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by

Dan TIBA

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Dan TIBA*)

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*) Department of Mathematics, National Institute for Scientific and Technical Creation, Bd.Pacii 220, Bucharest 79622, Romania

NECESSARY CONDITIONS FOR DISTRIBUTED CONTROL PROBLEMS WITH NONLINEAR STATE EQUATION

by Dan Tiba

Summary. We indicate necessary conditions for optimality in distributed control problems governed by hyperbolic variational inequalities of second order and by nonlinear differential systems with delay. These conditions are expressed by means of generalized gradients and are obtained by using an abstract approximating scheme of the control process.

1. INTRODUCTION

We shall study distributed control problems with convex cost criterion, governed by:

$$y_{tt} + A_0 y + \beta(y_t) \ni Bu \qquad \text{a.e. } \Omega \times]0,T[$$

$$y(0) = y_0 \;, \quad y_t(0) = v_0 \qquad \text{a.e. } \Omega$$

$$y|_\Gamma = 0 \qquad \text{a.e. } t \in]0,T[$$

where β is a maximal monotone operator in $R \times R$, A_0 is a second order elliptic operator in Ω a bounded domain in R^N with sufficiently smooth boundary Γ.

The scheme we develop can also be applied in the case of differential equations with delay:

$$y'(t) = A(y(t)) + Dy(t-h) + Bu(t) \qquad \text{a.e. } [0,T]$$

$$y(0) = y_0 \quad , \quad y(s) = \varphi(s) \qquad a.e. \ [-h, 0]$$

where D, B are matrices, $A : R^N \longrightarrow R^N$ is a Lipschitz function and $\varphi \in L^2(-h, 0; R^N)$.

Some results concerning existence, necessary conditions, approximation for nonlinear control problems appear in papers by Lions [7], Saguez [13], [14] , Mignot [9].

However our methods are closed related to those employed by Barbu [2], [3].

The plan of the paper is as follows: Section 2 contains some preliminary results which are useful in the sequel and are interesting by themselves.

In Section 3 we formulate an abstract control problem and an approximating process to derive the necessary conditions of optimality.

Section 4 specializes to problems governed by nonlinear hyperbolic systems the results from section 3 and section 5 to problems governed by nonlinear hereditary systems.

The following notation is used. If E is a Banach space, then $L^p(0, T; E)$, $1 \le p \le \infty$ is the space of p-integrable, E-valued functions, C(0,T;E) is the Banach space of continuous, E-valued functions and

$$W^{1,p}(0,T;E) = \{ y \in L^p(0,T;E) ; y' \in L^p(0,T;E) \},$$

is the usual Sobolev space.

Let $\varphi : E \to]-\infty, +\infty]$ be a convex, lower semicontinuous function. We denote by $\partial \varphi(x) \subset E'$ (the dual space) the set of all subgradients of φ at x :

$$\partial f(x)=\{x^*\in E' ;\ f(x)\leq f(y)+(x^*,x-y),\ \forall y\in E\}.$$

When f is Gâteaux differentiable, then $\partial f(x)$ is single valued, $\partial f(x)=\nabla f(x)$.

Consider F another Banach space and $K: E\times F\to[-\infty,+\infty]$ a closed, proper, saddle function (see Rockafellar [12]). The subdifferential al K is defined by:

$$\partial K(e,f)=[-\partial_e K(e,f),\ \partial_f K(e,f)] \quad \text{where:}$$

$$\partial_e K(e,f)=\{e^*\in E';\ K(u,f)\leq K(e,f)+(u-e,e^*),\ \forall u\}$$

$$\partial_f K(e,f)=\{f^*\in F';\ K(e,f)\leq K(e,v)+(f-v,f^*),\ \forall v\}$$

and it is a maximal monotone operator.

We denote by $D(f)$, $D(K)$, $D(\partial f)$, $D(\partial K)$ respectively the domains of $f, K, \partial f, \partial K$.

For a general background of convex analysis we cite the monographs Rockafellar [10], Precupanu-Barbu [4]. For generalized gradients we refer to paper Clarke [6] and to the recent survey of Rockafellar [11].

Finally we denote by $H^k(\Omega)$, $H_0^k(\Omega)$, $W^{k,p}(\Omega)$, $W_0^{k,p}(\Omega)$, $H^1(\Gamma)$, usual Sobolev spaces of real functions.

2. PRELIMINARIES

We give a result concerning continuous dependence of solution to hyperbolic variational inequalities of the right number. For other details we send to the books Barbu [1], Brezis [5].

Let V, H be Hilbert spaces with $V\subset H\subset V'$ the inclusion being continuous, dense.

Let $A : V \to V'$ be a linear continuous, symmetric operator, positive definite:

$$(2.1) \quad (Av, v) \geq \omega \|v\|^2 \quad , \quad \omega > 0$$

where (.,.) is the duality between V,V' (respectively the inner product in H) and $\|.\|$, $|.|$ denote the norms in V, H.

Function $\varphi : H \to]-\infty, +\infty]$ is convex, lower semi-continuous and proper.

We assume the following invariance condition:

(2.2) There is $h \in H$:

$$\varphi((I+\varepsilon A)^{-1}(u+\varepsilon h)) \leq \varphi(u) \quad , \quad \forall\, u \in D(\varphi) \quad , \quad \varepsilon > 0.$$

<u>THEOREM 2.1. Under the above hypotheses, if $v_0 \in V$,</u> $Ay_0 \in H$, $v_0 \in D(\varphi)$, $f \in L^2(0,T;V)$ <u>, problem:</u>

$$(2.3) \quad y''(t) + Ay(t) + \partial\varphi(y'(t)) = f(t) \quad a.e. \;]0,T[$$

$$(2.4) \quad y(0) = y_0 \quad , \quad y'(0) = v_0$$

<u>has a unique solution</u> $y \in C(0,T;V)$ <u>with</u> $y' \in C(0,T;H) \cap L^\infty(0,T;V)$ <u>and</u> $y'' \in L^2(0,T;H)$.

The proof can be found in one of the above cited monographs.

<u>THEOREM 2.2. Moreover assume that $V \subset H$ compact. Then if $f_n \to f$ weakly in</u> $L^2(0,T;V)$ <u>and</u> y_n, y <u>denote solutions to problem (2.3), (2.4) corresponding to f_n, f we have</u> $y_n \to y$ <u>in</u> $C(0,T;V)$ <u>and</u> $y_n' \to y'$ <u>in</u> $L^2(0,T;H)$ <u>strongly.</u>

PROOF.

We have :

(2.5) $$y_n'' + A y_n + \partial\varphi(y_n') = f_n \quad a.e. \ [0,T]$$

Let $[x,y] \in \partial\varphi \subset H \times H$.

Multiply (2.5) by $y_n'(t) - x$ and integrate over $[0,t]$:

$$\frac{1}{2}|y_n'(t) - x|^2 - \frac{1}{2}|v_0 - x|^2 + \frac{\omega}{2}\|y_n(t)\|^2 - \frac{1}{2}(Ay_0, y_0) -$$

$$- \int_0^t (Ay_n'(s), x)\,ds + \int_0^t (y, y_n'(s) - x)\,ds \le \int_0^t (f_n, y_n' - x)\,ds .$$

It yields $\{y_n'\}$ bounded in $L^\infty(0,T;H)$ and $\{y_n\}$ bounded in $L^\infty(0,T;V)$.

Hypotheses (2.2) gives:

(2.6) $$(A_\varepsilon y_n'(t), \partial\varphi(y_n'(t))) \geq ((I+\varepsilon A)^{-1} h, \partial\varphi(y_n'(t)))$$

where A_ε denotes the Yosida approximation of A_H , the realization of A in H.

From (2.5), by multiplication with $A_\varepsilon y_n'(t)$, using (2.6) and again (2.5) we infer:

$$\frac{1}{2}(A_\varepsilon y_n'(t), y_n'(t)) - \frac{1}{2}(A_\varepsilon v_0, v_0) + \frac{1}{2}(Ay_n(t), A_\varepsilon y_n(t)) -$$

$$- \frac{1}{2}(Ay_0, y_0) - ((I+\varepsilon A)^{-1} h, y_n'(t)) \le$$

$$\le C + \int_0^t (f_n(t), A_\varepsilon y_n'(t))\,dt.$$

Now we remind some properties of operator A, easily to be deduced from (2.1):

(2.7) $\|(I+\varepsilon A)^{-1} v\| \leq C\|v\|$, $\forall v \in V$

(2.8) $(A_\varepsilon v, Av) \geqslant |A_\varepsilon v|^2$

(2.9) $(A_\varepsilon y_n'(t), y_n'(t)) \geqslant \omega \|(I+\varepsilon A)^{-1} y_n'(t)\|^2$

(2.10) $\|A_\varepsilon y_n'(t)\|_* \leqslant \|A\| \cdot \|(I+\varepsilon A)^{-1} y_n'(t)\|$

where $\|\cdot\|_*$ is the norm in V' and $\|A\|$ is the norm of the linear continuous operator A.

From the preceding inequalities one can obtain

$|A_\varepsilon y_n(t)|$, $\|(I+\varepsilon A)^{-1} y_n'(t)\|$ uniformly bounded.

A standard argument now allows to make $\varepsilon \to 0$ and to prove

(2.11) $|Ay_n(t)|$, $\|y_n'(t)\|$ uniformly bounded.

Multiply (2.5) by $y_n''(t)$ and integrate over $[0, T]$:

$$\int_0^T |y_n''(t)|^2 dt - C\left(\int_0^T |y_n''(t)|^2 dt\right)^2 + \int_0^T (\partial f(y_n'(t)), y_n''(t)) dt \leq$$

$$\leq C\left(\int_0^T |y_n''(t)|^2 dt\right)^{1/2}.$$

It follows:

$$\int_0^T |y_n''(t)|^2 + f(y_n'(T)) - f(v_0) \leq C\left(\int_0^T |y_n''(t)|^2 dt\right)^{1/2}$$

Here C denotes some different constants. It yields $\{y_n''\}$ bounded in $L^2(0,T;H)$.

Because $A:V \to V'$ coercive and $V \subset H$ compact, then $D(A_H) \subset V$

compact and by (2.11) we obtain $y_n \to y$ in $C(0,T;V)$, extracting a convenient subsequence.

By extracting further subsequences if necessary we get:

$$
\begin{aligned}
y_n' &\to y' && \text{in } C(0,T;H) \text{ strongly,}\\
y_n' &\to y' && \text{in } L^{\infty}(0,T;V) \text{ weakly star,}\\
y_n'' &\to y'' && \text{in } L^2(0,T;H) \text{ weakly.}
\end{aligned}
$$

To pass to the limit we take $x \in L^2(0,T;H)$, such that $x(t) \in D(\varphi)$ a.e. Multiply (2.5) by $y_n'(t)-x(t)$ and integrate over $[0,T]$:

$$
\int_0^T (y_n''(t), y_n'(t)-x(t))\,dt + \int_0^T (Ay_n(t), y_n'(t)-x(t))\,dt +
$$

$$
+\int_0^T \varphi(y_n'(t))\,dt - \int_0^T \varphi(x(t))\,dt \leq \int_0^T (f_n(t), y_n'(t)-x(t))\,dt .
$$

We can make $n \to \infty$ because $\int_0^T \varphi(\cdot)$ is a lower semicontinuous functional on $L^2(0,T;H)$.

Because $x(t)$ is any element in $D(\varphi)$ it yields y to be the solution of (2.3), (2.4) and the convergence takes place on the initial sequence.

<u>REMARK</u>. The same result can be derived under weaker assumptions, for instance when $\varphi : V \to]-\infty,+\infty]$ convex, lower semicontinuous, proper function.

3. AN ABSTRACT APPROXIMATING PROCESS

Consider the following control problem:

(3.1) $$\min \int_0^T L(Sy(t), u(t))dt$$

with state equation:

(3.2) $y'(t)+My(t)+\nu y(t)=Fu(t)$ a.e. $[0,T]$

(3.3) $y(0)=y_0$.

Here we denote W, Z, X, Y Hilbert spaces with $Y\subset Z$ closed subspace with the topology given by the trace.

(3.4) $S:Z \longrightarrow X$ linear continuous operator,

(3.5) $F:W \longrightarrow Z$ linear continuous operator,

(3.6) $M:Z \longrightarrow Z$ maximal monotone operator.

We assume that $Y\supset R(F)$ (the range of F), ν is a fixed number (the interesting case is when ν is negative) and $y_0\in D(M)$.

(3.7) Function $L:X\times W \longrightarrow]-\infty,+\infty]$ is convex, lower semicontinuous proper, with finite Hamiltonian.

Let $M^\varepsilon: Z \rightarrow Z$, $\varepsilon > 0$, be a family of maximal monotone operators. We denote by $\theta_\varepsilon: L^2(0,T;Y) \longrightarrow L^2(0,T;Z)$ the correspondence $f \rightsquigarrow y$ given by:

$$y'(t)+M^\varepsilon y(t)+\nu y(t)=f(t) \quad \text{a.e. } [0,T]$$

$$y(0)=y_0$$

and θ when instead of M^ε is M.

Applications θ_ε, θ are well defined according to Theorem 2.1, p.124, Barbu [1].

We now list the main hypotheses of this section:

(a) $\theta_\varepsilon \circ F : L^2(0,T;W) \longrightarrow C(0,T;Z)$ is completely continuous, uniformly in ε.

(b) $S \circ \theta_\varepsilon : L^2(0,T;Y) \longrightarrow L^2(0,T;X)$ is

Gâteaux differentiable for every $\varepsilon > 0$.

(c) θ_ε approximates θ uniformly:

$$|\theta_\varepsilon(f)(t) - \theta(f)(t)|_Z \leq c\,\delta(\varepsilon) \quad , \quad \forall t \in [0,T],$$

where $\delta(\varepsilon) \longrightarrow 0$ when $\varepsilon \to 0$, for every f in $L^2(0,T;R(F))$.

Consider the following approximating control problem:

(3.8) $$\min \left\{ \int_0^T L^\varepsilon(Sy(t), u(t))\,dt + \frac{1}{2}\int_0^T \|u(t) - u^*(t)\|_W^2\,dt \right\}$$

with conditions:

(3.9) $y'(t) + M^\varepsilon y(t) + \nu y(t) = Fu(t)$ a.e. $[0,T]$

(3.10) $y(0) = y_0$,

where $L^\varepsilon = L_{\delta(\varepsilon)}$ is an Yosida regularization of convex function L (see Barbu-Precupanu [4]) and u^*, y^* are the optimal control and the optimal state in problem (3.1)-(3.3), which are supposed to exist.

<u>LEMA 3.1.</u> <u>PROBLEM</u> (3.8)-(3.10) <u>has solution</u> $[y_\varepsilon, u_\varepsilon] \in L^2(0,T;Z) \times L^2(0,T;W)$, <u>with</u> $y_\varepsilon = \theta_\varepsilon(Fu_\varepsilon)$.

<u>PROOF</u>

Functional (3.8) is coercive in u because $y = \theta_\varepsilon(Fu)$ admits a good evaluation in state equation (3.9). We have assured the weak lower semicontinuity because L^ε is Lipschitzian and $S \circ \theta_\varepsilon$ is completely continuous by (a).

LEMMA 3.2. For every $\varepsilon > 0$ there is $p_\varepsilon \in L^2(0,T;Y)$ such that:

$$p_\varepsilon = -[\nabla(S\circ\theta_\varepsilon)(Fu_\varepsilon)]^* \partial_1 L^\varepsilon(Sy_\varepsilon, u_\varepsilon) \tag{3.11}$$

$$F^* p_\varepsilon = \partial_2 L^\varepsilon(Sy_\varepsilon, u_\varepsilon) + u_\varepsilon - u^*. \tag{3.12}$$

PROOF

L^ε is Fréchet differentiable, $S\circ\theta_\varepsilon$ is Gâteaux differentiable. From the minimum condition we get:

$$\int_0^T (\partial_1 L^\varepsilon(Sy_\varepsilon, u_\varepsilon), \nabla(S\circ\theta_\varepsilon)(Fu_\varepsilon)Fv)_X\, dt +$$

$$+\int_0^T \langle \partial_2 L^\varepsilon(Sy_\varepsilon, u_\varepsilon) + u_\varepsilon - u^*, v\rangle_W\, dt = 0$$

for every $v \in L^2(0,T;W)$.

LEMMA 3.3. When $\varepsilon \longrightarrow 0$ we have:

(3.13) $y_\varepsilon \longrightarrow y^*$ in $C(0,T;Z)$ strongly,

(3.14) $u_\varepsilon \longrightarrow u^*$ in $L^2(0,T;W)$ strongly.

PROOF

From the minimum condition we have:

$$\int_0^T L^\varepsilon(Sy_\varepsilon, u_\varepsilon)\,dt + \frac{1}{2}\int_0^T \|u_\varepsilon - u^*\|_W^2\, dt \leq$$

(3.15)

$$\leq \int_0^T L^\varepsilon(S\circ\theta_\varepsilon(Fu^*), u^*)\,dt$$

But:

$$L^{\varepsilon}(S\circ\theta_{\varepsilon}(Fu^*)(t), u^*(t)) \leq L(Sy^*(t), u^*(t)) +$$

$$+ C\cdot\frac{|\theta_{\varepsilon}(Fu^*)(t)-\theta(Fu^*)(t)|^2}{2\,\delta(\varepsilon)}$$

from the definition of the Yosida regularization.

It yields:

$$(3.16)\quad \limsup_{\varepsilon\to 0}\left\{\int_0^T L^{\varepsilon}(Sy_{\varepsilon}, u_{\varepsilon})\,dt + \frac{1}{2}\int_0^T \|u^*-u_{\varepsilon}\|_W^2\,dt\right\} \leq$$

$$\leq \int_0^T L(Sy^*, u^*)\,dt$$

from hypothesis (c).

From the coercivity of the cost functionals, uniformly in ε, it results $\{u_{\varepsilon}\}$ to be bounded in $L^2(0,T;W)$. Therefore extracting a convenient subsequence, we get $u_{\varepsilon} \to u_0$ weakly.

We have the inequality:

$$|\theta_{\varepsilon}(Fu_{\varepsilon})-\theta(Fu_0)|_{C(0,T;Z)} \leq |\theta_{\varepsilon}(Fu_{\varepsilon})-\theta_{\varepsilon}(Fu_0)|_{C(0,T;Z)} +$$

$$+|\theta_{\varepsilon}(Fu_0)-\theta(Fu_0)|_{C(0,T;Z)} \leq |\theta_{\varepsilon}(Fu_{\varepsilon})-\theta_{\varepsilon}(Fu_0)|_{C(0,T;Z)} +$$

$$+\delta(\varepsilon) \longrightarrow 0$$

from hypotheses (a) and (c).

So $y_{\varepsilon} \to \tilde{y}=\theta(Fu_0)$ strongly in $C(0,T;Z)$.

Let $\varepsilon\to 0$ in (3.15). One obtains $u_0=u^*$ and $\tilde{y}=y^*$

by (3.16).

LEMMA 3.4. The Gâteaux differential supplies:

$$|[\nabla(S\circ\theta_\varepsilon)(w)v](t)|_X \le C\int_0^t |v(s)|_Y\, ds,$$

for every $w, v \in L^2(0,T;Y)$.

PROOF

$$\nabla(S\circ\theta_\varepsilon)(w)v = \lim_{\lambda\to 0}\frac{Sy_\lambda - Sy}{\lambda} \qquad \text{strongly in } L^2(0,T;X),$$

where:

$$y'(t)+M^\varepsilon y_\lambda(t)+\nu y_\lambda(t)=w(t)+\lambda v(t) \quad \text{a.e. } [0,T]$$
$$y'(t)+M^\varepsilon y(t)+\nu y(t)=w(t) \quad \text{a.e. } [0,T]$$

and $y_\lambda(0)=y(0)=y_0$.

We subtract the equations, multiply by $y_\lambda - y$ and integrate:

$$\frac{1}{2}|y_\lambda(t)-y(t)|^2 + \nu\int_0^t |y_\lambda(s)-y(s)|^2\, ds \le$$
$$\le \lambda\int_0^t (v(s),\, y_\lambda(s)-y(s))_Z\, ds.$$

Then $|y_\lambda(t)-y(t)|_Z \le C\lambda\int_0^t |v(s)|_Y\, ds$

and because S is linear continuous the proof is finished.

REMARK. Similarly we have the relation:

$$(3.17)\qquad |[\nabla(S\circ\theta_\varepsilon)(w)^* v](t)|_Y \le C\int_t^T |v(s)|_X\, ds,$$

for every $t\in[0,T]$, $w\in L^2(0,T;Y)$, $v\in L^2(0,T;X)$.

LEMMA 3.5. There is $p \in L^{\infty}(0,T;Y)$, $q \in L^2(0,T;X)$ such that, for $\varepsilon \to 0$:

(3.18) $p_\varepsilon \longrightarrow p$ weakly* in $L^{\infty}(0,T;Y)$,

(3.19) $\partial_1 L^\varepsilon(Sy_\varepsilon, y_\varepsilon) \longrightarrow q$ weakly in $L^1(0,T;X)$,

(3.20) $[q(t), F^* p(t)] \in \partial L(Sy^*(t), u^*(t))$ a.e. $[0,T]$.

PROOF

From the finite Hamiltonian hypothesis, we infer:

$$(\partial_1 L^\varepsilon(Sy_\varepsilon, u_\varepsilon), Sy_\varepsilon - Sy^* - \rho w)_X + \langle \partial_2 L^\varepsilon(Sy_\varepsilon, u_\varepsilon), u_\varepsilon - v_0 \rangle_W \geq$$

$$\geq L^\varepsilon(Sy_\varepsilon, u_\varepsilon) - L^\varepsilon(Sy^* + \rho w, v_0).$$ So:

$$-\rho(w, \partial_1 L^\varepsilon(Sy_\varepsilon, u_\varepsilon)) + (Sy_\varepsilon - Sy^*, \partial_1 L^\varepsilon(Sy_\varepsilon, u_\varepsilon)) \geq$$

$$\geq -\langle F^* p_\varepsilon + u^* - u_\varepsilon, u_\varepsilon - v_0 \rangle_W - L^\varepsilon(Sy^* + \rho w, v_0) + C,$$

for every $w \in X$, $|w|_X = 1$.

Then:

$$\frac{\rho}{2} |\partial_1 L^\varepsilon(Sy_\varepsilon(t), u_\varepsilon(t)|_X \leq (\|F^* p_\varepsilon(t)\|_W + \|u_\varepsilon(t) - u^*(t)\|_W \cdot$$

$$\cdot (C + \|u_\varepsilon(t)\|_W) + C \qquad a.e. [0,T].$$

But, form (3.17) we write:

$$|p_\varepsilon(t)|_Y \leq C \int_t^T |\partial_1 L^\varepsilon(Sy_\varepsilon(t), u_\varepsilon(t))|_X \, dt$$

and the Gronwall lemma gives:

$$|p_\varepsilon(t)|_Y \leq C \quad a.e. [0,T]$$

and next:

$$|\partial_1 L^\varepsilon(Sy_{y_\varepsilon}(t), u_\varepsilon(t))|_X \leq C(1+\|u_\varepsilon(t)-u^*(t)\|)\cdot(1+\|u^*(t)\|).$$

The Dunford-Pettis criterion shows:

$$\partial_1 L^\varepsilon(Sy_{y_\varepsilon}, u_\varepsilon) \longrightarrow q \qquad \text{wakly in } L^1(0,T;X).$$

We also have:

$$p_\varepsilon \longrightarrow p \quad \text{weakly}^* \quad \text{in } L^\infty(0,T;Y).$$

Relation (3.20) and $q \in L^2(0,T;X)$ can be derived by standard arguments because y_ε, u_ε converge strongly by Lemma 3.3.

4. HYPERBOLIC CONTROL PROBLEMS

In this section we treat control problems of the type:

(4.1) $$\min \int_0^T L(y,u)dt$$

with state equation:

(4.2) $y''(t)+Ay(t)+\partial\varphi(y'(t)) \ni Bu(t)$ a.e. $[0,T]$

(4.3) $y(0)=y_0$, $y'(0)=v_0$.

We shall consider the frame and the hypotheses of section 2 and apply the abstract scheme developed in section 3.

The state equation can be put in the form:

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} + \begin{bmatrix} 0 & -1 \\ A & \partial\varphi \end{bmatrix}\begin{bmatrix} y \\ v \end{bmatrix} \ni \begin{bmatrix} 0 \\ Bu(t) \end{bmatrix} \quad \text{a.e. } [0,T] \tag{4.4}$$

$$\begin{bmatrix} y(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix} \tag{4.5}$$

We take the spaces:

- $W=\{0\} \times U$, $\quad U$ a Hilbert space of controls,
- $Z= V \times H$, where V has the inner product induced by A, so it is identified with the dual,
- $X=H$, $\quad Y=\{0\} \times H$.

The corresponding operators are:

- $F=\begin{bmatrix} 0 \\ B \end{bmatrix}$, with $B:U \longrightarrow V$ linear continuous,
- $S: V \times H \longrightarrow H$ by $S\begin{bmatrix} y \\ v \end{bmatrix} = y$ is linear continuous,
- $M = \begin{bmatrix} 0 & -1 \\ A & \partial\varphi \end{bmatrix}$
- $M^{\varepsilon}: V\times H \longrightarrow V\times H$ is obtained replacing in M, $\partial\varphi$ by $(\partial\varphi)^{\varepsilon}$.

In this case, when $v_0 \in D(\varphi)$, equation (4.4), (4.5) and the approximate equation too, have strong unique solution according to Theorem 2.1.

In paper Barbu [2] a set of abstract hypotheses is imposed on family $\{\varphi^{\varepsilon}\}$ to ensure the desired properties.

To make everything less complicated, in the sequel we shall detail as an example the following control problem:

$$Min \int_0^T L(y(t), u(t))\, dt \tag{4.6}$$

with state equation:

$$\frac{\partial^2 y}{\partial t^2} - \Delta y + \beta\left(\frac{\partial y}{\partial t}\right) \ni Bu(t) \quad a.e.\ Q, \tag{4.7}$$

(4.8) $\quad y(0,x) = y_0(x) \qquad$ a.e. Ω,

(4.9) $\quad \frac{\partial y}{\partial t}(0,x) = v_0(x) \qquad$ a.e. Ω,

(4.10) $\quad y(t,x) = 0 \qquad$ a.e. $[0,T] \times \Gamma$,

where Ω is a bounded domain in R^N with smooth boundary Γ and $Q = [0,T] \times \Omega$.

In our notations we have $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, U is a Hilbert space of controls.

Operator $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ is the Laplacian with Dirichlet conditions and β is a maximal monotone graph in RxR. It yields $\beta = \partial j$ with $j : R \longrightarrow]-\infty, +\infty]$ a convex, lower semicontinuous, proper function.

Then $\varphi : L^2(\Omega) \longrightarrow]-\infty, +\infty]$ is given by:

(4.11)
$$\varphi(y) = \begin{cases} \int_\Omega j(y(x))\,dx & , y \in L^2(\Omega), j(y) \in L^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

As it is wellknown $\partial\varphi(y)(x) \in \beta(y(x))$ a.e.

Application φ^ε is obtained by:

(4.12)
$$j^\varepsilon(r) = \int_{-\infty}^{+\infty} j_\varepsilon(r - \varepsilon\theta)\rho(\theta)\,d\theta =$$
$$= \varepsilon^{-1}\int_{-\infty}^{+\infty} j_\varepsilon(\theta)\rho(\varepsilon^{-1}(r-\theta))\,d\theta .$$

Here j_ε denotes the Yosida regularization of j and ρ is a mollifier i.e. it is a C^∞ function with $\int_{-\infty}^{+\infty}\rho(\theta)\,d\theta = 1$, $\rho(\theta) \geq 0$, $\rho(\theta) = \rho(-\theta)$ and $\rho(\theta) = 0$ for $|\theta| \geq 1$.

Similarly we denote:

(4.13) $$\beta^{\varepsilon}(r) = \int_{-\infty}^{+\infty} \beta_{\varepsilon}(r - \varepsilon\theta)\rho(\theta)\,d\theta$$

where β_{ε} is the Yosida approximate of β, $\beta^{\varepsilon} = \partial j^{\varepsilon}$ and the following inequality is true:

(4.14) $$|\beta_{\varepsilon}(y - \varepsilon\theta)| \leq |\beta^{\varepsilon}(y)| + C.$$

The approximating state equation is:

(4.15) $$\frac{\partial^2 y}{\partial t^2} - \Delta y + \beta^{\varepsilon}\left(\frac{\partial y}{\partial t}\right) = Bu(t) \quad \text{a.e. } Q.$$

We start to verify hypotheses (a), (b), (c) from section 3.

Using the same argument as in the proof of Theorem 2.2 one can see that the estimates in (4.15) are independent of ε, so (a) can be deduced.

For (b) we put:

(4.16) $$\frac{\partial^2 y}{\partial t^2} - \Delta y + \beta^{\varepsilon}\left(\frac{\partial y}{\partial t}\right) = f \qquad \text{a.e. } Q,$$

(4.17) $$\frac{\partial^2 y_{\lambda}}{\partial t^2} - \Delta y_{\lambda} + \beta^{\varepsilon}\left(\frac{\partial y_{\lambda}}{\partial t}\right) = f + \lambda g \qquad \text{a.e. } Q,$$

(4.18) $$y(0) = y_{\lambda}(0) = y_0, \quad \frac{\partial y}{\partial t}(0) = \frac{\partial y_{\lambda}}{\partial t}(0) = v_0 \quad \text{a.e. } \Omega,$$

where $f, g \in L^2(Q)$.

We remark that we are interested in eq. (4.4), (4.5) but, because $f, g \in L^2(0,T;\{0\} \times H)$ in reality, then problem can be put in the above form.

We have

$$\nabla(S\circ\theta_\varepsilon)(f)g = \lim_{\lambda\to 0}\frac{y_\lambda - y}{\lambda} \quad \text{in } L^2(\Omega) \tag{4.19}$$

strongly. Subtract (4.16) and (4.17), multiply by $\frac{\partial y_\lambda}{\partial t} - \frac{\partial y}{\partial t}$ in the inner product of $L^2(\Omega)$ and integrate over $[0,t]$:

$$\frac{1}{2}\left|\frac{\partial y_\lambda}{\partial t}(t) - \frac{\partial y}{\partial t}(t)\right|^2_{L^2(\Omega)} + \frac{1}{2}\|y_\lambda(t) - y(t)\|^2_{H_0^1(\Omega)} \leq \tag{4.20}$$

$$\leq \lambda\int_0^t \left(g(s), \frac{\partial y_\lambda}{\partial t}(s) - \frac{\partial y}{\partial t}(s)\right)_{L^2(\Omega)} ds.$$

Then $\frac{\partial y_\lambda}{\partial t} \longrightarrow \frac{\partial y}{\partial t}$ in $C(0,T;L^2(\Omega))$ and $y_\lambda \longrightarrow y$ in $C(0,T;H_0^1(\Omega))$ strongly.

Moreover:

$$\left|\frac{\frac{\partial y_\lambda}{\partial t} - \frac{\partial y}{\partial t}}{\lambda}\right|, \quad \left\|\frac{y_\lambda(t) - y(t)}{\lambda}\right\| \quad \text{uniformly}$$

bounded on $[0,T]$ with respect of λ.

Because $H_0^1(\Omega)\subset L^2(\Omega)$ compact, extracting a convenient subsequence we have:

$$\frac{y_\lambda - y}{\lambda} \longrightarrow r \quad \text{in } C(0,T;L^2(\Omega)) \text{ for } \lambda\to 0.$$

We write:

$$\frac{y_\lambda''(t) - y''(t)}{\lambda} - \Delta\frac{y_\lambda(t) - y(t)}{\lambda} + \frac{\beta^\varepsilon(y_\lambda'(t)) - \beta^\varepsilon(y'(t))}{y_\lambda'(t) - y'(t)}\cdot$$

$$\cdot\frac{y_\lambda'(t) - y'(t)}{\lambda} = g(t) \quad \text{a.e. } Q.$$

We have omitted the variable x, and used y", y' instead of $\frac{\partial^2 y}{\partial t^2}$, $\frac{\partial y}{\partial t}$.

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We have $\dfrac{\mathcal{J}''_\lambda - \mathcal{J}''}{\lambda}$ bounded in $L^\infty(0,T;H^{-1}(\Omega))$

as $-\Delta \dfrac{\mathcal{J}_\lambda - \mathcal{J}}{\lambda}$ has the same property and

$\dfrac{\beta^\varepsilon(\mathcal{J}'_\lambda) - \beta^\varepsilon(\mathcal{J}')}{\lambda}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ because β^ε is Lipschitz of constant $1/\varepsilon$.

Using weak convergences, we get:

$$r''(t) - \Delta r(t) + \nabla\beta^\varepsilon((S\circ\theta_\varepsilon)(f)')\cdot r'(t) = g(t) \quad a.e.\ Q \tag{4.21}$$

$$r(0,x) = r'(0,x) = 0 \quad a.e.\ \Omega. \tag{4.22}$$

Problem (4.21), (4.22) has a unique solution and application $g \rightsquigarrow r$ is linear, continuous, hence (b) is true.

Now, we calculate, for (3.11), the adjoint of operator $\nabla(S\circ\theta_\varepsilon)$. We use the notation $[\nabla(S\circ\theta_\varepsilon)(f)]^*(q) = p$ and from $(r,q)_{L^2(Q)} = (g,p)_{L^2(Q)}$, integrating by parts, it yields:

$$p''(t) - \Delta p(t) - (\nabla\beta^\varepsilon((S\circ\theta_\varepsilon)(f)')\,p(t))' = q(t) \tag{4.23}$$

$$p(T) = 0 \quad , \quad p'(T) = 0. \tag{4.24}$$

We change the final conditions in initial ones and introduce the new unknown function $m' = p$, which verifies:

$$m''(t) - \Delta m(t) + \nabla\beta^\varepsilon((S\circ\theta_\varepsilon)(f)'(T-t)\cdot m'(t) = \tag{4.25}$$

$$= \int_0^t q(T-s)\,ds \quad a.e.\ Q$$

$$m(0,x) = m'(0,x) = 0 \quad a.e.\ \Omega. \tag{4.26}$$

As $\nabla\beta^\varepsilon((S\circ\theta_\varepsilon)(f)') \in L^\infty(0,T;L^2(\Omega))$ from β^ε Lipschitz, one can infer that problem (4.25), (4.26) has unique solution $m \in C(0,T;H_0'(\Omega))$, $m_t \in C(0,T;L^2(\Omega))$ and $m_{tt} \in L^\infty(0,T;H^{-1}(\Omega))$. That is (4.23), (4.24) has a unique ultra-weak solution (see Lions [8]) in distributions.

We continue by verifying (c). Let f be in $L^2(0,T;R(\beta))$ and $y_\varepsilon=(S\circ\theta_\varepsilon)(f)$, $y_\lambda=(S\circ\theta_\lambda)(f)$, so:

(4.27) $y''_\varepsilon(t)-\Delta y_\varepsilon(t)+\beta^\varepsilon(y'_\varepsilon(t))=f(t)$ a.e. Q,

(4.28) $y''_\lambda(t)-\Delta y_\lambda(t)+\beta^\lambda(y'_\lambda(t))=f(t)$ a.e. Q,

(4.29) $y_\varepsilon(0)=y_\lambda(0)=y_o$, $y'_\varepsilon(0)=y'_\lambda(0)=v_o$.

The following inequality will be used:

(4.30)
$$\begin{aligned}&(\beta^\varepsilon(y'_\varepsilon(t,x))-\beta^\lambda(y'_\lambda(t,x)))\cdot(y'_\varepsilon(t,x)-y'_\lambda(t,x))\geq\\ &\geq\int_{-\infty}^{+\infty}(\beta_\varepsilon(y'_\varepsilon(t,x)-\varepsilon\theta)-\beta_\lambda(y'_\lambda(t,x)-\lambda\theta))\cdot(\varepsilon\beta_\varepsilon(y'_\varepsilon(t,x)-\varepsilon\theta)-\\ &-\lambda\beta_\lambda(y'_\lambda(t,x)-\lambda\theta))\rho(\theta)d\theta+(\varepsilon-\lambda)\int_{-\infty}^{+\infty}(\beta_\varepsilon(y'_\varepsilon(t,x)-\varepsilon\theta)-\\ &-\beta_\lambda(y'_\lambda(t,x)-\lambda\theta))\theta\rho(\theta)d\theta.\end{aligned}$$

As in the proof of Theorem 2.2 we remark that $\beta^\varepsilon(y'_\varepsilon)$ is bounded in $L^2(Q)$ when $\varepsilon\to 0$.

Finally from (4.27)-(4.30) and (4.14), subtracting and multiplying by $y'_\varepsilon(t)-y'_\lambda(t)$ in $L^2(\Omega)$ and integrating over $[0,t]$, we get:

(4.31)
$$\frac{1}{2}|y'_\varepsilon(t)-y'_\lambda(t)|^2_{L^2(\Omega)}+\frac{1}{2}\|y_\varepsilon(t)-y_\lambda(t)\|^2_{H_0^1(\Omega)}\leq$$
$$\leq(\varepsilon+\lambda)C.$$

Then $y_\varepsilon\longrightarrow y$ in $C(0,T;H_0'(\Omega))$ strongly and $y'_\varepsilon\longrightarrow y'$ in $C(0,T;L^2(\Omega))$ strongly, where easily it can be obtained $y=(S\circ\theta)(f)$.

Making $\lambda \to 0$ we have the desired evaluation from (c) with $\delta(\varepsilon)=\varepsilon^{1/2}$.

The approximate cost functional we shall consider is

$$\int_0^T L^\varepsilon(y(t),u(t))dt+\frac{1}{2}\int_0^T \|u^*(t)-u(t)\|_U^2\,dt \tag{4.32}$$

where u^* is the optimal control and L^ε is the Yosida regularization of L of order $\delta(\varepsilon)$.

We can write the approximating optimality conditions for problem (4.32), (4.15). We denote y_ε, u_ε the optimal state and the optimal control in the approximate problem.

Then, according to section 3, there is $p_\varepsilon \in C(0,T;L^2(\Omega))$ such that:

(4.33) $y_\varepsilon''-\Delta y_\varepsilon+\beta^\varepsilon(y_\varepsilon')=Bu_\varepsilon$ a.e. Q,

(4.34) $-p_\varepsilon''+\Delta p_\varepsilon+(\nabla\beta^\varepsilon(y_\varepsilon')p_\varepsilon)'=q_\varepsilon$ a.e. Q,

$y_\varepsilon(0,x)=y_0(x)$, $y_\varepsilon'(0,x)=v_0(x)$ a.e. Ω,

$p_\varepsilon(T,x)=0$, $p_\varepsilon'(T,x)=0$ a.e. Ω,

(4.35) $y_\varepsilon|_\Gamma=0$, $p_\varepsilon|_\Gamma=0$,

(4.36) $[q_\varepsilon(t),\ B^*p_\varepsilon+u_\varepsilon(t)-u^*(t)]\in\partial L^2(y_\varepsilon(t),u_\varepsilon(t))$.

From section 3, we know:

$y_\varepsilon \to y^*$ in $C(0,T;H_0^1(\Omega))$ strongly,

$y_\varepsilon' \to y^{*\prime}$ in $C(0,T;L^2(\Omega))$ strongly,

$u_\varepsilon \to u^*$ in $L^2(0,T;U)$ strongly,

$p_\varepsilon \to p$ in $L^\infty(0,T;L^2(\Omega))$ weakly*,

$q_\varepsilon \to q$ in $L^1(0,t;L^2(\Omega))$ weakly and

(4.37) $$[q(t), B^* p(t)] \in \partial L(y^*(t), u^*(t)) \quad \text{a.e. } [0,T].$$

We write (4.34) in the form (4.25):

(4.38) $$m_\varepsilon''(t,x) - \Delta m_\varepsilon(t,x) + \nabla\beta^\varepsilon(y_\varepsilon'(T-t,x))\, m_\varepsilon'(t,x) = = -\int_0^t q_\varepsilon(T-s,x)\,ds \quad a.e.\ Q.$$

To pass to the limit in the above equation we suppose β to be locally Lipschitz and to verify:

(4.39) $$\sup\{|wy|;\ w\in\partial\beta(y)\} \le C(|\beta(y)| + y^2 + 1),\ \forall y\in R,$$

where $\partial\beta$ denotes the generalized gradient of β in the sense of Clarke.

From (4.39) it easy to obtain:

(4.40) $$|\nabla\beta^\varepsilon(y)\cdot y| \le C(|\beta^\varepsilon(y)| + y^2 + 1),\ \forall y\in R,$$

with C a positive constant, independent of ε.

Multiply (4.38) by m_ε' in $L^2(\Omega)$ and integrate over $[0,t]$. From $\nabla\beta^\varepsilon \geq 0$ we get:

$$\frac{1}{2}|m_\varepsilon'(t)|^2_{L^2(\Omega)} + \frac{1}{2}\|m_\varepsilon\|^2_{H_0^1(\Omega)} \le -\int_0^t q_\varepsilon\,ds\cdot m_\varepsilon'(t),$$

that is $\{m_\varepsilon'\}$ bounded in $L^\infty(0,T;L^2(\Omega))$ and $\{m_\varepsilon\}$ bounded in $L^\infty(0,T;H_0^1(\Omega))$.

From (4.33) as in the proof of Theorem 2.2 we see that $\{\beta^\varepsilon(y_\varepsilon')\}$ is bounded in $L^2(Q)$.

Let us consider for $\varepsilon > 0$ and n a fixed natural number, the set

$E_n^{\varepsilon} = \{(x,t)\in Q;\ \ |y_{\varepsilon}'(t,x)|\leq n\}.$

Then for $(x,t)\in E_n^{\varepsilon}$ we have

$$(4.41)\qquad |\nabla\beta^{\varepsilon}(y_{\varepsilon}'(t,x))|\leq C_n$$

independent of ε as β is locally Lipschitz.

Denote E a measurable subset of Q. Following Barbu [2] we write, by (4.40):

$$\left|\int_E m_{\varepsilon}'\cdot\nabla\beta^{\varepsilon}(y_{\varepsilon}')dxdt\right|\leq\int_{E\cap E_n^{\varepsilon}}|m_{\varepsilon}'|\cdot|\nabla\beta^{\varepsilon}(y_{\varepsilon}')|dxdt+$$

$$+\int_{E\setminus E_n^{\varepsilon}}|m_{\varepsilon}'|\cdot|\nabla\beta^{\varepsilon}(y_{\varepsilon}')|dxdt\leq C_n\int_E|m_{\varepsilon}'|\,dxdt+$$

$$+C\cdot\frac{1}{n}\int_{E-E_n^{\varepsilon}}|\beta^{\varepsilon}(y_{\varepsilon}')|\cdot|m_{\varepsilon}'|\,dxdt+C\cdot\frac{1}{n}+$$

$$+C\cdot\int_{E-E_n^{\varepsilon}}|y_{\varepsilon}'|\cdot|m_{\varepsilon}'|\,dxdt.$$

Taking into account that $\{m_{\varepsilon}'\}$ bounded in $L^{\infty}(0,T;L^2(\Omega))$ and $\{\beta^{\varepsilon}(y_{\varepsilon}')\}$ bounded in $L^2(Q)$ we obtain:

$$\left|\int_E m_{\varepsilon}'\cdot\nabla\beta^{\varepsilon}(y_{\varepsilon}')dxdt\right|\leq C\cdot\mu(E)^{1/2}\cdot C_n+$$

$$+C\cdot\frac{1}{n}+C\cdot\left(\int_{E-E_n^{\varepsilon}}|y_{\varepsilon}'|^2\right)^{\frac{1}{2}}$$

where C denotes several constant. Because $\{y_{\varepsilon}'\}$ is bounded in $L^{s}(Q)$ with some $s>2$ according to the Sobolev embedding theorems, so the last term of the sum is equicontinuous too.

Then, by the Dunford-Pettis criterion, we get:

$$\nabla\beta^{\varepsilon}(y_{\varepsilon}')\cdot m_{\varepsilon}' \longrightarrow h \qquad \text{weakly in } L^1(Q).$$

Extracting a convenient subsequence we derive:

$$\begin{aligned}
m_{\varepsilon} &\longrightarrow m && \text{in } C(0,T;L^2(\Omega)) \text{ strongly,}\\
m_{\varepsilon}' &\longrightarrow m' && \text{in } L^{\infty}(0,T;L^2(\Omega)) \text{ weakly}^*,\\
m_{\varepsilon}' &\longrightarrow m' && \text{in } C(0,T;H^{-1}(\Omega)) \text{ strongly,}\\
\nabla\beta^{\varepsilon}(y_{\varepsilon}')\cdot m_{\varepsilon}' &\longrightarrow h && \text{in } L^1(Q) \text{ weakly,}\\
m_{\varepsilon}'' &\longrightarrow m'' && \text{in } L^1(0,T;H^{-1}(\Omega)) \text{ weakly.}
\end{aligned}$$

Summarising to this point, we have:

THEOREM 4.1. Let $[y^*, u^*]\in W^{2,2}(0,T;L^2(\Omega)) \times L^2(0,T;U)$ be an optimal pair for problem (4.6)-(4.10). Then there exist functions $m\in L^{\infty}(0,T;H_0^1(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega))$, $q\in L^2(Q)$ and $h\in L^1(Q)$ which satisfy

$$\begin{aligned}
&y^{*''}-\Delta y^* +\beta(y^{*'})=Bu^* && \text{a.e. } Q,\\
&m'' -\Delta m + h = \int_0^t q && \text{a.e. } Q,\\
&y^*(0,x)=y_0(x),\ y^{*'}(0,x)=v_0(x) && \text{a.e. } \Omega,\\
&m(0,x)=m'(0,x)=0 && \text{a.e. } \Omega,
\end{aligned}$$

$$[q(t), -B^* m'(t)]\in \partial L(y^*(t), u^*(t)) \quad \text{a.e. } [0,T].$$

Furthermore y^*, u^*, m, q are limits of solutions y_{ε}, u_{ε}, m_{ε}, q_{ε} to approximate equations (4.33)-(4.36).

REMARK. From the Egorov theorem for every $\eta>0$, there is $Q_{\eta}\subset Q$ with mes $(Q-Q_{\eta})<\eta$ and $y_{\varepsilon}' \longrightarrow y^{*'}$ in $C(Q_{\eta})$. Because β is locally Lipschitz, we infer:

(4.42) $$|\nabla\beta^{\varepsilon}(y_{\varepsilon}'(t,x))| \le C \quad , \quad \forall (t,x)\in Q_{\eta},$$

so $\quad \nabla\beta^{\varepsilon}(y'_{\varepsilon}) \longrightarrow g \quad$ weakly* in $L^{\infty}(Q_{\eta})$.

Using Lemma 3 of Barbu [3] it yields:

(4.43) $\quad g(t,x) \in \partial\beta(y^{*\prime}(t,x)) \qquad$ a.e. Q_{η}.

It should be expected to obtain the relation:

(4.44) $\quad h(t,x) \in \partial\beta(y^{*\prime}(T-t,x)) \cdot m'(t,x) \qquad$ a.e. Q.

Indeed, in some outstanding cases this conclusion can be proven.

EXAMPLE 1. β is convex.

We are interested to pass to the limit in the product $\nabla\beta^{\varepsilon}(y'_{\varepsilon}) \cdot m'_{\varepsilon}$. Write

$$m'_{\varepsilon} = m_{\varepsilon}^{+} - m_{\varepsilon}^{-}$$

where m_{ε}^{+}, m_{ε}^{-} denote the positive and the negative part of m'_{ε} up to an additive constant. Extracting more subsequences we can suppose:

(4.45) $\quad m_{\varepsilon}^{+} \longrightarrow v^{+}, \quad m_{\varepsilon}^{-} \longrightarrow v^{-} \qquad$ weakly in $L^2(Q)$

(4.46) $m'=v^{+}-v^{-}$

and (adding a constant if necessary) m_{ε}^{+}, m_{ε}^{-}, v^{+}, v^{-} are all strict positive.

We give now a more precise calculation of $\nabla\beta^{\varepsilon}(y)$, $y \in R$, available for locally Lipschitz functions β.

(4.47) $$\beta^{\varepsilon}(\jmath)=\int_{-\infty}^{+\infty}\frac{I-(I+\varepsilon\beta)^{-1}}{\varepsilon}(\jmath-\varepsilon\theta)\,\rho(\theta)\,d\theta.$$

Since β is differentiable a.e. and
$\nabla(I+\varepsilon\beta)=1+\varepsilon\nabla\beta\geq 1$ a.e. then
$(I+\varepsilon\beta)^{-1}$ is differentiable a.e. and we obtain:

(4.48) $$\nabla\beta^{\varepsilon}(\jmath)=\int_{-\infty}^{\infty}\frac{1-\dfrac{1}{1+\varepsilon\nabla\beta(\jmath-\varepsilon\theta)}}{\varepsilon}\rho(\theta)\,d\theta=$$

$$=\int_{-1}^{1}\frac{\nabla\beta(\jmath-\varepsilon\theta)}{1+\varepsilon\nabla\beta(\jmath-\varepsilon\theta)}\rho(\theta)\,d\theta$$

When β is convex $\nabla\beta=\partial\beta$, the subdifferential of β in the points where β is differentiable, so:

(4.49) $$\nabla\beta^{\varepsilon}(\jmath)=\int_{-1}^{1}\frac{\partial\beta(\jmath-\varepsilon\theta)}{1+\varepsilon\partial\beta(\jmath-\varepsilon\theta)}\rho(\theta)\,d\theta.$$

Let Q_η be as in (4.42). We are interested in the weak convergence of $\nabla\beta^{\varepsilon}(\jmath_\varepsilon')\cdot m_\varepsilon^+$ in $L^2(Q_\eta)$. Consider f any in $L^2(Q_\eta)$. Then:

$$\int_{Q_\eta}\nabla\beta^{\varepsilon}(\jmath_\varepsilon')\cdot m_\varepsilon^+\cdot f\,dx\,dt=$$

$$=\int_{-1}^{1}\rho(\theta)\,d\theta\cdot\int_{Q_\eta}m_\varepsilon^+\,\partial_\beta(\jmath_\varepsilon'-\varepsilon\theta)\cdot f\cdot\frac{1}{1+\varepsilon\partial\beta(\jmath_\varepsilon'-\varepsilon\theta)}\,dx\,dt$$

As on Q_η we have for $\varepsilon\to 0$, $\{\jmath_\varepsilon'-\varepsilon\theta\}$ uniformly bounded, then $\partial\beta(\jmath_\varepsilon'-\varepsilon\theta)$ is bounded in $L^\infty(Q_\eta)$, so $1+\varepsilon\partial\beta(\jmath_\varepsilon'-\varepsilon\theta)$ converges uniformly to 1 on Q_η.

We have to study only the integral:

(4.50) $$\int_{Q_\eta} m_\varepsilon^+ \, \partial\beta(y_\varepsilon' - \varepsilon\theta) \cdot f \, dx \, dt, \quad \theta \in [-1,1] \quad \text{fixed.}$$

Consider the saddle function:

(4.51) $$K(m,y) = \begin{cases} m\,\beta(y) & m \geq 0 \\ -\infty & m < 0 \end{cases}$$

which is proper, closed. The maximal monotone operator ∂K in $R^2 x R^2$ is given by:

(4.52) $$\partial K(m,y) = [-\beta(y), \; m\,\partial\beta(y)], \quad \forall [m,y] \in D(\partial K).$$

Take the realization of ∂K in $L^2(Q_\eta) x L^2(Q_\eta)$

(4.53) $$\partial\tilde{K}(m,y)(t,x) = \partial K(m(t,x), y(t,x))$$

a.e. $(t,x) \in Q_\eta$

and for every $m, y \in L^2(Q_\eta)$, $m \geq 0$ a.e.

Operator $\partial\tilde{K}$ is maximal monotone in $L^2(Q_\eta) x L^2(Q_\eta)$.

Then:

(4.54) $$[-\beta(y_\varepsilon' - \varepsilon\theta), \; m_\varepsilon^+ \cdot \partial\beta(y_\varepsilon' - \varepsilon\theta)] \in \partial\tilde{K}(m_\varepsilon^+, y_\varepsilon' - \varepsilon\theta)$$

We remark that:

(4.55) $$[-\beta(y_\varepsilon' - \varepsilon\theta), \; m_\varepsilon^+ \partial\beta(y_\varepsilon' - \varepsilon\theta)] \longrightarrow [-\beta(y^{*\prime}), \tilde{h}]$$

weakly in $L^2(Q_\eta) x L^2(Q_\eta)$

(4.56) $$[m_\varepsilon^+, y_\varepsilon' - \varepsilon\theta] \longrightarrow [v^+, y_t^*]$$ weakly in $L^2(Q_\eta) \times L^2(Q_\eta)$.

We can also verify the following condition:

(4.57)
$$\lim_{\lambda,\varepsilon\to 0} \langle [m_\varepsilon^+, y_\varepsilon' - \varepsilon\theta] - [m_\lambda^+, y_\lambda' - \lambda\theta], [-\beta(y_\varepsilon' - \varepsilon\theta), m_\varepsilon^+ \partial\beta(y_\varepsilon' - \varepsilon\theta)] - [-\beta(y_\lambda' - \lambda\theta), m_\lambda^+ \partial\beta(y_\lambda' - \lambda\theta)] \rangle_{L^2(Q_\eta)\times L^2(Q_\eta)} =$$
$$= \lim_{\lambda,\varepsilon\to 0} \int_{Q_\eta} (m_\varepsilon^+ - m_\lambda^+)\cdot(\beta(y_\lambda' - \lambda\theta) - \beta(y_\varepsilon' - \varepsilon\theta))\,dx\,dt +$$
$$+ \lim_{\lambda,\varepsilon\to 0} \int_{Q_\eta} (y_\varepsilon' - \varepsilon\theta - y_\lambda' + \lambda\theta)(m_\varepsilon^+ \cdot \partial\beta(y_\varepsilon' - \varepsilon\theta) -$$
$$- m_\lambda^+ \cdot \partial\beta(y_\lambda' - \lambda\theta))\,dx\,dt = 0$$

since we have $\beta(y_\varepsilon' - \varepsilon\theta) \longrightarrow \beta(y^{*\prime})$ uniformly on Q_η and $y_\varepsilon' - \varepsilon\theta \longrightarrow y^{*\prime}$ uniformly on Q_η.

Applying a wellknown lemma (Barbu [1], p.42) we get

(4.58) $$[-\beta(y^{*\prime}), \tilde{h}] \in \partial\tilde{K}(v^+, y^{*\prime})$$ so,

(4.59) $$\tilde{h}(t,x) \in v^+(t,x)\cdot\partial\beta(y^{*\prime})$$ a.e. Q

In a similar way we obtain:

(4.60) $$\lim_{\varepsilon\to 0} m_\varepsilon^- \cdot \partial\beta(y_\varepsilon' - \varepsilon\theta) = \underset{\sim}{h}$$ weakly in $L^2(Q_\eta)$

and $\underset{\sim}{h}(t,x) \in v^-(t,x)\cdot\partial\beta(y^{*\prime})$ a.e. Q.

Then $h(t,x) = \lim_{\varepsilon\to 0} m_\varepsilon' \cdot \partial\beta(y_\varepsilon' - \varepsilon\theta) \in m'\cdot\partial\beta(y^{*\prime})(t,x)$ a.e. Q. So we have:

<u>THEOREM 4.2. Under hypotheses of Theorem 4.1, assume</u>

that β is convex. Then there exist functions $m \in L^{\infty}(0,T;H_0^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ and $q \in L^2(Q)$ which satisfy:

$$y^{*''} - \Delta y^* + \beta(y^{*\prime}) = Bu^* \quad \text{a.e. } Q,$$

$$m'' - \Delta m + \partial\beta(y^{*\prime}).m' = \int_0^t q \quad \text{a.e. } Q,$$

$$y^*(0,x) = y_0(x),\ y^{*\prime}(0,x) = v_0(x) \quad \text{a.e. } \Omega,$$

$$m(0,x) = m'(0,x) = 0 \quad \text{a.e. } \Omega$$

$$[q(t,x) - B^* m'(t,x)] \in \partial L(y^*(t,x), u^*(t,x)) \text{ a.e. } Q.$$

Furthermore y^*, u^*, m, q are limits of solutions y_ε, u_ε, m_ε, q_ε to approximate equations (4.33)-(4.36).

REMARK

A similar treatment can be carried out in the case β is a concave function on R.

EXAMPLE 2. β is differentiable function.

Let Q_η be as in (4.42). Then

$$\nabla\beta^\varepsilon(y_\varepsilon') \longrightarrow \nabla\beta(y^{*\prime}) \quad \text{weakly}^* \text{ in } L^\infty(Q_\eta)$$

on a convenient subsequence. It is easy to obtain

$$\nabla\beta^\varepsilon(y_\varepsilon'(t,x)) \longrightarrow \nabla\beta(y^{*\prime}(t,x)) \quad \text{a.e. } Q_\eta.$$

It yields $\nabla\beta^\varepsilon(y_\varepsilon') \longrightarrow \nabla\beta(y^{*\prime})$ strongly in $L^2(Q_\eta)$ from the Lebesgue theorem.

In this case we have:

$$\nabla\beta^\varepsilon(y_\varepsilon') \cdot m_\varepsilon' \longrightarrow \nabla\beta(y^{*\prime}) \cdot m' \quad \text{weakly in } L^1(Q_\eta)$$

so in (4.44) $h(t,x) \in \nabla\beta(y^{*\prime}(T-t,x)).m'(t,x)$ a.e. Q becomes true and a result similar to Theorem 4.2 can be stated.

REMARK. Since β is locally Lipschitz we know it is differentiable a.e. To apply the above argument it is sufficient to assume that the set:

$\tilde{Q}=\{(t,x)\in Q \mid \beta$ is differentible in $y^{*\prime}(t,x)\}$ has mes $(Q-\tilde{Q})=0$.

5. A control problem associated with an equation with delay

Consider the problem

$$(5.1) \quad Min \int_0^T L(y(t), u(t))\,dt$$

with state equation :

$$(5.2) \quad y'(t)=A(y(t))+Dy(t-h)+Bu(t) \qquad \text{a.e. } [0,T]$$

$$(5.3) \quad y(0)=y_o, \quad y(s)=\varphi(s) \qquad \text{a.e. } [-h,0].$$

Here D, B are autonomous matrices and $A:R^N \longrightarrow R^N$ is a Lipschitz function.

Application $L:R^N \times R^M \longrightarrow]-\infty,+\infty]$ is convex, lower semicontinuous, proper, with finite Hamiltonian. We suppose too:

$$(5.4) \quad y_o\in R^N, \quad \varphi \in L^2(-h,0;R^N).$$

The problem can be put in the frame of the abstract scheme from section 3 by:

$Z=R^N x L^2(-h,0;R^N)$; $\quad Y=R^N x\{0\}$;

$X=R^N$; $\quad W=R^M$.

Operator $M:Z \rightarrow Z$ is given by:

$$D(M)=\{[x_o,\varphi]\in R^N \times H^1(-h,0;R^N) \; ; \; \varphi(0)=x_o\}$$

$$M\begin{pmatrix} x_o \\ \varphi \end{pmatrix} = \begin{pmatrix} A(x_o)+D\varphi(-h) \\ \dot{\varphi} \end{pmatrix}$$

where $\dot{\varphi}$ denotes the derivative with respect of $\theta \in [-h,0]$. We keep notation φ' for the derivative with respect of $t \in [0,T]$.

Operator F is given by $[B,0]$ and S is the projection $S\begin{pmatrix} y(t) \\ y' \end{pmatrix} = y(t)$. Here pair $[y(t), y']$ plays the role of y in the abstract scheme from section 3.

It is known that operator M is ν- maximal dissipative in Z; then state equation (5.2), (5.3) has a unique strong solution. In this case we can infer even $y' \in L^2(0,T;R^N)$.

Application M^{ε} is obtained only using mollifiers, that is instead of A we put

$$(5.5) \quad A^{\varepsilon}(y) = \int_{R^N} A(y-\varepsilon x)\rho(x)dx, \qquad \varepsilon > 0$$

where ρ has the same properties as in section 4.

The family of operators A^{ε} is uniformly Lipschitz, so approximate equation (3.9) has unique, strong solution. Then applications θ, θ_{ε} are well defined.

Now we begin to verify hypotheses (a), (b), (c).

Let $u_k \longrightarrow u$ weakly in $L^2(0,T;R^M)$.

Equation (3.9) is equivalent with:

$$(5.6) \quad y_k'(t) = A^{\varepsilon}(y_k(t)) + Dy_k(t-h) + Bu_k(t) \qquad \text{a.e. } [0,T]$$

$$y_k(0) = y_0, \quad y_k(s) = \varphi(s) \qquad \text{a.e. } [-h,0].$$

Multiply by $y_k(t)$ and integrate over $[0,t]$:

$$\frac{1}{2}|y_k(t)|^2_{R^N} - \frac{1}{2}|y_0|^2 = \int_0^t A^{\varepsilon}(y_k(t))\cdot y_k(t)dt +$$

$$\int_0^t Dy_k(t-h)\cdot y_k(t) + \int_0^t Bu_k(t)\cdot y_k(t)dt.$$

We have the inequality:

$$|A^{\varepsilon}(y_k(t))-A(0)| = \left|\int_{R^N}\{A(y_k(t)-\varepsilon x)-A(0)\}\rho(x)\,dx\right| \leq$$
$$\leq \ell\int_{R^N}|y_k(t)-\varepsilon x|\rho(x)\,dx \leq \ell\,|y_k(t)| + \ell\cdot\varepsilon\int_{R^N}|x|\rho(x)\,dx$$

because A is Lipschitz of constant 1.

It yields:

$$\frac{1}{2}|y_k(t)|^2_{R^N} \leq C\left(1+\int_0^t|y_k(s)|^2ds+\left(\int_0^t|y_k(s)|^2ds\right)^{1/2}\right).$$

By a variant of Gronwall lemma we get $\{y_k\}$ uniformly bounded, so $\{A^{\varepsilon}(y_k)\}$ is uniformly bounded independent of ε. From equation (5.6) we see that $\{y_k'\}$ is bounded in $L^2(0,T;R^N)$ independent of ε (and of k, of course).

Extracting a convenient subsequence, we infer:

$$y_k \longrightarrow y \quad \text{in } C(0,T;R^N) \quad \text{strongly},$$
$$y_k' \longrightarrow y' \quad \text{in } L^2(0,T;R^N) \quad \text{weakly},$$

where y denotes solution to (5.6) corresponding to function u.

From the unicity of solution in (5.6) we see that the convergence takes place on the initial sequence and is independent of ε, that is (a).

For (b) we denote $\nabla\theta_{\varepsilon}(w)v = r$ with w, $v \in L^2(0,T;R^N)$. It is easy to verify that r exists (θ_{ε} is differentiable Gâteaux) and satisfies:

(5.7) $$r'(t) = \nabla A^{\varepsilon}(\theta_{\varepsilon}(w))\cdot r(t) + D r(t-h) + v(t)$$

a.e. $[0,T]$

(5.8) $r(0)=0$, $r(s)=0$ a.e. $s \in [-h,0]$.

We have to calculate the adjoint of operator $\nabla\theta_\varepsilon(w)$, i.e. $\nabla\theta_\varepsilon(w)^*: L^2(0,T;R^N) \longrightarrow L^2(0,T;R^N)$, linear, continuous.

Let $\nabla\theta_\varepsilon(w)^* q = p$, then $(r,q)=(v,p)$ where the scalar product is in $L^2(0,T;R^N)$.

Multiply (5.7) by p, integrate by parts and, after some calculation, we infer:

$$-p'(t) = \nabla A^\varepsilon(\theta_\varepsilon(w))^* p(t) + D^* p(t+h) + q(t) \qquad \text{a.e. } [0,T] \tag{5.9}$$

$$p(T)=0\ ,\ p(s)=0 \qquad \text{a.e. } s\in[T,T+h]. \tag{5.10}$$

Problem (5.9), (5.10) has unique strong solution since it can be put by $t \rightsquigarrow T-t$ in the form of a usual delay equation on interval $[-h,T]$.

We continue with assumption (c). Let us write:

$$y_\varepsilon'(t) = A^\varepsilon(y_\varepsilon(t)) + Dy_\varepsilon(t-h) + f(t) \qquad \text{a.e. } [0,T]\ ,$$

$$y'(t) = A(y(t)) + Dy(t-h) + f(t) \qquad \text{a.e. } [0,T]\ ,$$

$$y(0) = y_\varepsilon(0) = y_0\ ,\ y(s) = y_\varepsilon(s) = \varphi(s) \qquad \text{a.e. } [-h,0]$$

where f is in $L^2(0,T;R^N)$. Then, by:

$$|A^\varepsilon(y) - A(y)| = \left|\int_{R^N} (A(y-\varepsilon x) - A(y))\rho(x)dx\right| \le$$

$$\le l\cdot\varepsilon \int_{R^N} |x|\rho(x)dx = \varepsilon\cdot C$$

we get:

$$|y_\varepsilon(t)-y(t)| \le l\int_0^t |y_\varepsilon(s)-y(s)|ds + C\cdot T\cdot\varepsilon +$$

$$+ |D| \int_0^t |y_\varepsilon(s)-y(s)|ds.$$

So, we obtain:

(5.11) $\quad |y_\varepsilon(t)-y(t)| \le C\cdot\varepsilon$ for every $t\in[-h,T]$.

Then in the approximate control problem we have $\delta(\varepsilon)=\varepsilon$ and we can take $L^\varepsilon = L_\varepsilon$, the Yosida regularization of function L.

Section 3 provides the optimality system for the approximate control problem:

(5.12) $$\text{Min}\left\{\int_0^T L_\varepsilon(y,u)dt + \frac{1}{2}\int_0^T |u-u^*|^2 dt\right\}$$

subject to (5.2) with A^ε instead of A.

There is p_ε in $L^2(0,T;R^N)$ such that:

(5.13) $\quad y'_\varepsilon(t) = A^\varepsilon(y_\varepsilon(t)) + Dy_\varepsilon(t-h) + Bu_\varepsilon(t) \quad$ a.e. $[0,T]$,

(5.14) $\quad -p_\varepsilon(t) = \nabla A^\varepsilon(y_\varepsilon(t))^* p_\varepsilon(t) + D^* p_\varepsilon(t+h) - q_\varepsilon(t)$,

(5.15) $\quad [q_\varepsilon(t), B^* p_\varepsilon(t) + u_\varepsilon - u^*] \in \partial L_\varepsilon(y_\varepsilon(t), u_\varepsilon(t)) \quad$ a.e. $[0,T]$,

(5.16) $\quad y_\varepsilon(0) = y_0, \quad y_\varepsilon(s) = f(s) \quad$ a.e. $[-h,0]$,

(5.17) $\quad p_\varepsilon(T) = 0, \quad p_\varepsilon(s) = 0 \quad$ a.e. $[T, T+h]$.

We know that:

$y_\varepsilon \to y^*$ in $C(0,T;R^N)$ strongly,

$y'_\varepsilon \to y^{*\prime}$ in $L^2(0,T;R^N)$ strongly,

$u_\varepsilon \to u^*$ in $L^2(0,T;R^N)$ strongly,

$p_\varepsilon \to p$ in $L^\infty(0,T;R^N)$ weakly*,

$q_\varepsilon \to q$ in $L^1(0,T;R^N)$ weakly and

$$[q(t), B^* p(t)] \in \partial L(y^*(t), u^*(t)) \quad a.e.\ [0,T].$$

From (5.14) one sees $\{p'_\varepsilon\}$ bounded in $L^1(0,T;R^N)$, i.e. $p_\varepsilon \to p$ in $C(0,T;R^N)$ strongly. Then again from (5.14) we infer $p'_\varepsilon \to p'$ in $L^1(0,T;R^N)$ weakly.

Since $\nabla A^\varepsilon(y_\varepsilon(t))^*$, which denotes the transposed of matrix $\nabla A^\varepsilon(y_\varepsilon(t))$, is in $L^\infty(0,T;R^{N\times N})$, we get $\nabla A^\varepsilon(y_\varepsilon(t))^* \to g(t)$ weakly* in $L^\infty(0,T;R^{N\times N})$.

Next, using a variant of Lemma 3 from Barbu [3] we see that $g(t) \in \partial A(y^*(t))$ a.e. $[0,T]$, where ∂A is the generalized gradient in the sense of Clarke, of Lipschitz function A.

We can state the following result:

<u>THEOREM 5.1. Let</u> $[y^*, u^*]$ <u>be an optimal pair for problem</u> (5.1)-(5.3). <u>Then there exist functions</u> $p \in W^{1,1}(0,T;R^N)$ <u>and</u> $q \in L^1(0,T;R^N)$ <u>such that</u> :

$$y^{*\prime}(t) = A(y^*(t)) + Dy^*(t-h) + Bu^*(t) \quad a.e.\ [0,T],$$
$$-p'(t) = \partial A(y^*(t))^* \cdot p(t) + D^* p(t+h) - q(t),$$
$$[q(t), B^* p(t)] \in \partial L(y^*(t), u^*(t)) \quad a.e.\ [0,T],$$
$$y^*(0) = y_0, \quad y^*(s) = \varphi(s) \quad a.e.\ [-h,0],$$
$$p(T) = 0, \quad p(s) = 0 \quad a.e.\ [T, T+h].$$

<u>Moreover functions</u> y^*, u^*, p, q <u>are limits of functions</u> $y_\varepsilon, u_\varepsilon, p_\varepsilon, q_\varepsilon$ <u>which verify approximate equations</u> (5.13) - (5.17).

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