if

$$\mathrm{Re}\ (Ah,h) \geq 0 \quad \text{for every} \quad h \in H.$$

A pair of accretive operators $\{A,B\}$ is called a <u>dual pair</u> if

$$(Ah,k)=(h,Bk) \quad \text{for} \quad h \in \mathcal{D}(A) \quad \text{and} \quad k \in \mathcal{D}(B) .$$

A maximal dual pair of accretive operators is a selfexplained expression. In [7], Ch.IV, Proposition 4.2 it is proved that every dual pair of accretive operators can be extended to a maximal dual pair. The proof used Phillips idea [6] of Cayley transform. Using this method and Theorem 3.1 we give a labelling for all maximal dual pairs of accretive operators extending a given dual pair of accretive operators.

For an accretive operator A denote by T its Cayley transform, i.e. T is the contraction

$$(3.1)\qquad T=(A-I)(A+I)^{-1}:(A+I)\mathcal{D}(A)\longrightarrow(A-I)\mathcal{D}(A)\ .$$

We have that

$$(3.2)\qquad A=(I+T)(I-T)^{-1}\ .$$

In this correspondence any accretive extension of A corresponds to a contractive extension of T and conversely. In particular A is maximal accretive iff $(A+I)\mathcal{D}(A)=H$.

If A is maximal accretive, then A^* is maximal accretive, so any maximal dual pair of accretive operators is of the form $\{A,A^*\}$.

(For all this see [7], Ch.IV, Sec.4).

Let now $\{A,B\}$ be a dual pair of accretive operators with Cayley transforms T and S. Then we have

COROLLARY 3.1. *There exists a one-to-one correspondence between all maximal dual pairs of accretive operators extending* $\{A,B\}$ *and all the contractions from* $\mathcal{D}_{T^*}\ominus\overline{D_{T^*}(\mathcal{D}(S))}$ *into* $\mathcal{D}_{S^*}\ominus\overline{D_{S^*}(\mathcal{D}(T))}$.

Similar results can be stated for dissipative operators. Phillips noted also the connections between dual pairs of subspaces in a Kreĭn spaces and dual pairs of accretive operators;

this explains the analogy between Sections 2 and 3. Consequences of our main theorem to Kreĭn theory of selfadjoint extensions will be presented elsewhere.

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