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ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS
No.34/1981

BUCURESTI

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April 1981

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0. Introduction

We propose studying the so-called asymptotic dosing problem for first-order abstract differential equations of monotone type in Hilbert spaces.

The idea of writing of this paper originates from a paper of Turinici [6], who considered the asymptotic dosing problem for the finite - dimensional case. However, both our assumptions and the methods we use here are completely different from those of [6].

The main result we state in this paper, Theorem 1 below, relies on the well-known Opial's lemma and on a technique similar to that developed by Baillon and Haraux [1].

The last part of the paper (Section 2) is devoted to the study of existence of solutions to abstract evolution equations which include a mesure as an inhomogeneous term. This subject is closely related to the dosing problem.

1. Asymptotic dosing

Let H be a real Hilbert space with the inner product and the associated norm denoted by $(.,.)$ and $\|.\|$, respectively. Let A be a maximal monotone operator from H into itself, whose domain and range are denoted as usual by D(A) and R(A), respectively. Consider the sequence of Cauchy problems (P_n):

$$(P_o) \qquad \begin{cases} \dfrac{du_o}{dt}(t) + Au_o(t) \ni f(t), & o < t < T \\ u_o(o) = x \end{cases}$$

$$(P_n) \qquad \begin{cases} \dfrac{du_n}{dt}(t) + Au_n(t) \ni f(t), & nT < t < (n+1)T \\ u_n(nT) = u_{n-1}(nT) + d_n, \end{cases}$$

for n= 1,2,...,

where $T > o$ is fixed; $x \in \overline{D(A)}$ (the closure of D(A)); (d_n) is a given sequence in H;

(A_1) $f \in L^2_{loc}(o,\infty;H)$ and f is T-periodic. By "d/dt" we mean the ordinary derivative with respect to t.

As a first remark, the sequence (P_n) is well-defined only if each of the initial data of problems (P_n) belongs to $\overline{D(A)}$. The assumption

(A_2) $\qquad \overline{D(A)} = H$

guarantees this fact.

If in addition A is the subdifferential of a function $\varphi : H \longrightarrow]-\infty,+\infty]$ proper convex and lower-semicontinuous (one denotes $A = \partial\varphi$), then (see, e.g. [2, p.189]) each of the

problems (P_n) has a unique strong solution $u_n \in C([nT,(n+1)T];H) \cap W^{1,2}(nT+\delta, nT+T; H)$, for every $\delta \in]o,T[$, and $t^{\frac{1}{2}}du_n/dt \in L^2(nT,nT+T;H)$. We suppose the familiarity of the reader with the notation and the usual topologies of the function spaces we are introducing as well as with the concepts and fundamental results in the convex analysis and the theory of nonlinear monotone operators and evolution equations of monotone type developed in Hilbert spaces.

Let us define the function $u : [o,+\infty[\longrightarrow H$ by

$$(1.1) \qquad u(t) = \begin{cases} x, & \text{if} \quad t = o \\ u_n(t), & \text{if} \quad nT < t \leq nT + T, \quad n = 0,1,\ldots \end{cases}$$

Obviously u satisfies the problem

$$(1.2) \qquad \begin{cases} \frac{du}{dt}(t) + Au(t) \ni f(t), & \text{a.e.} \quad t > o \\ u(o) = x; \; u(nT^+) = u(nT^-) + d_n, & n = 1,2,\ldots \end{cases}$$

Suppose further that

(A_3) there exists $d \in H$, such that $\sum_{n=1}^{\infty} \| d_n - d \| < +\infty$,

and

(A_4) there exists at least one solution of the two-point boundary value problem

$$(1.3) \qquad \begin{cases} \frac{d\omega}{dt}(t) + A\omega(t) \ni f(t), & \text{a.e.} \quad t \in]o,T[\\ \omega(o) = \omega(T) + d. \end{cases}$$

The asymptotic dosing problem is that of finding of sufficient conditions in order that the sequence of functions $\{y_n : [o,T] \longrightarrow H \; ; \; y_n(t) = u(t+nT)\}$ converges, in a certain

sense, to a solution of (1.3). The convergence theorem below is the main result in this direction we prove in what follows. In fact, as a more general question, we investigate here the dosing effect combined with the periodic "continuous forcing" effect (see (A_1)). However, the result of Baillon and Haraux [1] on periodic "forcing" for such kind of equations cannot be derived in its general form, as a particular case of Theorem 1.

THEOREM 1. Let $A=\partial\varphi$ and assume that (A_1)-(A_4) hold. Then,

$$(1.4) \qquad \sup_{t\geq 0} \|u(t)\| < +\infty ,$$

and there exists a solution ω^* of (1.3) such that

$$(1.5) \qquad y_n(t) = u(t+nT) \longrightarrow \omega^*(t), \text{ as } n\to\infty , \ \forall\, t\in]0,T], \text{ weakly in } H,$$

and

$$(1.6) \qquad t^{3/4}\frac{dy_n}{dt} \longrightarrow t^{3/4}\frac{d\omega^*}{dt} , \text{ as } n\to\infty , \text{ strongly in } L^2(0,T;H)$$

In addition

$$(1.7) \qquad \varphi(y_n(t)) \longrightarrow \varphi(y(t)), \text{ as } n\to\infty, \text{ uniformly on } [\varepsilon,T] , \text{ for every } \varepsilon\in]0,T[$$

Proof. The proof makes use of Opial's lemma and the technique of Baillon and Haraux [1] . Let us denote by F the set of solutions of (1.3). Then, for every $\omega\in F$, we have

$$(1.8) \qquad \|y_n(t) - \omega(t)\| \leq \|y_n(0^+) - \omega(0)\|$$
$$\leq \|y_{n-1}(T) - \omega(T)\| + \|d_n - d\|$$
$$\leq \|y_{n-1}(t) - \omega(t)\| + \|d_n - d\|$$

$$\leq \| y_{n-1}(0^+) - \omega(0) \| + \| d_n - d \|$$

for every $t \in]0,T]$, $n=1,2,\ldots$

so that, by virtue of (A_3), (1.4) is satisfied. Moreover, (1.8) implies that, for each $t \in]0,T]$, the sequence $\{\| y_n(t) - \omega(t) \| - \sum_{i=1}^{n} \| d_i - d \|\}$ is nonincreasing. Therefore, for each $\omega \in F$, there exists a constant $C_\omega > 0$ (independent of t), such that

$$(1.9) \quad \begin{aligned} &\| y_n(t) - \omega(t) \| \longrightarrow C_\omega , \quad \forall t \in]0,T] \quad \text{and} \\ &\| y_n(0^+) - \omega(0) \| \longrightarrow C_\omega . \end{aligned}$$

In particular

$$(1.10) \quad \| y_n - \omega \|_{L^2(0,T;H)} \longrightarrow T^{\frac{1}{2}} C_\omega , \quad \forall \omega \in F .$$

By a standard reasoning (see [2, p. 191]) we deduce

$$(1.11) \quad \int_0^T t \left\| \frac{dy_n}{dt}(t) \right\|^2 dt \leq \int_0^T t \| f(t) \|^2 dt + \left(\| y_n(0^+) - q \| + \int_0^T \| f(t) \|^2 dt \right)^2, \text{ for some } q \in D(A).$$

Let (y_{n_k}) be an arbitrary subsequence of (y_n) such that

$$y_{n_k} \longrightarrow y, \text{ as } k \to \infty , \text{ weak-star in } L^\infty(0,T;H).$$

Then, it is obvious by (1.11) that

$$(1.12) \quad t^{\frac{1}{2}} \frac{dy_{n_k}}{dt} \xrightarrow[k \to \infty]{} t^{\frac{1}{2}} \frac{dy}{dt}, \text{ weakly in } L^2(0,T;H).$$

Moreover

$$(1.13) \quad y_{n_k}(t) \xrightarrow[k \to \infty]{} y(t), \text{ weakly in } H, \forall t \in]0,T] ,$$

because

$$t\, y_n(t) = \int_0^t \left[s \frac{dy_n}{ds}(s) + y_n(s) \right] ds, \quad \forall t \in]0,T].$$

In what follows we intend to show that $y \in F$. To this purpose let us remember a simple lemma due to Baillon and Haraux [1], which we need here for the particular case of subdifferential.

Lemma 1. Let $\psi : H \longrightarrow]-\infty, +\infty]$ be a proper convex and lower-semicontinuous function, where H is a real Hilbert space. If $h_k \in \partial\psi(z_k)$, $g \in \partial\psi(v)$, $h_k \longrightarrow h$ weakly, $z_k \longrightarrow z$ weakly, and $\lim_{k\to\infty} (h_k - g, z_k - v) = 0$, then $h \in \partial\psi(v)$ and $g \in \partial\psi(z)$.

Let us now define $\Phi : L^2(0,T:H) \longrightarrow]-\infty, \infty]$ by

$$\Phi(z) = \begin{cases} \int_0^T t^{\frac{1}{2}} \varphi(z)\, dt, & \text{if } t^{\frac{1}{2}} \varphi(z) \in L^1(0,T) \\ +\infty, & \text{otherwise} \end{cases}$$

It is easy to show that Φ is proper convex and lower semi-continuous and we have

$$(1.14) \quad w \in \partial\Phi(z) \iff w(t) \in t^{\frac{1}{2}} A z(t), \quad \text{a.e. } t \in]0,T[$$

The following calculation

$$0 \leqq - \int_0^T (y_n - \omega,\ t^{\frac{1}{2}} \frac{dy_n}{dt} - t^{\frac{1}{2}} \frac{d\omega}{dt})\, dt$$

$$= - \tfrac{1}{2} \int_0^T t^{\frac{1}{2}} \frac{d}{dt} \| y_n - \omega \|^2 \, dt$$

$$= - \tfrac{1}{2} T^{\frac{1}{2}} \| y_n(T) - \omega(T) \|^2 + 1/4 \int_0^T t^{-\frac{1}{2}} \| y_n - \omega \|^2 \, dt$$

$$\leqq \tfrac{1}{2} T^{\frac{1}{2}} \left[\| y_n(0^+) - \omega(0) \|^2 - \| y_n(T) - \omega(T) \|^2 \right], \ \forall \omega \in F$$

and (1.8) imply that

$$\lim_{n\to\infty} \int_0^T (y_n-\omega,\ t^{\frac{1}{2}}(f-\frac{dy_n}{dt}) - t^{\frac{1}{2}}(f-\frac{d\omega}{dt}))\,dt=0 . \tag{1.15}$$

Therefore Lemma 1 is applicable in the space $L^2(o,T;H)$ with $\psi=\Phi$, $h_k = t^{\frac{1}{2}}(f-\frac{dy_{n_k}}{dt})$, $z_k = y_{n_k}$, $g = t^{\frac{1}{2}}(f-\frac{d\omega}{dt})$, and $v=\omega$. So we conclude that

$$f-\frac{dy}{dt}\in A\omega \quad\text{and}\quad f-\frac{d\omega}{dt}\in Ay. \tag{1.16}$$

Using (1.16) and the well-known formula for the computation of $\frac{d}{dt}\varphi(y)$ (see e.g. [2, p. 189]) we may write

$$T^{3/2}\varphi(y(T)) - 3/2\,\Phi(y) = \int_0^T t^{3/2}(f-\frac{d\omega}{dt},\frac{dy}{dt})dt \tag{1.17}$$

$$= \int_0^T t^{3/2}\left[(f,\frac{dy}{dt})+\left\|\frac{d\omega}{dt}\right\|^2 - 2\,(\frac{dy}{dt},\frac{d}{dt})\right]dt .$$

Since we have

$$\int_0^T t^{\frac{1}{2}}(f-\frac{dy_{n_k}}{dt},\omega-y_{n_k})\,dt \le \Phi(\omega) - \Phi(y_{n_k})$$

and

$$\int_0^T t^{\frac{1}{2}}(f-\frac{d\omega}{dt},\ y-\omega)dt \le \Phi(y) - \Phi(\omega)$$

it follows that

$$\int_0^T t^{\frac{1}{2}}(f-\frac{d\omega}{dt},\ y-y_{n_k})dt + \int_0^T t^{\frac{1}{2}}(\frac{dy_{n_k}}{dt}-\frac{d\omega}{dt}, \tag{1.18}$$

$$y_{n_k} - \omega)dt \leq \Phi(y) - \Phi(y_{n_k}).$$

From (1.15), (1.18) and the lower-semicontinuity of Φ we obtain

$$\lim_{k\to\infty} \Phi(y_{n_k}) = \Phi(y). \tag{1.19}$$

On the other hand

$$T^{3/2}\,\varphi(y_{n_k}(T)) - 3/2\,\Phi(y_{n_k}) = \int_0^T t^{3/2}\left(f - \frac{dy_{n_k}}{dt}, \frac{dy_{n_k}}{dt}\right)dt, \tag{1.20}$$

Substracting (1.2o) from (1.17) and using (among other things) (1.19) we conclude that

$$\overline{\lim_{k\to\infty}} \int_0^T t^{3/2} \left\| \frac{dy_{n_k}}{dt} - \frac{d\omega}{dt} \right\|^2 dt =$$

$$\overline{\lim_{k\to\infty}}\; T^{3/2}\left[\varphi(y(T)) - \varphi(y_{n_k}(T))\right] = 0.$$

Therefore

$$t^{3/4}\,\frac{dy_{n_k}}{dt} \xrightarrow[k\to\infty]{} t^{3/4}\,\frac{d\omega}{dt}\,, \text{ strongly in } L^2(o,T;H) \tag{1.21}$$

and

$$\varphi(y_{n_k}(T)) \xrightarrow[k\to\infty]{} \varphi(y(T)). \tag{1.22}$$

Since $dy/dt = d\omega/dt$ (see (1.12) and (1.21)) it follows that $y \in F$. This along with (1.10) implies, by virtue of Opial's lemma (see, e.g. [1, p.107]) applied in the space $L^2(o,T;H)$, that there exists $\omega^* \in F$ such that (see also (1.4))

$$y_n \xrightarrow[n\to\infty]{} \omega^*, \text{ weak-star in } L^\infty(o,T;H).$$

Now, all the assertions of the theorem follow from the facts already proved above. We remark only that (1.22) can be analogously derived for every $t \in]o,T]$, and by Arzelà-Ascoli Criterion one obtains (1.7). The proof is now complete.

Remark 1. Let $A : H \longrightarrow H$ be a maximal monotone operator such that $A - aI$ is still monotone for some $a > o$. Assume in addition that $(A_1) - (A_3)$ hold. Hypothesis (A_4) is now automatically satisfied. Indeed if we denote by $u_o(t;x)$ the solution of problem (P_o) then the operator $\Gamma : H \longrightarrow H$, defined by $\Gamma x = u_o(T;x)+d$, is a contraction (with the constant e^{-aT}). By Banach's fixed point theorem there exists a unique fixed point of Γ, therefore problem (1.3) admits a unique solution ω^*. Performing a simple calculation one obtains

$$(1.23)\quad \|y_n(t) - \omega^*(t)\| \leqslant e^{-at}\left(\|y_{n-1}(t) - \omega^*(t)\| + \|d_n - d\|\right), \quad \forall t \in]o,T],$$

from which it follows that

$$(1.24)\quad \lim_{n\to\infty} \|y_n(t) - \omega^*(t)\| = o, \quad \forall t \in]o,T].$$

We remark that if $d_n = d$, $\forall n$ then

$$(1.25)\quad \|y_n(t) - \omega^*(t)\| \leqslant e^{-nat}\|u_o(t) - \omega^*(t)\|, \quad \forall t \in]o,T].$$

It should be noticed that, except for this particular case in which A is strong monotone, we failed in the attempt to find some reasonable conditions assuring the existence in

the boundary value problem (1.3).

Example

Let Ω denote an open and bounded subset of R^N whose boundary $\partial\Omega$ is smooth enough. Let β be a maximal monotone graph in RxR such that $0 \in \beta(0)$ and $D(\beta)=R$. Therefore there exists a continuous convex function $j : R \longrightarrow R$ such that $\beta = \partial j$. We consider the following problem (see [2, p.202])

$$(1.26)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \beta(u) \ni 0 & \text{on }]0,\infty[\times \Omega \\ u(t,x) = 0 & \text{on }]0,\infty[\times \partial\Omega \\ u(0,x) = u_0(x) & \text{on } \Omega \end{cases}$$

We remember that the function $\varphi : L^2(\Omega) \longrightarrow]-\infty, +\infty]$ defined by

$$\varphi(u) = \begin{cases} \frac{1}{2}\displaystyle\int_\Omega |\text{grad } u|^2 \, dx + \int_\Omega j(u)dx, \text{ if} \\ \qquad u \in H_0^1(\Omega) \text{ and } j(u) \in L^1(\Omega) \\ +\infty \quad \text{, otherwise,} \end{cases}$$

is convex and lower-semicontinuous on $L^2(\Omega)$. The operator $A = \partial\varphi : L^2(\Omega) \longrightarrow L^2(\Omega)$ is defined by

$$Au = -\Delta u + \beta(u)$$

and the closure of $D(A)$ in $L^2(\Omega)$ is the whole of $L^2(\Omega)$, because $D(\beta) = R$ (see [2, p. 89]).

Problem (1.26) can be understood as the following Cauchy problem

$$(1.27)\quad \begin{cases} \frac{d}{dt}\, u(t,.) + Au(t,.) \ni o, \quad t > o, \text{ in } L^2(\Omega) \\ u(o,.) = u_o . \end{cases}$$

It is well-known that, if a "forcing" does not exist, $u(t,.)$ converges exponentially in $L^2(\Omega)$, as $t \to \infty$, to o, the unique stationary solution of the problem.

Let us imagine a dosing process with some $T > o$ and $d \in L^2(\Omega)$, $d \neq o$. Then, according to Remark 1 (whose assumptions are obviously satisfied) $u(t+n\,T,.)$ approaches $\omega^*(t,.)$, $t \in]o,T]$, as $n \to \infty$, where ω^* is the unique solution of the problem

$$(1.28)\quad \begin{cases} \frac{d}{dt}\, \omega(t,.) + A\omega(t,.) \ni o, \quad o < t < T, \text{ in } L^2(\Omega) \\ \omega(o,.) = \omega(T,.) + d(.), \end{cases}$$

and certainly $\omega^* \neq o$.

This example may be of physical interest. Indeed, if (1.26) is interpreted as the heat equation, then a dosing process as described above can assure the preserving of a nonzero temperature u, when the time t tends to ∞. In other words, a dosing process generated by an impulsive distributed heat source - can achieve an effect which is somehow similar to that achieved by a periodic "continuous" heat injection.

2. Evolution equations with a measure as an inhomogeneous term

It is easy to see that the function u given by (1.1), restricted to some interval $[o,b]$, is of bounded variation ($u \in BV(o,b;H)$) and satisfies the following Cauchy problem

$$(2.1) \qquad \begin{cases} du + Au \ni \mu, & \text{in } M(o,b;H), \\ u(o) = x, \end{cases}$$

where $M(o,b;H)$ represents the dual space of $C([o,b];H)$, $\mu = dg$, with

$$g(t) = g_1(t) + \int_o^t f(s)ds,$$

g_1 being a simple function ($g_1 = o$ on $[o,T]$, $g_1 = d_1$ on $]T,2T]$, $g_1 = d_2$ on $]2T, 3T]$ and so on, such that the interval $[o,b]$ to be completely covered); for some $v \in BV(o,b;H)$ we denote by dv the measure generated by v by means of the Stieltjes integral associated to it, that is

$$(2.2) \qquad dv(h) = \int_o^b (h(t),dv(t)), \quad \forall\, h \in C([o,b];H).$$

We also recall that for every $\mu \in M(o,b;H)$ there is a function $v \in BV(o,b;H)$ such that $\mu = dv$ (see (2.2)).

It should be noticed that in the general case when $\mu = dg$ is arbitrary in $M(o,b;H)$ it is fairly difficult to investigate the existence of solutions to (2.1). The rest of the paper is intended for presenting of some facts relating to this problem.

Making the change $z = u - g$ Eq. (2.1) formaly reduces to a "time dependent" equation:

we obtain by (2.7) and (2.9) that

$$(2.10)\quad \| z_\lambda + g - (I+\lambda A)^{-1}(z_\lambda + g) \|_{L^2(0,b;H)} \leq C\lambda .$$

Throughout the text C represents a general positive constant. Next, by (2.7) and (2.10) it follows that there is $z \in L^\infty(0,b;H)$ such that (on some subsequences)

$$(2.11)\quad z_\lambda \longrightarrow z \quad , \text{ as } \lambda \to 0, \text{ weak-star in } L^\infty(0,b;H)$$

and

$$(2.12)\quad (I+\lambda A)^{-1}(z_\lambda + g) \longrightarrow z+g \quad , \text{ as } \lambda \to 0,$$

weakly in $L^2(0,b;H)$.

Eq. (2.6) (where $z = z_\lambda$) can be written in the equivalent form

$$(2.13)\quad z_\lambda + g \in \partial\varphi_\lambda^*\left(-\frac{dz_\lambda}{dt}\right), \text{ a.e. on }]0,b[,$$

where φ_λ^* is the conjugate function associated to φ_λ. Using the definition of subdifferential we have, in virtue of (2.13),

$$\varphi_\lambda^*\left(-\frac{dz_\lambda}{dt}\right) \leq \varphi_\lambda^*(y) - (z_\lambda + g)\frac{dz_\lambda}{dt} + y),$$

a.e. $t \in]0,b[$, and $\forall y \in H$,

which implies

$$(2.14)\quad \int_0^b \varphi_\lambda^*\left(-\frac{dz_\lambda}{dt}\right) dt \leq C .$$

On the other hand, the definition of conjugate function [2, p. 52] leads us to

$$(2.15)\quad \varphi_\lambda^*\left(-\frac{dz_\lambda}{dt}\right) \geq -(w_0 + \rho w_\lambda , \frac{dz_\lambda}{dt}) - \varphi_\lambda(w_0 + \rho w_\lambda),$$

where $w_o \in \text{Int } D(\varphi)$ and w_λ is the function defined as follows

$$w_\lambda = \begin{cases} -\dfrac{dz_\lambda/dt}{\|dz_\lambda/dt\|}, & \text{if } \dfrac{dz_\lambda}{dt} \neq o \\ 0, & \text{if } \dfrac{dz_\lambda}{dt} = o . \end{cases}$$

As φ is continuous at w_o, by (2.15) combined with (2.14), where $\rho > o$ is chosen to be small enough, it follows that

$$\left\{ \frac{dz_\lambda}{dt} \right\} \text{ is bounded in } L^1(o,b;H).$$

Therefore there exists a measure $\nu = dz_1$ in $M(o,b;H)$, $z_1 \in BV(o,b;H)$, such that, on some subsequence,

$$(2.16) \qquad \frac{dz_\lambda}{dt} \longrightarrow \nu \quad , \text{ as } \lambda \to o, \text{ weak-star in } M(o,b;H).$$

Let $p \in H$ such that (some subsequence of) $\{z_\lambda(b)\}$ converges weakly to p, as $\lambda \to o$. Passing to the limit in the equality

$$\int_0^b \left(\frac{dz_\lambda}{dt}, h\right)dt = (z_\lambda(b), h(b)) - (z_o, h(o)) - \int_0^b \left(z_\lambda, \frac{dh}{dt}\right)dt, \quad \forall\, h \in W^{1,1}(o,b;H)$$

one obtains

$$\nu(h) = (p, h(b)) - (z_o, h(o)) - \int_0^b \left(z, \frac{dh}{dt}\right)dt, \quad \forall\, h \in W^{1,1}(o,b;H).$$

On the other hand,

$$\nu(h) = (z_1(b), h(b)) - (z_1(o), h(o)) - \int_o^b \left(z_1, \frac{dh}{dt}\right)dt, \quad \forall\, h \in W^{1,1}(o,b;H).$$

From the last two relations it follows that z_1 is a function from the equivalence class z (with respect to the equality a.e. on $[o,b]$), and

$$z_1(b) = p, \quad z_1(o) = z_o .$$

Summarising, we have

$$z(o) = z_o, \quad \gamma = dz \ , \text{ and}$$

(2.17)

$$z_\lambda(b) \longrightarrow z(b), \text{ as } \lambda \to o, \text{ weakly in H, on some subsequence,}$$

where z was identified with z_1.

Now, passing to the limit in (2.8), after this inequality was integrated from o to b, we obtain exactly (2.5) by means of (2.12), (2.16) and (2.17). Thus the proof is complete.

Remark 3. In general, the solution of (2.3) in the sense of Proposition 2 is not unique, as the following simple example shows

$$(2.18)\begin{cases} dz + \partial I_K(z+g) \ni o, & o < t < 2 \ , \\ z(o) = 0, \end{cases}$$

where K is the real interval $[1, +\infty[$, I_K is the associated indicator function (i.e., $I_K(y)=o$, for $y \in K$, and $I_K(y) = +\infty$, for $y \in \ ]-\infty, 1[$), and $g : [o,2] \longrightarrow \ ]-\infty, +\infty[$ is defined by

$$g(t) = \begin{cases} 1 & \text{if} \quad o \leqslant t < 1 \\ o & \text{if} \quad 1 \leqslant t \leqslant 2 . \end{cases}$$

Then, for instance, the following functions belong to the set

of solutions to (2.18) in the sense of Proposition 2:

$$z_\xi(t) = \begin{cases} 0, & \text{if } 0 \leq t < 1 \\ \xi, & \text{if } 1 \leq t \leq 2, \end{cases}$$

for every $\xi \in [1, 2]$.

We conclude by noticing that in certain particular cases the solutions of (2.3) can be more regular. For example, if $D(A)=H$ and A is bounded on bounded sets then, starting with the approximate problems (2.6), it is easy to see that (2.3) has a unique strong solution $z \in W^{1,\infty}(0,b;H)$, so (2.1) has the unique solution $u = z + g$.

Acknowledgement

The author is indebted to Dr.C.Ursescu from University of Iaşi for some stimulating conversation during the preparation of this paper.

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