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by

Silviu Teleman

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*) Department of Mathematics, National Institute for Scientific and Technical Creation, Bd.Pacii 220, 79622 Bucharest, Romania

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The aim of this Note is to prove some new topological properties of the boundary measures; namely, roughly speaking, that the boundary measures are inner regular (i.e., by closed compact measurable subsets, where compactness, closedness is meant with respect to the Choquet topology). Stronger results are obtained for the pure states space of a C^*- algebra.

1. Let E be any Hausdorff locally convex topological real vector space and $K \subset E$ any non-empty compact convex subset. We shall use the notations introduced in [6], as well as many of the results we have presented there.

We recall that for any bounded function $f: K \to \mathbb{R}$ the function $\bar{f}: K \to \mathbb{R}$ is defined by

$$\bar{f} = \inf \{h;\ f \leq h,\ h \in A(K)\},$$

where the infimum is computed point-wise. Then $\bar{f}$ is the smallest concave upper semicontinuous function majorizing f (see [6], p.12; [5], §3; [4], Ch.XI, D18).

<u>PROPOSITION I. For an bounded upper semi-continuous function $f: K \to \mathbb{R}$ we have</u>

$$\bar{f}(x)= \sup\{\mu(f);\ \mu \sim \varepsilon_x\}, \qquad x\in K.$$

PROOF. In this equality μ runs over the compact convex set $\mathcal{M}_x^1(K)$ of all Radon probability measures μ , whose barycenter $b(\mu)=x$.

a) If we define

$$\varphi(x)= \sup\{\mu(f);\ \mu\sim \varepsilon_x\}, \quad x\in K,$$

then φ obviously is bounded and it is easy to prove that φ is concave; on the other hand, it is easy to prove that $f\leq\varphi$.

b) φ is upper semi-continuous. Indeed, let us first remark that, since the mapping

$$\mathcal{M}_+^1(K)\ni\mu\mapsto\mu(f_0)$$

on the set $\mathcal{M}_+^1(K)$ of all Radon probability measures on K, is continuous, for any $f_0\in C(K;\mathbb{R})$, the mapping

(1) $$\mathcal{M}_+^1(K)\ni\mu\mapsto\mu(f)$$

is upper semi-continuous. Let now $\alpha\in\mathbb{R}$ and define $L_\alpha=\{x\in K; \varphi(x)\geq\alpha\}$. Let $(x_i)_{i\in I}$ be a net in L_α and assume that $x_i\to\check{x}$ in K. Let $\varepsilon>0$ be given. Then we have

$$\varphi(x_i)\geq\alpha>\alpha-\varepsilon,$$

and, therefore, for any $i\in I$, there exists a $\mu_i\in\mathcal{M}_{x_i}^1(K)$, such that

(2) $$\mu_i(f) > \alpha - \varepsilon \quad , i \in I.$$

Passing to a subnet, if necessary, we can assume that $\lim\limits_{i\in I} \mu_i = \mu$ exists in $\mathcal{M}_+^1(K)$. From (2) and from the fact that the mapping in (1) is upper semi-continuous, we infer that

(3) $$\mu(f) \geq \alpha - \varepsilon ;$$

since we have $\lim\limits_{i\in I} b(\mu_i) = b(\mu)$, from (3) we infer that $\varphi(x) \geq \alpha - \varepsilon$, for any $\varepsilon > 0$, and, therefore $\varphi(x) \geq \alpha$. It follows that $x \in L_\alpha$, and this shows that L_α is closed; i.e., φ is upper semi-continuous.

c) If $h \in A(K)$ and $f \leq h$, then

$$\mu(f) \leq \mu(h) = h(b(\mu)) = h(x),$$

for any $\mu \in \mathcal{M}_x^1(K)$, $x \in K$. We infer that

$$\varphi(x) \leq h(x), \quad h \in A(K), \quad x \in K;$$

and, therefore, we have

(4) $$\varphi \leq \bar{f} ;$$

since $\bar{f}$ is the smallest concave upper semi-continuous function majorizing f, from (4) we immediately infer that $\varphi = \bar{f}$, and the Proposition is proved.

<u>REMARK</u>. Proposition 1 is a slight extension of Proposition 3.1 from [5] , where it is stated for a continuous function f.

COROLLARY. *For any bounded upper semi-continuous function $f:K \to \mathbb{R}$ we have*

$$\bar{f}(x)=f(x), \qquad x \in \operatorname{ex} K.$$

PROOF. This is an immediate consequence of the preceding Proposition and of H.Bauer's Theorem (see [5], Proposition 1.4; [6], Proposition 1.3).

PROPOSITION 2. *For any bounded upper semi-continuous function $f:K \to \mathbb{R}$ and any decreasing net $(f_\alpha)_{\alpha\in A}$ of bounded upper semi-continuous functions on K, such that $f_\alpha \downarrow f$ point-wise on K, we have $\bar{f}_\alpha \downarrow \bar{f}$ point-wise on K.*

PROOF. It is obvious that $(\bar{f}_\alpha)_{\alpha\in A}$ is a decreasing net, such that $\bar{f} \leq \lim\limits_{\alpha\in A} \bar{f}_\alpha$. Let then $\varepsilon > 0$ and $x \in K$ be given; there exists a $h \in A(K)$, such that

(i) $h(x) < \bar{f}(x) + \varepsilon$ and $f(y) < h(y), \ \forall\, y \in K.$

Let $K_\alpha = \{y; f_\alpha(y) - h(y) \geq 0\}$, $\alpha \in A$; since we have

$$\inf\{f_\alpha(y) - h(y);\ \alpha \in A\} < 0, \qquad y \in K,$$

we infer that $K_\alpha \downarrow \emptyset$ and, therefore, we can find an $\alpha_0 \in A$, such that $K_{\alpha_0} = \emptyset$ (because the sets K_α are compact).

We infer that $f_{\alpha_0}(y) \leq h(y)$, $y \in K$, and, therefore, we have

$$\mu(f_{\alpha_0}) \leq \mu(h) = h(b(\mu)), \qquad \mu \in \mathcal{M}_+^1(K).$$

From Proposition 1 and from (1) we infer that

$$\bar{f}_{\alpha_0}(x) \leq h(x) < \bar{f}(x) + \varepsilon,$$

and this implies that

$$\inf\{\bar{f}_\alpha(x);\ \alpha \in A\} \leq \bar{f}(x), \quad x \in K.$$

The Proposition is proved.

<u>COROLLARY.</u> <u>For any bounded upper semi-continuous function $f: K \to \mathbb{R}$ and any measure</u> $\mu \in \mathcal{M}_+^1(K)$, <u>which is maximal with respect to the Choquet order relation, we have</u>

$$\mu(\bar{f}) = \mu(f).$$

<u>PROOF.</u> There exists a decreasing net $(f_\alpha)_{\alpha \in A}$ of continuous functions $f_\alpha: K \to \mathbb{R}$, such that $f_\alpha \downarrow f$ point-wise on K. If $\mu \in \mathcal{M}_+^1(K)$ is maximal with respect to the Choquet order relation, then, we have

$$(1) \qquad \mu(\bar{f}_\alpha) = \mu(f_\alpha), \quad \alpha \in A$$

(see [5], Proposition 4.2.; [6], Lemma 1.2).

From (1) and from Proposition 2, by taking into account the τ - continuity of the measure μ, we infer that $\mu(\bar{f})=\mu(f)$, and the Corollary is proved.

LEMMA 1. *Let X be any compact space,* $F' \subset X$ *a* G_δ- *subset and* μ *a positive Radon measure on X. Then, for any* $\varepsilon > 0$ *there exists a compact Baire measurable subset* $D \subset F'$, *such that*

$$\mu(F') - \varepsilon < \mu(D).$$

PROOF. Let $F' = \bigcap_{n=0}^{\infty} G_n$, where $G_n \subset X$ are open subsets. Since μ is regular, there exists a compact subset $D_o \subset F'$, such that

$$\mu(F') - \varepsilon < \mu(D_o).$$

For any $n \in N$ we can find a continuous function $f_n : X \to [0,1]$ such that

$$f_n(x)=1, \text{ for } x \in D_o, \text{ and } f_n(x)=0, \text{ for } x \in \complement G_n.$$

Let $X_n = \{x \in X;\ f_n(x)=1\}$. Then $D = \bigcap_{n=0}^{\infty} X_n$ is a compact Baire measurable subset of X, such that $D_o \subset D \subset \bigcap_{n=0}^{\infty} G_n = F'$, and the Lemma is proved.

We shall denote by $\mathcal{B}_o(X)$ the σ- algebra of the Baire measurable subsets of the topological space X, i.e., the σ- algebra of subsets of X, which is generated by the set of all closed G_δ-subsets of X, whereas $\mathcal{B}(X)$ will stand for the σ-algebra of the Borel measurable subsets of X, which is generated by the set of all closed subsets of X.

When several topologies are considered on X, a special mark will indicate the topology to which these σ - algebras correspond.

2. For any function $f:K \to \mathbb{R}$ we shall denote $z(f)=\{x\in K;\ f(x)=0\}$ and $u(f)=\{x\in K;\ f(x)=1\}$.

Let now $F\subset K$ be a compact subset of K. Then χ_F is an upper semi-continuous function, whereas $\overline{\chi}_F$ is a concave upper semi-continuous function. It is easy to see that $F'=z(\overline{\chi}_F)$ is, therefore, a measure extremal G_δ- subset of K (see [6], p.26 and p.39).

LEMMA 2. $u(\overline{\chi}_F)=\overline{co}(F)$.

PROOF. From $\chi_F\leq\overline{\chi}_F\leq 1$ we infer that

$$F= u(\chi_F)\subset u(\overline{\chi}_F). \tag{1}$$

On the other hand, $u(\overline{\chi}_F)$ is a compact convex subset of K; therefore, from (1) we infer that

$$\overline{co}(F)\subset u(\overline{\chi}_F).$$

Let now $x_o\in K\setminus\overline{co}(F)$. Since the mapping $\mathcal{M}^1_{x_o}(K)\ni\mu\mapsto\mu(F)$ is upper semi-continuous on the compact space $\mathcal{M}^1_{x_o}(K)$, we infer that there exists a $\mu_o\in\mathcal{M}^1_{x_o}(K)$, such that

$$\mu_o(F)= \sup\{\mu(F);\ \mu\sim\varepsilon_{x_o}\}.$$

If we had $x_o\in u(\overline{\chi}_F)$, then, with Proposition 1, we would infer that $\mu_o(F)=1$, and, therefore, $x_o=b(\mu_o)\in\overline{co}(F)$, a contradiction. It follows that $x_o\notin u(\overline{\chi}_F)$ and, therefore,

$$u(\overline{\chi}_F)\subset\overline{co}(F).$$

The Lemma is proved.

PROPOSITION 3 . For any compact subset $F\subset K$ we have

a) $F\cap F'=\emptyset$ and $F\cup F'\supset \operatorname{ex} K$;

b) $\mu(F)+\mu(F')=1$, for any Radon probability measure $\mu\in\mathcal{M}_+^1(K)$, which is maximal for the Choquet order relation.

PROOF. a) If $x\in F$, then $\overline{\chi}_F(x)=1$, and therefore, $x\notin F'$ (as above, we have $F'=z(\overline{\chi}_F)$); for any $x\in \operatorname{ex} K$, if $x\notin F$, we have $\overline{\chi}_F(x)=\chi_F(x)=0$, by the Corollary to Proposition 1; it follows that $x\in F'$.

b) Let $\mu\in\mathcal{M}_+^1(K)$ be a Choquet maximal Radon probability measure on K. By the Corollary to Proposition 2 we have

$$\mu(F)=\mu(\chi_F)=\mu(\overline{\chi}_F), \tag{1}$$

and, therefore, if we take into account Lemma 2 above and Proposition 1.10, b) from [6], we infer that

$$\begin{aligned}\mu(\overline{\chi}_F)&=\mu(u(\overline{\chi}_F))+\mu((1-\chi_{u(\overline{\chi}_F)})\overline{\chi}_F)=\\&=\mu(\overline{co}(F))+\mu((1-\chi_{u(\overline{\chi}_F)})\overline{\chi}_F)=\\&=\mu(F)+\mu((1-\chi_{u(\overline{\chi}_F)})\overline{\chi}_F).\end{aligned} \tag{2}$$

From (1) and (2) we infer that

$$\mu((1-\chi_{u(\overline{\chi}_F)})\overline{\chi}_F)=0,$$

and this implies that $\mu(F')=1-\mu(F)$. The Proposition is proved.

3. Let now $D\subset K$ be a Baire measurable subset and $h_n\in A(K)$, $n\in N$, a sequence of affine continuous real functions on

K, such that D be $\{h_n;\ n\in N\}$ - measurable; it follows that

$$x_o\in D,\ x\in K,\ \text{and}\ h_n(x)=h_n(x_o),\ n\in N \Rightarrow x\in D$$

(such a sequence can always be found; see [6], 1.5). Let $\mu\in\mathcal{M}_+^1(K)$ be a Choquet maximal Radon probability measure on K. By virtue of Lemma 1.2 from [6], we have

$$\mu(\overline{h_n^2})=\mu(h_n^2),\ n\in N.$$

We infer that, for any $n\in N$, there exists a sequence $(h_{nm})_{m\in N}$, $h_{nm}\in A(K)$, $m\in N$, such that

$$\overline{h_n^2}\le h_{nm},\ m\in \mathbb{N} \quad \text{and} \quad \mu(h_n^2)=\inf_{m\in\mathbb{N}}\mu(h_{n_0}\wedge\ldots\wedge h_{nm})$$

If we denote $\mathcal{F}_0=\{h_n;\ n\in\mathbb{N}\}$ and $\mathcal{F}_1'=\{h_{nm};\ m,n\in\mathbb{N}\}$, $\mathcal{F}_1=\mathcal{F}_0\cup\mathcal{F}_1'$, then $\mathcal{F}_1$ is a countable subset of A(K), and for any $h\in\mathcal{F}_1$ we can find a sequence $(h_n')_{n\in N}$, $h_n'\in A(K)$, $n\in N$, such that

$$h^2\le h_n',\ n\in\mathbb{N},\ \text{and}\ \mu(h^2)=\inf_{n\in\mathbb{N}}\mu(h_0'\wedge\ldots\wedge h_n').$$

Let $\mathcal{F}_2'=\{h_n';\ h\in\mathcal{F}_1,\ n\in\mathbb{N}\}$ and $\mathcal{F}_2=\mathcal{F}_1\cup\mathcal{F}_2'$.

By induction, we can find an increasing sequence $(\mathcal{F}_n)_{n\in N}$ of countable subsets of A(K), such that for any $h\in\mathcal{F}_n$ there exists a sequence $(h_n')_{n\in N}$ in $\mathcal{F}_{n+1}$, such that $h^2\le h_n'$, $n\in N$, and

$$\mu(h^2)=\inf_{n\in\mathbb{N}}\mu(h_0'\wedge\ldots\wedge h_n').$$

Let $\mathcal{F}=\bigcup_{n\ge 0}\mathcal{F}_n$. Then $\mathcal{F}$ is a countable subset of A(K) such that

1) D is $\mathcal{F}$-measurable;

2) for any $h\in\mathcal{F}$ there exists a sequence $(h'_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$, such that $h^2\leq h'_n$, $n\in\mathbb{N}$, and

$$(1)\qquad \mu(h^2)=\mu(\overline{h^2})=\inf_n \mu(h'_0\wedge h'_1\wedge\ldots\wedge h'_n).$$

We shall now consider the affine continuous mapping $\theta: K\to\mathbb{R}^{\mathcal{F}}$, given by

$$\theta(x)=(h(x))_{h\in\mathcal{F}}\ ,\qquad x\in K.$$

Since $\mathbb{R}^{\mathcal{F}}$ is metrizable, we infer that $K_o=\theta(K)$ is a metrizable compact convex subset of $\mathbb{R}^{\mathcal{F}}$. Of course, $\theta(D)$ is a Baire measurable subset of $\theta(K)$ and $D=\theta^{-1}(\theta(D))$.

We shall now consider the Radon probability measure $\mu_o=\theta_*(\mu)$ on K_o.

If we denote by p_h the projection in $\mathbb{R}^{\mathcal{F}}$ (or its restriction to K_o) which corresponds to $h\in\mathcal{F}$, then we have

$$h=p_h\circ\theta\ ,\qquad h\in\mathcal{F}.$$

Since $(p_h)_{h\in\mathcal{F}}$ is a total set of affine continuous real functions on K_o, from Corollary 1 to Proposition 1.3 from [6] we infer that

$$(2)\qquad \operatorname{ex} K_o=\{y\in K_o;\ p_h^2(y)=\overline{p_h^2}(y),\ h\in\mathcal{F}\}.$$

From (1) we now infer that

$$\mu(h^2)=\mu_o(p_h^2)\leq\mu_o(\overline{p_h^2})\leq\mu(\overline{h^2}),\quad h\in\mathcal{F},$$

and, therefore,

(3) $$\mu_o(p_h^2) = \mu_o(\overline{p_h^2}), \quad h \in \mathcal{F}.$$

From (2) and (3) we infer that

$$\mu_o(ex\,K) = 1,$$

and, therefore, μ_o is a Choquet maximal Radon probability measure on K_o.

We have, therefore, the following Approximation Theorem.

THEOREM 1. *Let $D \subset K$ be a Baire measurable subset and $\mu \in \mathcal{M}_+^1(K)$ a Choquet maximal Radon probability measure on K. Then there exists a (metrizable) compact convex set $K_o \subset R^N$ and an affine continuous surjective mapping $\theta : K \to K_o$, such that*

a) *$\theta(D)$ is a Baire measurable subset of K_o;*

b) *$\theta^{-1}(\theta(D)) = D$;*

c) *$\theta_*(\mu)$ is a Choquet maximal Radon probability measure on K_o.*

We can now prove the following

COROLLARY. *Let $D \subset K$ be a Baire measurable subset. Then, for any $\varepsilon > 0$, there exists a compact extremal Baire measurable subset $D_o \subset D$, such that $\mu(D) - \varepsilon < \mu(D_o)$.*

PROOF. With the notations of the preceding Theorem, by Ulam's Theorem (see [1], Ch.1, Theorem 1.4), there exists a compact subset $A_o \subset \theta(D) \cap ex\,K_o$, such that $\theta_*(\mu)(\theta(D)) - \varepsilon < \theta_*(\mu)(A_o)$. Let $D_o = \theta^{-1}(A_o)$. Then $D_o \subset D$ and D_o is a compact extremal Baire measurable subset of K, such that

$$\mu(D_o) = \theta_*(\mu)(A_o) > \theta_*(\mu)(\theta(D)) - \varepsilon = \mu(D) - \varepsilon.$$

The Corollary is proved.

REMARK. The preceding Theorem obviously holds for any sequence $\{D_n\}_{n\geq 0}$ of Baire measurable subsets of K and any sequence $(\mu_n)_{n\geq 0}$ of Choquet maximal Radon probability measures.

THEOREM 2. Let $H\subset K$ be a G_δ-subset and $\mu\in\mathcal{M}_+^1(K)$ a Choquet maximal Radon probability measure on K. Then, for any $\varepsilon>0$, there exists a compact extremal Baire measurable subset $D_1\subset H$, such that $\mu(D_1)>\mu(H)-\varepsilon$.

PROOF. By Lemma 1 there exists a compact Baire measurable subset $D\subset H$, such that

$$\mu(D)>\mu(H)-\frac{\varepsilon}{2}.$$

Let K_o and θ correspond to D and μ, as in the preceding Theorem. Then we have

$$\theta_*(\mu)(\theta(D)\cap ex K_o)=\theta_*(\mu)(\theta(D))=\mu(\theta^{-1}(\theta(D)))=$$
$$=\mu(D).$$

Since ex K_o is a Polish space, by Ulam's Theorem (see [1], Ch.1, Theorem 1.4) there exists a compact subset $D_o\subset\theta(D)\cap ex\ K_o$ such that

$$\theta_*(\mu)(D_o)>\theta_*(\mu)(\theta(D)\cap ex K_o)-\frac{\varepsilon}{2}.$$

If we denote $D_1=\theta^{-1}(D_o)$, then the set D_1 defined in this manner has all the required properties. The Theorem is proved.

We recall that the Choquet topology on ex K is that for which $\{F \cap \text{ex } K;\ F \subset K \text{ compact, extremal}\}$ is the set of all closed subsets of the topology (see [2], Ch.II.2; [6], p.27).

We shall specify by C-closed, C-open, etc., the various topological epithets corresponding to the Choquet topology.

Let $\mathcal{B}_o(K)$ be the σ- algebra of all Baire measurable subsets of K and $\mathcal{A}_o(\text{ex } K)=\{D\cap \text{ex } K;\ D\in\mathcal{B}_o(K)\}$.

For any Choquet maximal Radon probability measure $\mu\in\mathcal{M}^1_+(K)$ one can induce the boundary measure $\check{\mu}:\mathcal{A}_o(\text{ex } K)\to[0,1]$, given by

$$\check{\mu}(D\cap \text{ex } K)=\mu(D), \qquad D\in\mathcal{B}_o(K).$$

We have proved in [6] that for any C-closed subset $A\subset \text{ex} K$ we have

$$\check{\mu}^*(A)=\mu(F),$$

for any compact extremal subset $F\subset K$, such that $F\cap \text{ex } K=A$ (see [6], Proposition 1.11).

We shall now prove the following Inner Regularity Theorem

THEOREM 3. Let $G\subset \text{ex } K$ be any C-open subset of ex K. Then

$$\check{\mu}_*(G)=\sup\{\check{\mu}(A);\ A\subset G,\ \text{C-closed and } A\in\mathcal{A}_o(\text{ex } K)\}.$$

PROOF. Let $F\subset K$ be any compact extremal subset of K, such that $G=(\text{ex } K)\setminus F$. We then have

$$\check{\mu}_*(G)=1-\check{\mu}^*(F\cap \text{ex } K)=1-\mu(F),$$

the second equality being a consequence of Proposition 1.11 from [6].

If we denote $F'=\mathcal{Z}(\overline{\chi_F})$, then, by Proposition 3, we have $\mu(F')=\check{\mu}_*(G)$. Let $\varepsilon>0$ be given. By Theorem 2, there exists a compact extremal Baire measurable subset $D_1\subset F'$, such that

$$\mu(F')<\mu(D_1)+\varepsilon;$$

we infer that $A_1=D_1\cap \text{ex } K$ is a C-closed $\mathcal{A}_0(\text{ex } K)$-measurable subset of G, such that

$$\check{\mu}_*(G)<\check{\mu}(A_1)+\varepsilon.$$

The Theorem is proved.

In [6] we have proved that any C-Baire measurable subset of ex K (with respect to the Choquet topology; i.e., any set belonging to the smallest σ- algebra of subsets of ex K, containing all C-closed (C-G_δ)-subsets of ex K) is $\check{\mu}$ - measurable (see [6], Theorem 1.5). We shall now prove that the boundary measure $\check{\mu}$ is inner regular on the σ- algebra $\mathcal{B}_0(\text{ex } K;\ C)$ of all C-Baire measurable subsets of ex K.

<u>THEOREM 4</u>. <u>For any</u> $B\in\mathcal{B}_0(\text{ex } K;C)$ <u>and any</u> $\varepsilon>0$ <u>there exists a C-closed subset</u> $A\subset \text{ex } K$, <u>such that</u>

$$A\subset B,\ A\in\mathcal{A}_0(\text{ex } K)\ \underline{\text{and}}\ \check{\mu}(B)-\varepsilon<\check{\mu}(A).$$

<u>PROOF</u>. a) Let $A\subset \text{ex } K$ be any C-closed (C-G_δ)-subset of ex K. Then A and $G=(\text{ex } K)\setminus A$ are $\check{\mu}$ - measurable, and there exists an increasing sequence $(F_n)_{n\geq 0}$ of compact extremal subsets of K,

such that $\bigcup_{n=0}^{\infty}(F_n \cap \text{ex } K)=G$. Let $F\subset K$ be any compact extremal subset of K, such that $F\cap \text{ex } K=A$.

Let $H=K\setminus(\bigcup_{n=0}^{\infty} F_n)$. Then H is a G_δ-subset of K; by Theorem 2, given $\varepsilon>0$, there exists a compact extremal Baire measurable subset $D_1\subset H$, such that $\mu(H)-\varepsilon<\mu(D_1)$.

On the other hand, from $F\cap F_n=\emptyset$, $n\in \mathbb{N}$, we infer that $F\subset H$ and

$$\mu(F)+\mu(F_n)=\check{\mu}^*(F\wedge \text{ex} K)+\check{\mu}^*(F_n\wedge \text{ex} K), \quad n\geq 0;$$

we obtain that

$$\mu(F)+\mu(\bigcup_{n\geq 0} F_n)=\check{\mu}^*(F\wedge \text{ex} K)+\check{\mu}^*(G)=1,$$

and, therefore, we have

$$\mu(F)=\mu(H).$$

If we denote $A_1=D_1\cap \text{ex } K$, then $A_1\subset A$, A_1 is C-closed and $A_1\in A_o(\text{ex } K)$.

b) If $G\subset \text{ex } K$ is any open $(C\text{-}F_\sigma)$-subset of ex K, then it is $\check{\mu}$ - measurable, by virtue of Theorem 1.5 from [6] and the set $A\subset G$ required by the Theorem exists by virtue of Theorem 3.

c) Let $\mathcal{B}_1$ be the set of all subsets B of ex K, such that B and $(\text{ex } K)\setminus B$ have the property required by Theorem 4. Then, by a) and b), any C-closed $(C\text{-}G_\delta)$-subset of ex K belongs to $\mathcal{B}_1$ and, since $\mathcal{B}_1$ is easily shown to be a σ- algebra of subsets of ex K, we obviously have that $\mathcal{B}_o(\text{ex } K;\ C)\subset\mathcal{B}_1$. The Theorem is proved.

According to Theorem 1.5 from [6] we have

$$\mathcal{B}_o(\text{ex } K;\ C) \subset \mathcal{A}_o(\text{ex } K)^{\sim}_{\check{\mu}},$$

where the right-hand member is the completion of $\mathcal{A}_o$(ex K) with respect to $\check{\mu}$.

4. The preceding results can be strengthened as follows.

Let us consider the σ-algebra $\mathcal{B}_1$(ex K) of subsets of ex K, generated by all the sets of the form $D \cap$ ex K, where $D \subset K$ is a compact extremal Baire measurable subset. Of course, we have

(*) $\qquad \mathcal{B}_1(\text{ex } K) \subset \mathcal{A}_o(\text{ex } K),$

and also

(**) $\qquad \mathcal{B}_1(\text{ex } K) \subset \mathcal{B}(\text{ex } K; C),$

where $\mathcal{B}$(ex K;C) denotes the σ- algebra of all the Borel measurable subsets of ex K, with respect to the Choquet topology.

We shall denote by $\mathcal{B}_1(\text{ex } K)^{\sim}_{\check{\mu}}$ the Lebesgue completion of the σ-algebra $\mathcal{B}_1$(ex K), with respect to the restriction of $\check{\mu}$ to $\mathcal{B}_1$(ex K).

We have the following <u>Regularity Theorem</u>.

<u>THEOREM 5.</u> a) $\mathcal{B}_1(\text{ex } K)^{\sim}_{\check{\mu}} = \mathcal{A}_o(\text{ex } K)^{\sim}_{\check{\mu}}$.

b) <u>For any</u> $A \in \mathcal{A}_o(\text{ex } K)^{\sim}_{\check{\mu}}$ <u>and any</u> $\varepsilon > 0$ <u>there exists a C-closed set</u> $A_o \in \mathcal{A}_o$(ex K), <u>such that</u>

$$A_o \subset A \text{ and } \check{\mu}(A) - \varepsilon < \check{\mu}(A_o).$$

<u>PROOF.</u> a) From (*) we immediately obtain that

(1) $\mathcal{B}_1(\text{ex } K)^{\sim}_{\check{\mu}} \subset \mathcal{A}_o(\text{ex } K)^{\sim}_{\check{\mu}}$.

Let now $A_1 \in \mathcal{A}_o(\text{ex } K)^{\sim}_{\check{\mu}}$ and $\varepsilon > 0$ be given. Then there exists an $A_o \in \mathcal{A}_o(\text{ex } K)$, such that

(2) $A_o \subset A_1$ and $\check{\mu}(A_o) = \check{\mu}(A_1)$.

Let $D_o \in \mathcal{B}_o(K)$ be a Baire measurable subset of K, such that $D_o \cap \text{ex } K = A_o$. By the Corollary to Theorem 1, there exists a compact extremal Baire measurable subset $D \subset D_o$, such that $\mu(D_o) - \varepsilon < \mu(D)$. We then have:

$$\check{\mu}(A_o) - \varepsilon < \check{\mu}(D \cap \text{ex } K),$$

and $A = D \cap \text{ex } K \in \mathcal{B}_1(\text{ex } K)$.

By a standard argument we infer that there exists a set $A \in \mathcal{B}_1(\text{ex } K)$, such that

$$A \subset A_1 \text{ and } \check{\mu}(A) = \check{\mu}(A_1),$$

where we have also taken into account (2).

A similar argument, applied to $\complement A_1$, yields a set $A' \in \mathcal{B}_1(\text{ex } K)$, such that

$$A_1 \subset A' \text{ and } \check{\mu}(A') = \check{\mu}(A_1).$$

We infer that $A_1 \in \mathcal{B}_1(\text{ex } K)^{\sim}_{\check{\mu}}$ and, therefore, we have

(3) $\mathcal{A}_o(\text{ex } K)^{\sim}_{\check{\mu}} \subset \mathcal{B}_1(\text{ex } K)^{\sim}_{\check{\mu}}$.

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From (1) and (3) we infer statement a) of the Theorem.

b) For any compact extremal Baire measurable subset $D\subset K$, property b) obviously holds for the set $A=D\cap \mathrm{ex}\, K$, with $A_o=A$.

Let us now consider the set $(\mathrm{ex}\, K)\setminus A$. We have $(\mathrm{ex}\, K)\setminus A=(K\setminus D)\cap \mathrm{ex}\, K$, and $K\setminus D$ is a Baire measurable subset of K. By the Corollary to Theorem 1, there exists a compact extremal Baire measurable subset $D_o\subset K\setminus D$, such that $\mu(K\setminus D)-\varepsilon<\mu(D_o)$. Then we have $A_o=D_o\cap(\mathrm{ex}\, K)\subset (\mathrm{ex}\, K)\setminus A$, and $\check{\mu}((\mathrm{ex}\, K)\setminus A)-\varepsilon<\check{\mu}(A_o)$. The set A_o meets the requirements from statement b) of the Theorem.

Let now $\mathcal{B}'$ be the set of all subsets, $S\in\mathcal{B}_1(\mathrm{ex}\, K)$, of ex K, such that for any $\varepsilon>0$ there exists compact extremal Baire measurable subsets D_o, $D_1\subset K$, such that $D_o\cap \mathrm{ex}\, K\subset S$, $D_1\cap \mathrm{ex}\, K \subset (\mathrm{ex}\, K)\setminus S$ and $\check{\mu}(S)-\varepsilon<\check{\mu}(D_o\cap \mathrm{ex}\, K)$, $\check{\mu}((\mathrm{ex}\, K)\setminus S)-\varepsilon<\check{\mu}(D_1\cap \mathrm{ex}\, K)$. Then $\mathcal{B}'$ obviously is a σ - algebra of subsets of ex K, containing all the generators of $\mathcal{B}_1(\mathrm{ex}\, K)$, by virtue of the preceding argument. We first infer that $\mathcal{B}'=\mathcal{B}_1(\mathrm{ex}\, K)$ and then, by an easy argument, that property b) holds for any $A\in\mathcal{B}_1(\mathrm{ex}\, K)^{\sim}_{\check{\mu}}$. Part a) of the Theorem now concludes the proof.

5. In this Section we shall consider the case of the quasi-states space $E_o(\mathcal{C})$ of an arbitrary C^*-algebra $\mathcal{C}$, i.e.,

$$E_o(\mathcal{C})=\{f\in\mathcal{C}^* ;\ f\geq 0,\ \|f\|\leq 1\},$$

endowed with the $\sigma(\mathcal{C}^*;\mathcal{C})$ - topology. Then $E_o(\mathcal{C})$ is a compact convex set, in $(\mathcal{C}^*;\sigma(\mathcal{C}^*;\mathcal{C}))$, and $\mathrm{ex}\, E_o(\mathcal{C})=P(\mathcal{C})\cup\{0\}$, where $P(\mathcal{C})$ denotes the set of all pure states of $\mathcal{C}$.

Let $\mu\in\mathcal{M}_+^1(E_o(\mathcal{C}))$ be a maximal orthogonal Radon probability measure, such that $\|b(\mu)\|=1$. By Henrichs' Theorem (see [3], p.106; and also [6], Theorem 3.10) μ is maximal for the Choquet order relation, and, therefore, the foregoing Theory can be applied to μ. We shall make extensive use of the results of [6]. According to Proposition 3.2 from [6] we have $\check{\mu}^*(\{0\})=0$ and, therefore, $\check{\mu}^*(P(\mathcal{C}))=1$.

Moreover, $\{0\}\subset P(\mathcal{C})\cup\{0\}$ is a C-closed subset of $P(\mathcal{C})\cup\{0\}$; hence, $P(\mathcal{C})$ is a C-open subset of $P(\mathcal{C})\cup\{0\}$.

According to Theorem 5.2 from [6], the probability measure $\check{\mu}:\mathcal{A}_o(P(\mathcal{C})\cup\{0\})\to[0,1]$ can be extended to a probability measure

$$\tilde{\mu}:A_1(P(\mathcal{C})\cup\{0\})\to[0,1],$$

defined on the σ - algebra $A_1(P(\mathcal{C})\cup\{0\})$, generated by $\mathcal{A}_o(P(\mathcal{C})\cup\{0\})$ and $\mathcal{B}(P(\mathcal{C})\cup\{0\};C)$.

Since we have

$$\mathcal{B}_1(P(\mathcal{C})\cup\{0\})\subset\mathcal{B}(P(\mathcal{C})\cup\{0\};C),$$

we infer that we also have

$$\mathcal{A}_o(P(\mathcal{C})\cup\{0\})_{\tilde{\mu}}^{\sim}=\mathcal{B}_1(P(\mathcal{C})\cup\{0\})_{\tilde{\mu}}^{\sim}\subset$$
$$\subset\mathcal{B}(P(\mathcal{C})\cup\{0\};C)_{\tilde{\mu}}^{\sim},$$

where we have also applied Theorem 5.

<u>THEOREM 6.</u> a) $\mathcal{A}_o(P(\mathcal{C})\cup\{0\})_{\tilde{\mu}}^{\sim}\subset\mathcal{B}(P(\mathcal{C})\cup\{0\};C)_{\tilde{\mu}}^{\sim}$.

b) <u>For any</u> $A\in\mathcal{B}(P(\mathcal{C})\cup\{0\};C)_{\tilde{\mu}}^{\sim}$ <u>and any</u> $\varepsilon>0$ <u>there exists</u> <u>a</u> C-<u>closed subset</u> $A_o\subset P(\mathcal{C})\cup\{0\}$, <u>such that</u>

$$A_o\subset A \text{ and } \tilde{\mu}(A)-\varepsilon<\tilde{\mu}(A_o).$$

<u>PROOF.</u> In order to develop the proof, we have to recall some notations and results that we have used and obtained in

[6].

Namely, let $f_o = b(\mu)$, and let $\pi_{f_o} : \mathcal{C} \to \mathcal{L}(H_{f_o})$ be the corresponding cyclic representation, according to the GNS-construction; let $\xi^o_{f_o} \in H_{f_o}$ be the corresponding cyclic vector, and $\mathcal{C}_\mu \subset (\pi_{f_o}(\mathcal{C}))'$ the maximal abelian von Neumann algebra, corresponding to μ (see [6], Theorem 3.4). Let $\mathcal{B} \subset L(H_{f_o})$ be the C^*- algebra generated by $\pi_{f_o}(\mathcal{C})$ and $\mathcal{C}_\mu$, and let $\alpha \in \mathcal{M}'_+(E(\mathcal{B}))$ be the central measure on $E(\mathcal{B})$, corresponding to the vector state $g_o = \omega_{\xi^o_{f_o}} | \mathcal{B} \in E(\mathcal{B})$. Since

$$\mathcal{C}_\mu \subset \mathcal{B} \qquad \text{and} \qquad \mathcal{B}' = \mathcal{C}_\mu,$$

α is a maximal orthogonal measure on $E(\mathcal{B})$, representing g_o and, by C.F.Skau's Theorem, it is the greatest Radon probability measure on $E(\mathcal{B})$, representing g_o, for the Choquet order relation (see [6], Proposition 3.5); and also Chapter 5).

The mapping $\tau : E(\mathcal{B}) \to E_o(\mathcal{C})$ is defined by

$\tau(g) = g \circ \pi_{f_o}$, $g \in E(\mathcal{B})$, whereas $\sigma = \tau | P(\mathcal{B})$.

We have $\sigma(P(\mathcal{B})) \subset P(\mathcal{C}) \cup \{0\}$ (see [6], Chapter 3), and $\tau_*(\alpha) = \mu$, $\sigma_*(\check{\alpha}) = \check{\mu}$ (see [6], Lemma 3.8 and Proposition 3.10).

i) For any compact extremal subset $F \subset E_o(\mathcal{C})$ the set $\tau^{-1}(F) \subset E(\mathcal{B})$ is a compact extremal subset of $E(\mathcal{B})$, and we have

$$\tilde{\mu}(F \cap (P(\mathcal{C}) \cup \{0\})) = \tilde{\alpha}(\sigma^{-1}(F \cap (P(\mathcal{C}) \cup \{0\}))) =$$

$$= \tilde{\alpha}(\tau^{-1}(F) \cap P(\mathcal{B})) = \check{\alpha}^*(\tau^{-1}(F) \cap \operatorname{supp} \check{\alpha}) =$$

$$= \alpha(\tau^{-1}(F)\cap F_0) = \alpha(\tau^{-1}(F)) = \tau_*(\alpha)(F) = \mu(F),$$

where we denoted by F_0 the smallest compact extremal subset of $E(\mathcal{B})$, containing suppα, whereas supp$\check{\alpha} = F_0 \cap P(\mathcal{B})$.

Let us now remark that property b) obviously holds for any C-closed subset $A \subset P(\mathcal{C}) \cup \{0\}$. Let now $G = (P(\mathcal{C}) \cup \{0\}) \setminus F$ be any C-open subset of $P(\mathcal{C}) \cup \{0\}$, where $F \subset E_0(\mathcal{C})$ is any compact extremal subset of $E_0(\mathcal{C})$. By virtue of the Inner Regularity Theorem (see Theorem 3, above), there exists a C-closed subset $A_0 \subset G$, such that $\check{\mu}_*(G) - \varepsilon < \tilde{\mu}(A_0)$.

On the other hand, we have

$$\check{\mu}_*(G) = 1 - \check{\mu}^*((P(\mathcal{C})\cup\{0\})\setminus G) = 1 - \mu(F) =$$
$$= 1 - \tilde{\mu}(F \cap (P(\mathcal{C})\cup\{0\})) = \tilde{\mu}(G),$$

where we have taken into account equality (1).

ii) If we now denote by $\mathcal{B}'$ the set of all $A \in \mathcal{B}(P(\mathcal{C})\cup\{0\}; C)^{\sim}_{\tilde{\mu}}$, such that property b) holds for A and for $(P(\mathcal{C})\cup\{0\})\setminus A$, it is easy to prove that $\mathcal{B}'$ is a σ- algebra. Since, by virtue of i), $\mathcal{B}'$ contains any C-closed subset of $P(\mathcal{C})\cup\{0\}$, we infer that $\mathcal{B}(P(\mathcal{C})\cup\{0\}; C) \subset \mathcal{B}'$ and, therefore, $\mathcal{B}' = \mathcal{B}(P(\mathcal{C})\cup\{0\}; C)^{\sim}_{\tilde{\mu}}$, since $\mathcal{B}'$ is easily shown to be complete.

In this manner, property b) in the statement of the Theorem is proved. Part a) of the Theorem was already proved just before.

<u>REMARK</u>. The preceding theory can be viewed as a non-commutative extension of the theory of Radon measures.

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