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ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS
No.23/1981

BUCURESTI

A GENERALIZATION OF THE KREIN RESOLVENT FORMULA AND APPLICATIONS TO THE THEORY OF SELF-ADJOINT EXTENSIONS OF SEMI-BOUNDED SYMMETRIC OPERATORS

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March 1981

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Abstract.

A generalisation to the case of arbitrary (equal) deficiency indices of a formula of Krein [7] which gives the resolvent of an arbitrary self-adjoint symmetric operator A, in terms of a fixed self-adjoint extension A_0 and of self-adjoint operators (abstract boundary conditions operators) acting in the deficiency subspace of $(A-\lambda_0)$ for some $\lambda_0 \in \rho(A_0)$ is proved (Theorem 1).

Combined with a distinguished property of the Friedrichs extension (Theorem 2) the resolvent formula gives almost immediately the basic results of the Krein-Vishik-Birman theory [4-6] of the self-adjoint extensions of positive symmetric operators, as well as new results saying that the Friedrichs extension is the limit, in some sense, of sequences of self-adjoint extensions which are not bounded from below.

1. Introduction.

Let T_0 be the closure in $L^2([0,\infty))$ of $-d^2/dx^2$ defined on $C_0((0,\infty))$. T_0 has deficiency indices (1,1) and its self-adjoint extentions T_α are indexed by the boundary conditions

$$(1.1) \qquad \frac{d}{dx} f(x)\Big|_{x=0} = \alpha f(0) \qquad \alpha \in (-\infty, \infty]$$

The Friedrichs extention T_F of T_0 corresponds to the boundary condition $f(0)=0$,which,at the formal level is obtained by taking the limit $\alpha \to \infty$ in (1.1). Indeed, one can verify that

$$(1.2) \qquad \lim_{\alpha \to \infty} \| (T_F+1)^{-1} - (T_\alpha+1)^{-1} \| = 0$$

Again at the formal level, one can see that T_F is also obtained by taking the limit $\alpha \to -\infty$ in (1.1) and indeed, again one can verify with some work that

$$(1.3) \qquad \lim_{\alpha \to -\infty} \| (T_F+1)^{-1} - (T_\alpha+1)^{-1} \| = 0$$

Now for $\alpha<0$, $\sigma(T_\alpha)=\{\lambda_\alpha\}\cup[0,\infty)$ where $\lim_{\alpha\to-\infty}\lambda_\alpha=-\infty$ .So in the sense of (1.3) T_F is the limit of self-adjoint extentions of T_0, which are not(uniformly) bounded from below.

The same phenomenon has been observed recently, in the study of regularisations of the one-dimentional Schrodinger operator [2,3] .The initial motivation of this paper was to see whether this phenomenon (the fact that the Friederichs extension of a semi-bounded symmetric operator is in some sense, the limit of some sequences of self-adjoint extensions which are not uniformly bounded from below) is a generic one or is related to the concrete structure of the above examples. The fact that this is a generic phenomenon is the content of our Theorems 6 and 7.

Since the classical Krein-Birman theory [4,5,6] of self-adjoint extentions of semi-bounded symmetric operators is concerned mainly with self-adjoint extensions bounded from below, it is more or less clear that the above problem cannot be easily settled in the framework of this theory and new tools are nedeed. Our main tool is a generalisation to the case of arbitrary (equal) deficiency indices of a formula of Krein [7,8] which gives the resolvent of an <u>arbitrary</u> self-adjoint extension of a symmetric operator A, in termd of the resolvent of a <u>fixed</u> self-adjoint extension of A and of self-adjoint operators (abstract boundary conditions operators) acting in the deficiency space of $(A-\lambda_0)$ for some $\lambda_0 \in \varrho(A_0)$. The point of this formula is that it almost immediately gives the spectral properties of a self-adjoint extension in terms of the spectral properties of the corresponding boundary conditions operator and viceversa. As already said, for finite deficiency indices, this formula has been proved by Krein [7,8] . Moreover, a formula with essentially the same structure has beem announced [9] for semi-bounded self-adjoint extentions of semi-bounded symmetric operators.

In full generality the result seems to be new(although we cannot exclude the fact that it is known as folk-lore within the Krein's school).Finaly,we would like to point out that although we shall prove below the formula only for self-adjoint extensions of symmetric operators,one can along the same lines give the formula for generalised resolvents of an arbitrary symmetric operator.We hope to come back to these questions in a future publication.

Our next result,having again preparatory character,but we believe,interesting in itself,gives a distinguished property of the Friedrichs extension and is contained in Theorem 2.It concerns the behaviour of the resolvent of the Friederichs extension as the argument is going to $-\infty$.

The basic results of the Krein-Vishik-Birman theory (see Theorems 3,4,5) as well as our new results in the theory of self-adjoint extensions of semi-bounded symmetric operators (see Theorems 6,7,8) are easy consequences of Theorems 1 and 2.

2.A generalised Krein resolvent formula.

In this section we shall prove the following theorem:

Theorem 1. Let A be a closed symmetric operator in a separable Hilbert space $\mathcal{H}$,with domain $\mathcal{D}(A)$ and equal(finite or infinite) deficiency indices.Let A_0 be a fixed self-adjoint extension of A, $\lambda_0 \in \rho(A_0)$,and $P(\lambda_0)$ be the orthogonal projection on $\mathcal{H} \ominus (A-\lambda_0)\mathcal{D}(A)$ .Then denoting

$$(2.1) \quad E(\lambda_1, \lambda_2) = (A_0 - \lambda_2)(A_0 - \lambda_1)^{-1} \; ; \; \lambda_1, \lambda_2 \in \rho(A_0)$$

$$(2.2) \qquad F(\lambda_1,\lambda_2)=(\lambda_1-\lambda_2)E(\lambda_1,\bar{\lambda}_2)+(\lambda_2-\bar{\lambda}_2)/2$$

the following assertions hold:

i. The formula

$$(2.3) \qquad R_\lambda=(A_0-\lambda)^{-1}-E(\lambda,\lambda_0)Q\left[n+QF(\lambda,\lambda_0)Q\right]^{-1}QE(\lambda,\lambda_0)$$

gives a one to one correspondence between all the self-adjoint extensions of A and all the pairs (Q, n) where Q is an orthogonal projection smaller than $P(\lambda_0)$ and n is a self-adjoint operator in $Q\mathcal{H}$.

ii. Suppose $\lambda_0, \lambda_0' \in \varrho(A_0)$ and let $(Q,n),(Q',n')$ representing the same self-adjoint extension of A, via the formula (2.3) written in λ_0 and λ_0' respectively. Then

$$(2.4) \qquad Q'\mathcal{H}=E(\bar{\lambda}_0', \bar{\lambda}_0)Q\mathcal{H}$$

$$(2.5) \qquad n'=Q'E(\lambda_0,\lambda_0')\left[n+\mathrm{Re}\,QF(\lambda_0',\lambda_0)Q\right]E^*(\lambda_0,\lambda_0')Q'$$

During the proof of Theorem 1, we shall use the following elementary fact. Let P be an orthogonal projec ion in $\mathcal{H}$. t

In the orthogonal sum representation of $\mathcal{H}$, $\mathcal{H}=P\mathcal{H}\oplus(1-P)\mathcal{H}$ all bounded operators, B, have a matrix representation

$$B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}$$

For an operator $d: P\mathcal{H}\to P\mathcal{H}$ we shall denote by D the operator

$$D=\begin{pmatrix} d & 0 \\ 0 & 0\end{pmatrix}$$

and for a operator B: $\mathcal{H} \to \mathcal{H}$ we shall denote $b_P = b_{11}: P\mathcal{H} \to P\mathcal{H}$

Lemma 1. Let B: $\mathcal{H} \to \mathcal{H}$, $d: P\mathcal{H} \to P\mathcal{H}$ be bounded operators in $\mathcal{H}$ and $P\mathcal{H}$ respectively. The operators $1+BD$, $1+DB$ have bounded inverse in $\mathcal{H}$ if and only if $1+b_P d$, $1+d b_P$ respectively, have bounded inverse in $P\mathcal{H}$. Moreover

$$D(1+BD)^{-1} = \begin{pmatrix} d(1+b_P d)^{-1} & 0 \\ 0 & 0 \end{pmatrix}; \quad (1+DB)^{-1}D = \begin{pmatrix} (1+d b_P)^{-1} d & 0 \\ 0 & 0 \end{pmatrix}$$

Proof. Verify that

$$(1+BD)^{-1} = \begin{pmatrix} (1+b_P d)^{-1} & 0 \\ -b_{21} d (1+b_P d)^{-1} & 0 \end{pmatrix}$$

and the similar formula for $(1+DB)^{-1}$.

Proof of Theorem 1. We shall consider first nonreal λ_o. Let A_α, A_o be self-adjoint extensions of A and $\lambda, \lambda_o \in \varrho(A_o) \cap \varrho(A_\alpha)$. Denoting

$$(2.6) \qquad C_\alpha(\lambda) = (A_o - \lambda)^{-1} - (A_\alpha - \lambda)^{-1}$$

and using the resolvent equation one obtains

$$(2.7) \qquad C_\alpha(\lambda)\left[1 - (\lambda - \lambda_o)\left((A_o - \lambda_o) - C_\alpha(\lambda_o)\right)\right] = \\ = \left[1 + (\lambda - \lambda_o)(A_o - \lambda)^{-1}\right] C_\alpha(\lambda_o)$$

Let $\tilde{A}$ be the restriction of A_o and A_α on $\mathcal{D}(A_o)\cap\mathcal{D}(A_\alpha)$. Then $\tilde{A}$ is a closed symmetric extension of A, so that if $P(\lambda)$ and $Q_\alpha(\lambda)$ are the orthogonal projections on $\mathcal{H}\ominus(A-\lambda)\mathcal{D}(A)$ and $\mathcal{H}\ominus(\tilde{A}-\lambda)\mathcal{D}(\tilde{A})$ then $Q_\alpha(\lambda)\leq P(\lambda)$. From the fact that A_o and A_α are self-adjoint extensions of A it follows that [8 Ch. VIII §106]

$$(2.8)\qquad C_\alpha(\lambda)=Q_\alpha(\bar{\lambda})\,C_\alpha(\lambda)\,Q_\alpha(\lambda)$$

Moreover, from the definition of $\tilde{A}$ it follows that $C_\alpha(\lambda)$ is injective on $Q_\alpha(\lambda)\mathcal{H}$. The next observation is the fact that for $\lambda_1,\lambda_2\in\varrho(A_o)$

$$(2.9)\qquad (1-Q_\alpha(\bar{\lambda}_1))\,E(\lambda_1,\lambda_2)\,Q_\alpha(\bar{\lambda}_2)=0$$

wherefrom it follows that $E(\lambda_1,\lambda_2)$ maps bicontinuously $Q_\alpha(\bar{\lambda}_2)\mathcal{H}$ onto $Q_\alpha(\bar{\lambda}_1)\mathcal{H}$. Due to the fact that

$$(2.10)\qquad 1-(\lambda-\lambda_o)\left((A_o-\lambda_o)^{-1}-C_\alpha(\lambda_o)\right)=(A_\alpha-\lambda)(A_\alpha-\lambda_o)^{-1}$$

the l.h.s. of equation (2.10) is invertible and one can rewrite (2.7) as

$$(2.11)\qquad C_\alpha(\lambda)=E(\lambda,\lambda_o)C_\alpha(\lambda_o)\left[1+(\lambda-\lambda_o)E(\lambda,\lambda_o)C_\alpha(\lambda_o)\right]^{-1}E(\lambda,\lambda_o)$$

Denote by $d_\alpha : Q_\alpha(\lambda_o)\mathcal{H}\to Q_\alpha(\lambda_o)\mathcal{H}$ the operator

$$(2.12)\qquad d_\alpha=E(\bar{\lambda}_o,\lambda_o)\,C_\alpha(\lambda_o)\Big|_{Q_\alpha(\lambda_o)\mathcal{H}}$$

and by $D_\alpha : \mathcal{H} \to \mathcal{H}$

$$D_\alpha = E(\bar{\lambda}_0, \lambda_0)\, C_\alpha(\lambda_0)$$

Then d_α is injective and the matrix representation of D_α (according to the decomposition $\mathcal{H} = Q_\alpha(\lambda_0)\mathcal{H} \oplus (1 - Q_\alpha(\lambda_0))\mathcal{H}$ is

$$D_\alpha = \begin{pmatrix} d_\alpha & 0 \\ 0 & 0 \end{pmatrix}$$

Using Lemma 1 one can rewrite (2.11) as follows

$$(2.13) \quad \begin{aligned} C_\alpha(\lambda) &= \\ &= E(\lambda, \bar{\lambda}_0) Q_\alpha(\lambda_0) d_\alpha \left[1 + (\lambda - \lambda_0)\, e_{Q_\alpha}(\lambda, \bar{\lambda}_0) d_\alpha\right]^{-1} Q_\alpha(\lambda_0) E(\lambda, \lambda_0) \end{aligned}$$

Since d_α is injective one can define n_α by

$$(2.14) \qquad n_\alpha = d_\alpha^{-1} - i\, Jm\, \lambda_0$$

Starting from $C_\alpha^*(\lambda_0) = C_\alpha(\bar{\lambda}_0)$ and using (2.11) for $C_\alpha(\bar{\lambda}_0)$ one can verify that

$$(2.15) \qquad D_\alpha^* = D_\alpha\left[1 - (\lambda - \lambda_0) D_\alpha\right]^{-1} = D_\alpha (A_\alpha - \lambda_0)(A_\alpha - \bar{\lambda}_0)^{-1} E(\lambda_0, \bar{\lambda}_0)$$

wherefrom it follows that d_α^* is also injective and then $(d_\alpha^{-1})^* = (d_\alpha^*)^{-1}$. Combining (2.14) and (2.15) one sees that n_α is self-adjoint. Combining (2.2), (2.13) and (2.14) one obtains (2.3). Conversely, suppose R_λ be defined by (2.3). One can verify directly that

$$(2.16) \qquad F(\lambda, \lambda_0) = \tfrac{1}{2}\left[(\lambda - \lambda_0) E(\lambda, \bar{\lambda}_0) + (\lambda - \bar{\lambda}_0) E(\lambda, \lambda_0)\right]$$

wherefrom

$$(2.17)\quad \operatorname{Im} F(\lambda,\lambda_0) \equiv \frac{1}{2i}\left(F(\lambda,\lambda_0) - F^*(\lambda,\lambda_0)\right) = (\operatorname{Im}\lambda)\,|E(\lambda,\lambda_0)|^2$$

It follows that $[n+f_Q(\lambda,\lambda_0)]^{-1}$ exists (at least)for non-real λ .Defining $d: Q\mathcal{H} \to Q\mathcal{H}$ by

$$d = (n + i\operatorname{Im}\lambda_0)^{-1}$$

and taking into account (2.3) and Lemma 1 one obtains

$$(2.18)\quad R_\lambda = (A_0-\lambda)^{-1} - E(\lambda,\bar\lambda_0)D[1+(\lambda-\lambda_0)E(\lambda,\bar\lambda_0)D]^{-1}E(\lambda,\lambda_0) =$$

$$= (A_0-\lambda)^{-1} - E(\lambda,\bar\lambda_0)[1+(\lambda-\lambda_0)DE(\lambda,\bar\lambda_0)]^{-1}DE(\lambda,\lambda_0)$$

(remind that $D = QdQ$ is viewed as an operator in $\mathcal{H}$).

Using (2.18) one can verify that R_λ satisfy the equations

$$(2.19)\quad R_\lambda - R_{\lambda_0} = (\lambda-\lambda_0)R_\lambda R_{\lambda_0} = (\lambda-\lambda_0)R_{\lambda_0}R_\lambda$$

The next observation is that

$$(2.20)\quad \overline{\mathcal{D}(A_0)} \cap P(\lambda_0)\mathcal{H} = \{0\}$$

Indeed,suppose that $h\in\mathcal{D}(A_0)\cap P(\lambda_0)\mathcal{H}$ .For all $k\in\mathcal{D}(A)$

$$(k,(A_0-\lambda_0)h) = ((A_0-\bar\lambda_0)k,h) = 0$$

i.e. $(A_0-\lambda_0)h=0$ which implies $h=0.$ *Now* (2.3), (2.9) *and* (2.20) *implies*

$$(2.21)\quad \ker R_{\lambda_0} = \{0\}$$

From (2.16) it follows that $F^*(\lambda,\lambda_0) = F(\bar\lambda,\lambda_0)$ wherefrom

(2.22) $\mathcal{R}_\lambda^* = \mathcal{R}_{\bar\lambda}$

The properties (2.19),(2.21),(2.22) are sufficient to imply that there exists a self-adjoint operator T such that for nonreal λ [8, Chap VI §75; 10, Chap VIII §1]

(2.23) $\mathcal{R}_\lambda = (T-\lambda)^{-1}$

Finaly, taking into account that (see (2.9))

$$P(\lambda_0)\,E(\lambda,\lambda_0)\,(1-P(\lambda)) = 0$$

it follows from (2.3) and (2.23) that

$$(T-\lambda)^{-1}\Big|_{(A-\lambda)\mathcal{D}(A)} = (A_0-\lambda)^{-1}\Big|_{(A-\lambda)\mathcal{D}(A)}$$

wherefrom

(2.24) $T \supset A$

which finishes the proof of i. for nonreal λ_0.

Let now $\lambda_0' \in \varrho(A_0)$ be arbitrary (real or nonreal). Let A_α be a self-adjoint extension of A and Q_α' be the orthogonal projection on $\mathcal{H}\ominus(A_0-\lambda_0')(\mathcal{D}(A_0)\cap\mathcal{D}(A_\alpha))$. Then (see (2.9))

$$Q_\alpha'\,E(\bar\lambda_0',\bar\lambda_0)\,Q_\alpha = E(\bar\lambda_0',\bar\lambda_0)\,Q_\alpha$$

(2.25)

$$Q_\alpha\,E(\lambda_0',\lambda_0)\,Q_\alpha' = Q_\alpha\,E(\lambda_0',\lambda_0)$$

A simple, but a little bit tedious calculation, using the definition and the properties of $F(\lambda,\lambda_0)$ and (2.25), shows that if n_α' is given by (2.5) then for nonreal λ

$$E(\lambda,\bar\lambda_0)\,Q_\alpha\,[n_\alpha + Q_\alpha F(\lambda,\lambda_0)Q_\alpha]^{-1}\,Q_\alpha\,E(\lambda,\lambda_0) =$$

$$= E(\lambda,\bar\lambda_0')\,Q_\alpha'\,[n_\alpha' + Q_\alpha' F(\lambda,\lambda_0')Q_\alpha']\,Q_\alpha'\,E(\lambda,\lambda_0')$$

which proves (2.3) for arbitrary $\lambda_0' \in \varrho(A_0)$ and at the same time, the second point of the thoerem.

3. Applications to the theory of self-adjoint extensions of semi-bounded symmetric operators.

We shall start by listing a few observations of technically preparatory character.

1° Let $\mathcal{H}$ be a finite-dimensional Hilbert space, $D \subset \mathbb{C}$ be the domain containing $I = [a, b] \subset \mathbb{R}$ and $F(\lambda): \mathcal{H} \to \mathcal{H}$ an operator valued function satisfying

a. $F(\lambda)$ is analytic in D.

b. $F^*(\bar{\lambda}) = F(\lambda)$ for $\lambda \in I$

c. $\frac{dF}{d\lambda} > 0$ on I

Then, all the eigenvalues of $F(\lambda)$ are continuous and stricly increasing functions of λ on I.

2° Let N be a semi-bounded self-adjoint operator, $F(\lambda)$ a bounded self-adjoint operators valued function continuous for $\lambda \in I \subset \mathbb{R}$. Then $\inf \sigma(N + F(\lambda))$ is a continuous function of λ on I.

3° Let A_n be a sequence of bounded self-adjoint operators satisfying the conditions:

i. $0 \leq A_{n+1} \leq A_n$

ii. $s\text{-}\lim_{n\to\infty} A_n = 0$

iii. There exist $\delta_n \geq 0$, $\lim_{n\to\infty} \delta_n = \delta$

such that the essential spectrum of A_n is contained in $[0\ \delta_n]$.

Then

$$(3.1) \qquad \lim_{n\to\infty} \|A_n\| \leq \delta$$

Proof. Suppose

$$(3.2)\qquad \lim_{n\to\infty} \|A_n\| = \lim_{n\to\infty} (\sup \sigma(A_n)) > \delta .$$

Then for arbitrary $\varepsilon > 0$ there exist $n_1, p < \infty$ such that for $n \geq n_1$, A_n has precisely p eigenvalues(counting multiplicities) in $[\delta+\varepsilon, \infty)$ .Let P_n be the spectral projection of A_n corresponding to $[\delta+\varepsilon, \infty)$ and ψ_1, $\|\psi_1\|=1$ an eigenvector of A_{n_1} corresponding to its largest eigenvalue, λ_1. It is not hard to see that for $n > n_1$

$$(3.3)\qquad \|P_n \psi_1\|^2 \geq 1/2$$

Indeed,suppose (3.3) is not true for some $n_2 > n_1$ .Then writing

$$(3.4)\qquad \psi_1 = g + P_{n_2}\psi_1 , \qquad g \in (1-P_{n_2})\mathcal{H}$$

and using i. one can verify that

$$(3.5)\qquad (g, A_{n_1} g) \geq \lambda_1 (1 - 2\|P_{n_2}\psi_1\|^2) + (\delta+\varepsilon)\|P_{n_2}\psi_1\|^2 =$$
$$= (\delta+\varepsilon)\|g\|^2 + (\delta+\varepsilon-\lambda_1)(2\|P_{n_2}\psi_1\|^2 - 1) \geq (\delta+\varepsilon)\|g\|^2$$

On the other hand,from i. for all $\psi \in P_{n_2}\mathcal{H}$

$$(3.6)\qquad (\psi, A_{n_1}\psi) \geq (\psi, A_{n_2}\psi) \geq (\delta+\varepsilon)\|\psi\|^2$$

From (3.4-6) and the min-max principles it follows that A_{n_1} has at least $p+1$ eigenvalues in $[\delta+\varepsilon, \infty)$ which is a contradiction. From (3.3) and (3.4)

$$(3.7) \quad \lim_{n\to\infty} (\psi_1, A_n \psi_1) \geq (\delta+\varepsilon)/2$$

which contradicts ii. and the Lemma is proved.

From now on A will be a closed symmetric operator satisfying

$$(3.8) \quad m(A) = \inf_{f\in \mathcal{D}(A)} (f, Af)/\|f\|^2 = 1$$

In what follows, A_F denotes the Friedrichs extension of A [11], P denotes the orthogonal projection on $\mathcal{H} \ominus A\mathcal{D}(A)$ and Q an orthogonal projection smaller than P.

Consider the analytic valued function $f_Q(\lambda): Q\mathcal{H} \to Q\mathcal{H}$

$$(3.9) \quad f_Q(\lambda) = Q F(\lambda, 0)\Big|_{Q\mathcal{H}} = \lambda Q A_F (A_F - \lambda)^{-1}\Big|_{Q\mathcal{H}}; \quad \lambda \in \varrho(A_F)$$

For $\lambda \in (-\infty, 1)$, $f_Q(\lambda)$ is self-adjoint and

$$(3.10) \quad \sigma(f_Q(\lambda)) \subset [\lambda/(1-\lambda), \lambda]$$

Moreover, on $(-\infty, 1)$, $f_Q(\lambda)$ is strictly increasing

$$(3.11) \quad \frac{d}{d\lambda} f_Q(\lambda) \geq \min\{1, 1/(1-\lambda)^2\}$$

From (3.10) it follows that for $\lambda \in (-\infty, 0) \cup (0,1)$, $f_Q^{-1}(\lambda)$ exists and (3.11) implies [11] that it is a monotonically decreasing function of λ.

Our next theorem describes a distinguished property of the Friedrichs extension.

Theorem 2. For all $Q \leq P$

(3.12) i. $$s\text{-}\lim_{\lambda \to -\infty} f_Q^{-1}(\lambda) = 0$$

ii. If, in addition A_F^{-1} is compact then

(3.13) $$\lim_{\lambda \to -\infty} \| f_Q^{-1}(\lambda) \| = 0$$

Proof. i. If G_μ is the spectral measure of A_F, then

$$(f, a A_F (A_F + a)^{-1} f) = \int_1^\infty \frac{a\mu}{a+\mu} d(G_\mu f, f) \; ; \quad a > 0$$

and the monotone convergence theorem implies that

$$\lim_{a \to \infty} (f, a A_F (A_F + a)^{-1} f) < \infty$$

is equivalent with the fact that $f \in \mathcal{D}(A_F^{1/2})$. On the other hand [11] $\mathcal{D}(A_F^{1/2}) \cap P\mathcal{H} = \{0\}$ wherefrom, for $f \in Q\mathcal{H}$

(3.14) $$\lim_{a \to \infty} (f, a A_F (A_F + a)^{-1} f) = \infty$$

The first point of the theorem follows from (3.14) by standard arguments. Indeed, since for $\lambda < 0$, $f_Q^{-1}(\lambda)$ is decreasing and negative, it follows that it has a weak limit as $\lambda \to -\infty$ and then [11] a strong one. Denote

(3.15) $$B = s\text{-}\lim_{\lambda \to -\infty} f_Q^{-1}(\lambda)$$

Since $B \leq 0$, for all $\varepsilon > 0$, $\lambda < 0$ [11]

(3.16) $$(-B + \varepsilon)^{-1} \geq (-f_Q^{-1}(\lambda) + \varepsilon)^{-1}$$

Suppose now $B \neq 0$ .Then it exists $g \in Q\mathcal{H}$ such that $\lim\limits_{\varepsilon \to 0} (g, (-B+\varepsilon)^{-1} g) = b < \infty$ .Then

$$(3.17) \quad b \geq \lim_{\varepsilon \to 0} \left(g, \left(-f_Q^{-1}(\lambda) + \varepsilon\right)^{-1} g \right) = -\left(g, f_Q(\lambda) g\right)$$

Taking the limit $\lambda \to -\infty$ in (3.17) one contradicts (3.14) so that $B = 0$.

ii. If $(A_F - \lambda)^{-1}$ is compact, it follows that $f_Q(\lambda) - \lambda = \lambda^2 Q (A_F - \lambda)^{-1} \restriction_{Q\mathcal{H}}$ is compact and then $\lambda^{-1} - f_Q^{-1}(\lambda) = $ $= f_Q^{-1}(f_Q(\lambda) - \lambda)/\lambda$ is compact. In other words, the essential spectrum of $-f_Q^{-1}(\lambda)$ is contained in $[0, -\frac{1}{\lambda}]$ and (3.13) follows from 3°.

Taking $A_0 = A_F, \lambda_0 = 0$ the formula (2.3) for an arbitrary self-adjoint extension A_α of A writes as

$$(3.18) \quad (A_\alpha - \lambda)^{-1} = (A_F - \lambda)^{-1} - E(\lambda, 0) Q_\alpha [n_\alpha + f_{Q_\alpha}(\lambda)]^{-1} Q_\alpha E(\lambda, 0)$$

Let $m_\alpha(\lambda): Q_\alpha\mathcal{H} \to Q_\alpha\mathcal{H}$ be defined by

$$m_\alpha(\lambda) = n_\alpha + f_{Q_\alpha}(\lambda)$$

The following is a direct consequence of Theorem 1.

Theorem 3. Let $\lambda_1 \in \varrho(A_F)$ .Then $\lambda_1 \in \varrho(A_\alpha)$ if and only if $0 \in \varrho(m_\alpha(\lambda_1))$

The following two theorems are among the basic results of the Krein-Vishik-Birman theory [4-6,12] .

Theorem 4.

i. $\sigma(A_\alpha) \subset [\lambda_1, \infty)$ if and only if

$$(3.19) \qquad m_\alpha(\lambda_1) \leq 0$$

ii. If A_F^{-1} is compact then A_α is bounded from below if and only if $-n_\alpha$ is bounded from below.

Proof. i. Suppose $m_\alpha(\lambda_1) \leq 0$. Then from (3.11) for all $\lambda < \lambda_1$, $m_\alpha(\lambda) \leq -\int_\lambda^{\lambda_1} \min\{1, 1/(1-t)^2\}\,dt$ so that due to Theorem 3, $\lambda \in \varrho(A_\alpha)$.

Suppose now $\sigma(A_\alpha) \subset [\lambda_1, \infty)$. Then [11] for all $\lambda < \lambda_1$, $(A_F-\lambda)^{-1} \leq (A_\alpha-\lambda)^{-1}$ which implies via Theorem 1 that $m_\alpha(\lambda) \leq 0$ which together with 2° finishes the proof of i.

ii. Suppose $-n_\alpha$ is semibounded. From Theorem 2ii $f_Q(\lambda) \leq -\beta(\lambda)$; $\lim_{\lambda\to-\infty} \beta(\lambda) = -\infty$ i.e. for $\lambda \to -\infty$, $m_\alpha(\lambda) < 0$ i.e. $\lambda \in \varrho(A_\alpha)$. Conversely, if $\sigma(A_\alpha) \subset [\lambda_1, \infty)$ then by the first point of the theorem $n_\alpha + f_{Q_\alpha}(\lambda_1) \leq 0$ and the proof is finished.

Suppose now $\sigma(A_\alpha) \subset [1, \infty)$ so that n_α has a bounded inverse. Define $c_\alpha : P\mathcal{H} \to P\mathcal{H}$ as the orthogonal sum of n_α^{-1} on $Q_\alpha\mathcal{H}$ and zero on $(P-Q_\alpha)\mathcal{H}$. The following is a direct consequence of Theorem 4.

Theorem 5. Let

$$f_P^{-1} = s\text{-}\lim_{\lambda\to 1} f_P^{-1}(\lambda) \tag{3.20}$$

Then $\sigma(A_\alpha) \subset [1, \infty)$ if and only if

$$-f_P^{-1} \leq c_\alpha \leq 0 \tag{3.21}$$

In particular, A has a unique self-adjoint extension with the spectrum included in $[1, \infty)$ if and only if $f_P^{-1} = 0$

The next three theorems are, besides Theorems 1 and 2 the main new results of our paper.

Theorem 6. Let A_ℓ be a sequence of self-adjoint extensions of A satisfying:

$$(3.22)\quad (-a_\ell, 1) \subset \rho(A_\ell)\ ;\ \lim_{\ell\to\infty} a_\ell = \infty$$

$$(3.23)\quad A_F^{-1} - A_\ell^{-1} \geq 0$$

Then

$$(3.24)\quad \text{i. } s\text{-}\lim_{\ell\to\infty} (A_F^{-1} - A_\ell^{-1}) = 0$$

ii. If one of the following is true :

a. $\dim P < \infty$

b. A_F^{-1} is compact

then

$$(3.25)\quad \lim_{\ell\to\infty} \| A_F^{-1} - A_\ell^{-1} \| = 0$$

Proof. Without loss of generality we can take $Q_\ell = P$. Indeed if $Q_\ell < P$ one can consider $\tilde{A}_\ell$ such that

$$A_F^{-1} - \tilde{A}_\ell^{-1} = (A_F^{-1} - A_\ell^{-1}) + \ell^{-1}(P - Q_\ell)$$

From (3.22) it follows that n_ℓ has a bounded inverse and (3,23) implies $n_\ell > 0$. From (3.22) and Theorem 3, for all $\lambda \in (-a_\ell, 0)$

$$(3.26)\quad 0 \in \rho(n_\ell + f_P(\lambda))$$

which is equivalent to

$$(3.27)\quad 0 \in \rho(n_\ell^{-1} + f_P^{-1}(\lambda)) \qquad \text{all } \lambda \in (-a_\ell, 0)$$

Since for $\lambda<0, |\lambda|$ sufficiently small $\sigma(n_q^{-1}+f_P^{-1}(\lambda))\subset(-\infty,0)$ from (3.26) and 2°

$$\sigma(n_q^{-1}+f_P^{-1}(-a_q))\subset(-\infty,0)$$

i.e.

$$0\leq n_q^{-1}\leq -f_P^{-1}(-a_q) \tag{3.27}$$

Taking into account that $n_q^{-1}=A_F^{-1}-A_q^{-1}$,Theorem 2 and (3.27) the proof is finished.

Theorem 7. Suppose $\dim P\mathcal{H}=m<\infty$ and A_q be a sequence of self-adjoint extensions of A with the property that there exists $\{a_q\}_1^\infty$, $a_q>1$, $\lim\limits_{q\to\infty} a_q=\infty$ such that A_q has m eigenvalues (counting multiplicities) in $(-\infty,-a_q)$. Then

$$\lim_{q\to\infty}\|A_F^{-1}-A_q^{-1}\|=0 \tag{3.28}$$

Proof. Since A_q can have at most m eigenvalues in $(-\infty,1)$ [8] it follows that $0\in\varrho(A_q)$ .From 1° ,(3.11) and the fact that A_q must have m eigenvalues in $(-\infty,-a_q)$ it follows that $Q_q=P$ and $n_q>0$ .Then one can apply Theorem 6 ii.

Let now $a\in(-\infty,1)$ and $A_K(a)$ be the self-adjoint extension of A corresponding to the pair $(P,-f_P(a))$ .It is easy to see that $A_K(a)$ is nothing but the "soft" extension of Krein [4] corresponding to the point a, i.e. $A_K(a)$ is minimal among the self-adjoint extensions of A having spectrum in $[a,\infty)$ .The following is a direct consequence of Theorems 6 and 2.

Corrolary.

(3.29) i. $s\text{-}\lim_{a\to-\infty}\left(A_F^{-1}-A_K^{-1}(a)\right)=0$

ii. If one of the following is true

a. A_F^{-1} is compact

b. $\dim P<\infty$

Then

(3.30) $\lim_{a\to-\infty}\|A_F^{-1}-A_K^{-1}(a)\|=0$

We shall end up noting the following relations between the spectral properties of A_α and n_α.

Theorem 8.

i. Suppose n_α has discrete spectrum (i.e. has only finitely degenerated eigenvalues having no finite points of accumulation). Then the spectrum of A_α included in $[a,b]\subset\varrho(A_F)$ is discrete.

ii. Suppose A_F^{-1} is compact and suppose $[a,b]\subset\varrho(A_F)\cap\varrho(-n_\alpha)$ Then the spectrum of A_α included in $[a,b]$ is discrete.

Proof.

i. There exists $\delta\in\mathbb{R}$ such that $(n_\alpha+\delta)^{-1}$ exists and is bounded. Then

$$n_\alpha+f_{Q_\alpha}(\lambda)=(n_\alpha+\delta)\left[1+(n_\alpha+\delta)^{-1}\left(f_{Q_\alpha}(\lambda)-\delta\right)\right]$$

Since $(n_\alpha+\delta)^{-1}$ is compact and $f_{Q_\alpha}(\lambda)$ is a ~~x~~ bounded operators valued analytic function in $\mathbb{C}\setminus\sigma(A_F)$ the first point of the theorem follows from Theorem 3, and the analytic Fredholm alternative [1]

ii. Let $\lambda\in[a,b]$. Then $n_\alpha+f_{Q_\alpha}(\lambda)=$

$$=n_\alpha+\lambda+\lambda^2Q_\alpha(A_F-\lambda)^{-1}\Big|_{Q_\alpha\mathcal{H}}=(n_\alpha+\lambda)\left[1+\lambda^2(n_\alpha+\lambda)^{-1}Q_\alpha(A_F-\lambda)^{-1}\Big|_{Q_\alpha\mathcal{H}}\right]$$

and again one can apply the analytic Fredholm alternative.

4.Acknowledgements.

Useful disscusions with Drs. N.Angelescu,P.Dita and V.Georgescu are acknowledged.

5.References.

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