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by

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ON A LINEAR THEORY OF RECURSIVE-ENUMERABLE SETS

SORIN ISTRAIL

Using rational-like operations $(\cdot, \cup, *)$ with matrices, we say that a 0-1 matrix $M=(e_{w,w'})_{w,w' \in K^*}$ is rational if it belongs to the rational closure of a finite set of 0-1 matrices.

We show that a language $L \subset K^*$ is recursive-enumerable (r.e.) if and only if its characteristic vector $d = (d^{(w)})_{w \in K^*}$ is a component of the minimal solution of a linear-rational system of equations, i.e. of the form

$$\left\{ X_i = M_{i1} X_1 + \ldots + M_{it} X_t + X_{i(o)}, \quad 1 \leq i \leq t \right.$$

with all M_{ij} rational and $X_{i(o)}$ with finitely many 1's.

A similar result holds for the family of context-sensitive sets (CS) in connexion with systems defined in terms of non-singular rational matrices.

Normal form theorems (two equations in two unknowns suffice) for r.e. and CS sets are presented together with some pictorial representations of rational matrices (exponential band, block diagonal).

As an application of the above characterizations, the families of r.e. and CS sets are organized as algebraic theories (similar with the ADJ (1976) construction for context-free sets).

0. NOTATIONS.

In the paper we work with matrices and vectors (columns) with element from $\{o,1\}$. For a column d, d^T stands for its transpose.

We use + (or $\sum$) as follows: a) when applied to vectors or matrices it has the usual meaning of addition; b) when used on the set $\{0,1\}$ it has the disjunctive meaning, i.e. $1+1 = 1+o = o+1 = 1$, $o+o = o$.

"$*$" stands for multiplication of numbers or matrices.

We refer the book (Salomaa (1973)) for all unexplained notations and results in formal languages theory.

1. LINEAR-RATIONAL SYSTEMS OF EQUATIONS

Let V be an alphabet, and for any $L \subset V^*$, we define its characteristic vector as an infinite V^*- indexed o-1 column Charv (L) = $(d^{(u)})_{u\in V^*}$ defined as usual by

$$d^{(u)} = \begin{cases} 1 , & u\in L \\ o, & u\notin L \end{cases}$$

The inverse function Lang, associates to a V^*- indexed o-1 column $d = (d^{(u)})_{u\in V^*}$, the language whose characteristic vector is d, i.e.

$$\mathrm{Lang}\ (d) = \{ w \mid d^{(w)} = 1 \}.$$

Of course, for any d and L as above, we have

$$\mathrm{Charv}\ (\mathrm{Lang}\ (d)) = d$$

$$\mathrm{Lang}\ (\mathrm{Charv}\ (L)) = L.$$

We shall denote by $\hat{0}$ the vector Charv($\emptyset$), where $\emptyset$ is the empty set. For a o-1 finite column C, we shall denote C^o its o-completion to infinity, i.e. if $C = (C^{(\lambda)},\ldots, C^{(u)})^T$ then $C^o = (C^{(\lambda)},\ldots,C^{(u)},o,o,\ldots)^T$.

Let us consider a collection of <u>finite</u> o-1 matrices, all having t+1 column:

$$m_j = (C_{oj},\ldots,C_{tj})\ ,\quad 1 \leq j \leq p.$$

We shall associate to the set $m_1,\ldots,m_p$ a collection of <u>infinite</u> o-1 matrices, all indexed by $K^* \times K^*$, where $K=\{a_1,\ldots,a_t\}$ is an alphabet.

This collection will be called the rational closure of $m_1,\ldots,m_p$, denoted $\mathcal{M}\{m_1,\ldots,m_p\}$.

Let be $\mathcal{K}$ the family of $K^* \times K^*$ -indexed o-1 matrices. Any element M of $\mathcal{K}$ is given by its "column form" as follows:

$$M = (d_w)_{w\in K^*},\ \text{where } d_w = (d_w^{(\lambda)},\ldots,d_w^{(u)},\ldots)^T_{u\in K^*}$$

for any $M = (d_w)_{w\in K^*} \in \mathcal{K}$, let us consider the set of "places" of non-zero columns: $E = \{w \mid w\in K^*,\ d_w \neq \hat{0}\}$.

Because we are interested in the occurences of non-zero columns, we shall write sometimes M_E instead of M.

We define the "rational" operations on $\mathcal{K}$:

product " . ", union " $\cup$ " and star " $*$ ".

Let be $M_E = (d_w)_{w\in K^*}$, $M'_{E'}=(d'_{w'})_{w'\in K^*}$ belonging to $\mathcal{K}$.

We define:

$$M_E \cdot M'_{E'} = M''_{EE'}$$

$$M_E \cup M'_{E'} = M''_{E\cup E'}$$

$$(M_E)^* = M''_{E^*}$$

where:

I. $M''_{EE'} = \left(\sum_{u=ww'} d_w \bullet d'_{w'} \right)_{u \in K^*}$, where

$$d_w \bullet d'_{w'} = (d_w^{(\lambda)}, \ldots, d_w^{(v)}, \ldots)^T \bullet (d'^{(\lambda)}_{w'}, \ldots, d'^{(v')}_{w'}, \ldots)^T$$

$$= \left(\sum_{y=y_1 y_2} d_w^{(y_1)} \bullet d'^{(y_2)}_{w'} \right)^T_{y \in K^*}$$

We remark that $d_w = \hat{0}$ or $d'_{w'} = \hat{0}$ is equivalent to $d_w \bullet d'_{w'} = \hat{0}$.

Therefore, $\sum_{u=ww'} d_w \bullet d'_{w'} \neq \hat{0}$ iff $u \in EE'$.

An equivalent definition for $d_w \bullet d'_{w'}$ is

$$d_w \bullet d'_{w'} = \text{Charv}(\text{Lang}(d_w)\text{Lang}(d'_{w'})).$$

II. $M''_{E \cup E'} = (d_w + d'_w)_{w \in K^*}$, where

$$d_w + d'_w = (d_w^{(\lambda)}, \ldots, d_w^{(v)}, \ldots)^T + (d'^{(\lambda)}_w, \ldots, d'^{(v)}_w, \ldots)^T$$

$$= (d_w^{(\lambda)} + d'^{(\lambda)}_w, \ldots, d_w^{(v)} + d'^{(v)}_w, \ldots)^T$$

Also, equivalent, we have:

$$d_w + d'_w = \text{Charv}(\text{Lang}(d_w) \cup \text{Lang}(d'_w)).$$

III. $M''_{E^*} = \bigcup_{n \geqslant 0} M''_{E^n}$, with $M''_{E^0} = (\hat{0}, \hat{0}, \ldots)$ and

for all $n > 0$, $M''_{E^n} = \underbrace{M_E.(M_E.\ \ldots (M_E) \ldots)}_{n}$.

DEFINITION 1

The rational closure of a finite set of (t+1)-columns o-1 matrices, $m_1, \ldots, m_p$, denoted $\mathcal{M}\{m_1, \ldots, m_p\}$ is defined as follows:

(1) if $m_j = (C_{\lambda j}, C_{1j}, \ldots, C_{tj})$, $1 \leqslant j \leqslant p$ then for all i,j,

$, 1 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant p$ we have

$$M^j_{\{\lambda\}} = (C^o_{\lambda j}, \hat{\theta}, \hat{\theta}, \ldots) \in \mathcal{M}\{m_1, \ldots, m_p\} \text{ and}$$

$$M^j_{\{a_i\}} = (\underbrace{\hat{\theta}, \ldots, \hat{\theta}}_{i-1}, C^o_{ij}, \hat{\theta}, \ldots) \in \mathcal{M}\{m_1, \ldots, m_p\};$$

(2) if $M_E, M'_{E'} \in \mathcal{M}\{m_1, \ldots, m_p\}$, $E, E' \subset K^*$, $K = \{a_1, \ldots, a_t\}$,

$$M_E \cdot M'_{E'}, \; M_E \cup M'_{E'}, \; (M_E)^* \in \mathcal{M}\{m_1, \ldots, m_p\}.$$

DEFINITION 2.

A matrix M is called rational if there exist two positive integers t and p such that M is a $K^* \times K^*$ - indexed o-l matrix belonging to $\mathcal{M}\{m_1, \ldots, m_p\}$ for some (t+1)-columns, finite o-l matrices $m_1, \ldots, m_p$.

A column d is called rational if $d = \mathcal{C}^o$ for some finite o-l vector C.

We shall introduce a class of systems of "linear" equations which use rational matrices and columns.

DEFINITION 3

A linear-rational system of equations is given by

$$\left\{ X_i = M^{(i1)} X_1 + \ldots + M^{(it)} X_t + X_{i(o)} \right., \quad 1 \leq i \leq t$$

where for some $m_1, \ldots, m_p$ and all i, j, $1 \leq i, j \leq t$, $M^{(ij)} \in \mathcal{M}\{m_1, \ldots, m_p\}$, and $X_{i(o)}$ is rational.

We shall call a linear sum as the above right sides, i.e.

$$f_i = \sum_{j=1}^{t} M^{(ij)} X_j + X_{i(o)}, \text{ a rational polynomial.}$$

Before to use the linear-rational systems for the characterization of the characteristic vectors of r.e. sets, two remarks are in order.

REMARK 1. Each linear-rational system of equations has a unique minimal solution, given as usual by $X^{MIN} = (X_1^{MIN}, \dots, X_t^{MIN})$ where for $1 \leq i \leq t$:

$$X_i^{MIN} = \bigcup_{n \geq 0} f_i(X_1^{(n)}, \dots, X_t^{(n)}), \quad X_i^{(o)} = X_i(o)$$

and for all $n \geq 1$

$$X_i^{(n+1)} = f_i(X_1^{(n)}, \dots, X_t^{(n)}).$$

If we do not impose restrictions on matrices in systems of equations, the characteristic vector of any subset of K^* can appear as minimal solution of a system.

REMARK 2. Any $d \in CV(K^*)$ is the minimal solution of an equation

$$X = MX + Charv(K^*).$$

where M is given by: $M = (x_{w,w'})_{w,w' \in K^*}$ with

$$x_{w,w'} = \begin{cases} 1 & , \quad w = w' \text{ and } d^{(w)} = 1 \\ o & , \quad \text{otherwise.} \end{cases}$$

Then we have:

$$d = X^{MIN} \ (= X^{(1)}).$$

2. A CHARACTERIZATION THEOREM

In this section we discuss the relation between linear-rational systems of equations and the characteristic vectors of r.e. sets.

Let $m_1, \dots, m_p$ be (t+1)-columns, finite o-1 matrices and $K = \{a_1, \dots, a_t\}$.

PROPOSITION 1.

For any $M \in \mathcal{M}\{m_1, \dots, m_p\}$, $M = (d_w)_{w \in K^*}$, there exists a non deterministic Turing machine T_M "computing" M, i.e. beginning

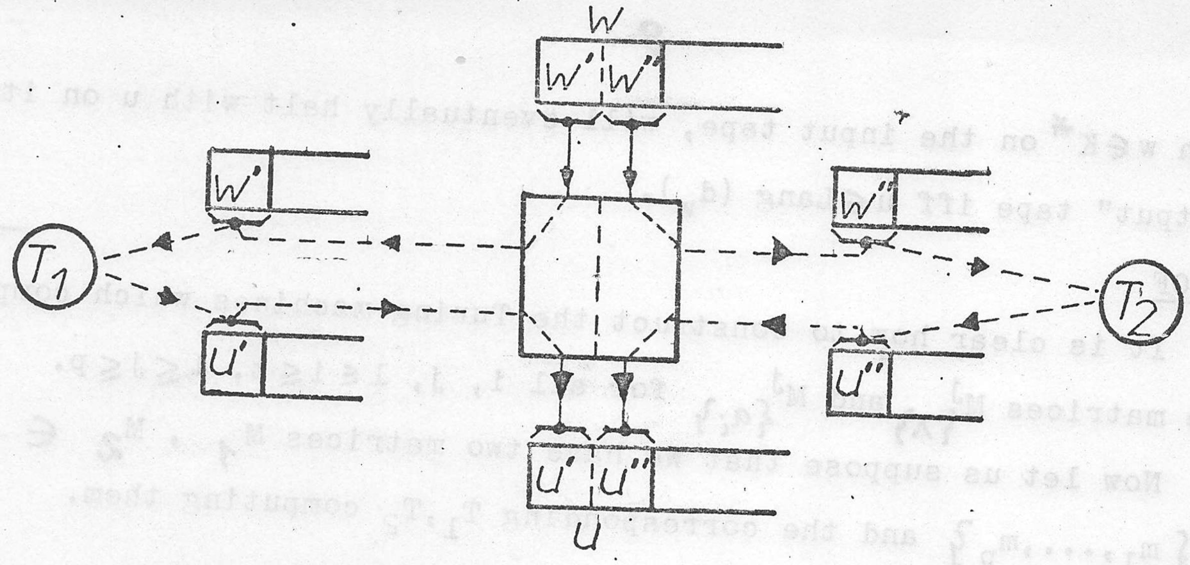

Figure 1 The machine $T_{M1 \cdot M2}$

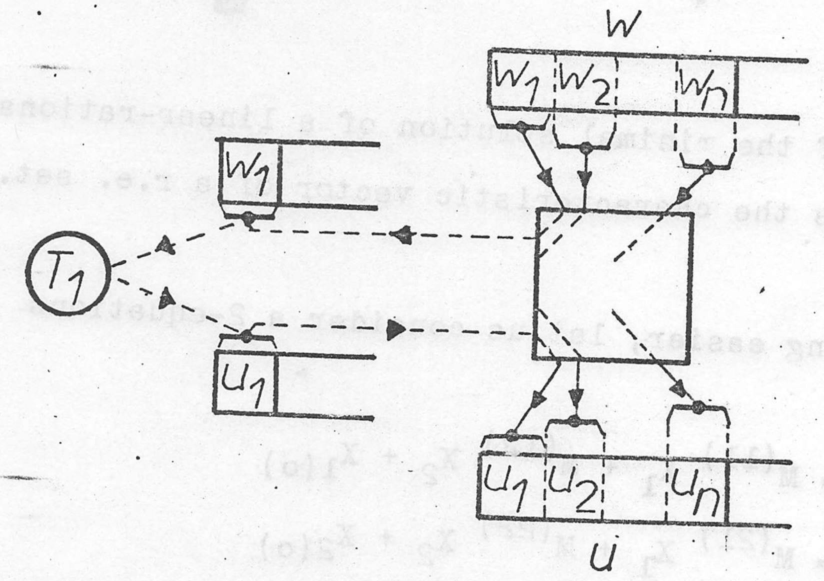

Figure 2 The machine $T_{(M1)^*}$

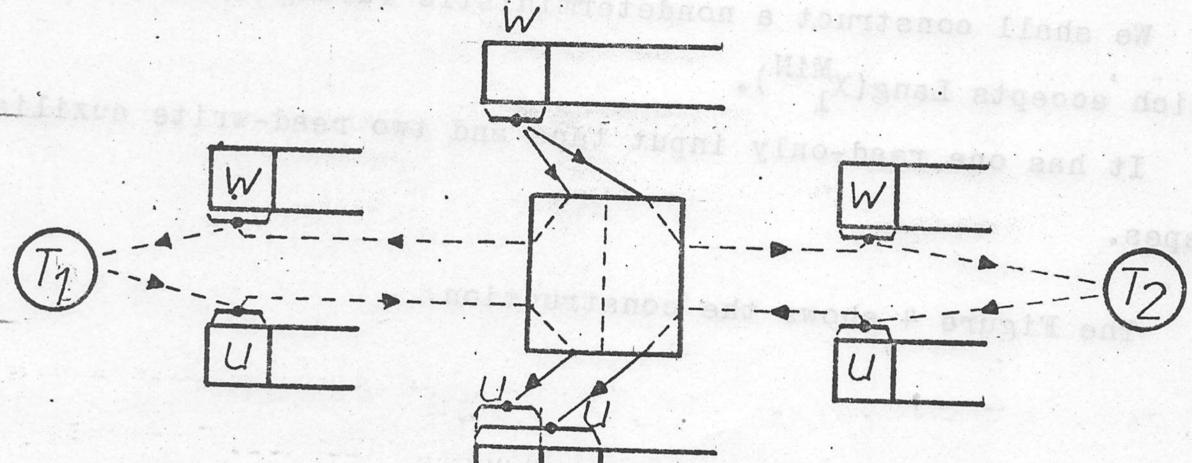

Figure 3 The machine $M1 \cup M2$

with $w \in K^*$ on the input tape, will eventually halt with u on its "output" tape iff $u \in \text{Lang}\ (d_w)$.

PROOF

It is clear how to construct the Turing machines which compute the matrices $M^j_{\{\lambda\}}$ and $M^j_{\{a_i\}}$ for all i, j, $1 \leq i \leq t$, $1 \leq j \leq p$.

Now let us suppose that we have two matrices M_1, $M_2 \in \mathcal{M}\{m_1, \ldots, m_p\}$ and the corresponding T_1, T_2 computing them.

The Figures 1, 2, 3 show the structure of the machines

$$T_{M_1 \cdot M_2},\ T_{(M_1)^*} \text{ and } T_{M_1 \cup M_2}\ .$$

■

PROPOSITION 2

Every component of the minimal solution of a linear-rational system of equations is the characteristic vector of a r.e. set.

PROOF

To make the writing easier, let us consider a 2-equations system

$$\begin{cases} X_1 = M^{(11)}\ X_1 + M^{(12)}\ X_2 + X_{1(o)} \\ X_2 = M^{(21)}\ X_1 + M^{(22)}\ X_2 + X_{2(o)} \end{cases}$$

Let be T_{11}, T_{12}, T_{21}, T_{22} the Turing machines given by the Proposition 1 which "compute" the corresponding matrices.

We shall construct a nondeterministic Turing machine $T^{(1)}$ which accepts $\text{Lang}(X_1^{MiN})$.

It has one read-only input tape and two read-write auxiliary tapes.

The Figure 4 shows the construction.

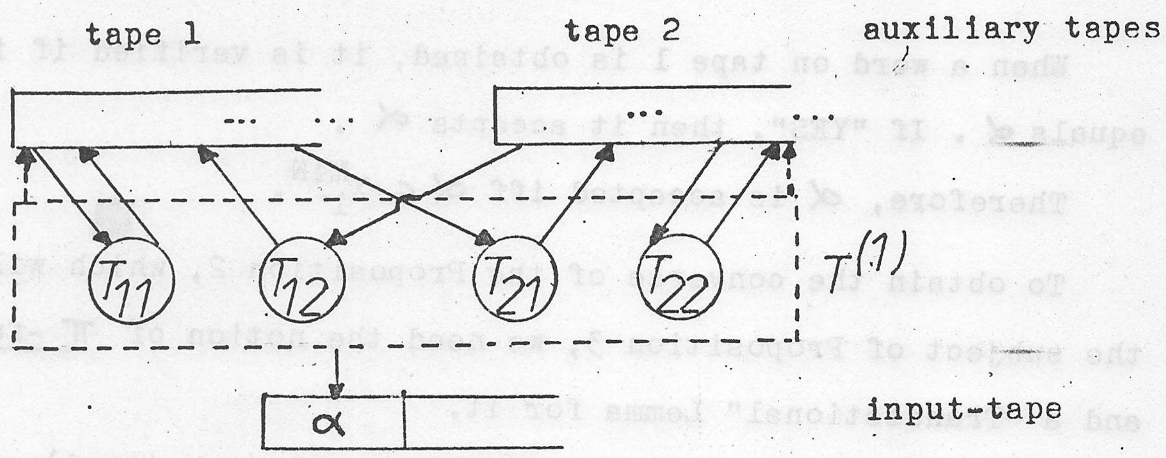

Figure 4. The machine $T^{(1)}$.

When a word α is placed on the input tape, $T^{(1)}$ begin by the initialization of the auxiliary tapes:

Initial step. Nondeterministicaly selects a word from each finite set Lang $(X_{1(o)})$ and Lang $(X_{2(o)})$ and writes them respectively on tape 1 and tape 2.

Next $T^{(1)}$ will repeatedly perform the following Basic step, until the machine will halt, or else the task is continued for ever.

Basic step. Suppose that on tape 1 and tape 2 there are two words w_1 and w_2. The machine behaves as follows:

I. Tests if $\alpha = w_1$. If "YES" it accepts α and halts, otherwise continues by II.

II. Nondeterministicaly it selects one member of each set $\{T_{11}, T_{12}\}$ and $\{T_{21}, T_{22}\}$. If they are T_{1i} and T_{2j} then take w_1 respectively w_2 as inputs and write their outputs respectively on tape i and tape j.

$\square$

Let us remark that when a word α is on the input-tape, $T^{(1)}$ simulates in paralell on tape i the work of the system of equations in order to obtain the words of Lang(X_i^{MIN}), for i=1,2.

When a word on tape 1 is obtained, it is verified if it equals α. If "YES", then it accepts α.

Therefore, α is accepted iff $\alpha \in X_1^{MIN}$.

To obtain the converse of the Proposition 2, which will be the subject of Proposition 3, we need the notion of <u>π-function</u> and a "Translational" Lemma for it.

(π-functions are introduced in Istrail (submitted) as a basic step is studying primitive actions of generative devices. In Istrail (1979) they are implicit used to obtain, generalizations of the ALGOL-like theorem).

<u>DEFINITION 4.</u>

Let $K = \{a_1, \ldots, a_t\}$ be an alphabet. A <u>π-function</u> over K is a pair $\pi = (h,R)$, where h is a finite substitution $h: 2^{K^*} \longrightarrow 2^{K^*}$ and R a regular set over K.

It is a function $\pi : 2^{K^*} \longrightarrow 2^{K^*}$ defined by $\pi(L) = h(L \cap R)$.

We shall associate to π a rational matrix M^{π} as follows.

$M^{\pi} = (e_{w,w'})_{w,w' \in K^*}$, where

$$e_{w,w'} = \begin{cases} 1 & , \quad w \in \pi(w') = h(\{w'\} \cap R) \\ o & , \quad \text{otherwise} \end{cases}$$

That is, if $M^{\pi} = (d_w)_{w \in K^*}$ (in columns form) then

a) $M^{\pi} = M_R^{\pi}$ (i.e. $d_w \neq \hat{O}$ iff $w \in R$), and

b) for $w \in R$, $d_w = \mathrm{Charv}(h(w))$.

<u>LEMMA 1.</u>

For every π-function over K, M^{π} belongs to $\mathcal{M}\{m\}$, for some finite $(|K| + 1)$ - columns matrix m.

<u>PROOF.</u>

Let be $\pi = (h,R)$ aver $K = \{b_1, \ldots, b_t\}$. We intend to construct a finite matrix m^h such that $M^\pi \in \mathcal{M}\{m^h\}$.

In the finite set $\bigcup\{\, h(b_i) \mid 1 \leq i \leq t\}$, let $\overline{w}$ be the word, maximal with respect to the lexicographic order, and, suppose that in the lexicographic enumeration (with 1 assigned for λ, 2 for b_1, etc.) it is assigned to $\overline{w}$ the number p.

Then we construct m^h as a $p \times (t+1)$ matrix:

$$m^h = (C_\lambda, C_1, \ldots, C_t),\ \text{where}$$

$C_\lambda = (1,0,\ldots,0)^T$ and for all i, $1 \leqslant i \leqslant t$, $C_i = (C_i^{(\lambda)}, \ldots, C_i^{(u)}, \ldots$
$\ldots, C_i^{(\overline{w})})^T$, with

$$C_i^{(u)} = \begin{cases} 1 & , \ u \in h(b_i) \\ 0 & , \ \text{otherwise} \end{cases}$$

We have that $M^\pi \in \mathcal{M}\{m^h\}$.

Indeed, let us consider

$$M_\lambda = (C_\lambda^0, \hat{\theta}, \hat{\theta}, \ldots) \text{ and}$$

$$M_{\{b_i\}} = (\underbrace{\hat{\theta}, \ldots, \hat{\theta}}_{i-1}, C_i^0, \hat{\theta}, \ldots) \text{ for all } i, 1 \leqslant i \leqslant t.$$

Now, the structure of R, viewed as a regular expression, provides us with a sequence of rational operations with the above matrices such that

$$M_R = M^\pi \in \mathcal{M}\{m^h\}.$$

If $f: K^* \longrightarrow 2^{K^*}$ is a function, we associate to it the matrix $M^f = (d_w)_{w \in K^*}$, where $d_w = \mathrm{Charv}(f(w))$, for all $w \in K^*$.

The function f is extended to 2^{K^*} by $f(L) = \bigcup\{f(w) \mid w \in L\}$, for all $L \subset K^*$.

<u>TRANSLATIONAL LEMMA</u>

The operation $f \rightsquigarrow M^f$ provides us with the following two "translations":

(1) The composition of functions "o" translates to matrix multiplication "$\star$", i.e.

$$M^{f o f'} = M^{f} \star M^{f'}$$

(2) f applied to L translates to M^f multiplied by Charv(L), i.e.

$$\mathrm{Charv}(f(L)) = M^{f} \star \mathrm{Charv}(L).$$

We prove only (1). Let be $M^f = (e_{w,w'})_{w,w' \in K^*}$, $M^{f'} = (e'_{w,w'})_{w,w' \in K^*}$ and $M^f \star M^{f'} = (e''_{w,w'})_{w,w' \in K^*}$.

We have $e''_{w,w'} = \sum_{w'' \in K^*} e_{w,w''} \star e'_{w'',w'}$ and $e''_{w,w'} = 1$ iff there exists $w'' \in K^*$ such that $e_{w,w''} = 1 = e'_{w'',w'}$. By the definition of matrices it follows $e_{w,w''} = 1$ iff $w \in f(w'')$ and $e'_{w'',w'} = 1$ iff $w'' \in f'(w')$.

Now it is clear that

$e''_{w,w'} = 1$ iff $w \in f o f'(w')$ which yields (1).

The converse of Proposition 2 is also true.

PROPOSITION 3

For every Turing machine M accepting the language L(M), there exists a linear-rational system of equations such that Charv(L(M)) is the first component of the minimal solution of the system.

PROOF

We consider a Turing machine M as a rewriting system (Salomaa (1973)) with Q the set of states V_T the tape alphabet, $Q_1 \subseteq Q$ the final states set and F a set of rules. Let be x_0, y two new symbols and $\bar{Q}$, $\bar{V}_T$ the "barred" version of Q and respectively V_T; $\# \in V_T$ is the boundary marker and B is the blank symbol.

We shall simulate a two-tracks tape, with two-placed symbols. Namely we consider the alphabet

$$V = \left\{ \binom{e}{d} \;\middle|\; e \in V_T,\ d \in V_T \cup \overline{V}_T \right\}.$$

We shall use the notations $\binom{V_T}{V_T} = \left\{ \binom{e}{a} \;\middle|\; a, e \in V_T \right\}$ and $\binom{V_T}{a} = \left\{ \binom{e}{a} \;\middle|\; e \in V_T \right\}$.

For simplicity, a finite substitution h will be specified by a set of context-free rules given only for the letters, for which the substitution is not identity.

For example $\left[\binom{e}{a} \to \binom{e}{a},\ \binom{e}{a} \to \binom{e}{\overline{a}},\ s \to s,\ s \to \overline{s} \;\middle|\; a, e \in V_T,\ s \in Q \right]$ is the substitution h_o given by

$$h_o\left(\binom{e}{a}\right) = \left\{ \binom{e}{a}, \binom{e}{\overline{a}} \right\},\quad h_o(s) = \{s, \overline{s}\}$$

and

$$h_o\left(\binom{e}{\overline{d}}\right) = \left\{ \binom{e}{\overline{d}} \right\} \text{ for } e \in V_T,\ \overline{d} \in \overline{V}_T.$$

Note that $[\ldots](X)$ means $h_o(X)$.

We shall associate an equation to each type of rules given by the machine.

(1) <u>Overprint</u>. $s\,a \to s'b$ is simulated by:

$$X = \left[\overline{s} \to s',\ \binom{e}{\overline{a}} \to \binom{e}{b} \;\middle|\; e \in V_T \right] \left(h_o(X) \cap \binom{V_T}{V_T}^* \overline{s} \binom{V_T}{\overline{a}} \binom{V_T}{V_T}^* \right)$$

(2) <u>Move-right</u>: $sac \to as'c$ is simulated by:

$$X = \left[\overline{s} \to \binom{e}{a},\ \binom{e}{\overline{a}} \to s',\ \binom{e}{\overline{c}} \to \binom{e}{c} \;\middle|\; e \in V_T \right] \left(h_o(X) \cap \binom{V_T}{V_T}^* \overline{s} \binom{V_T}{\overline{a}} \binom{V_T}{\overline{c}} \binom{V_T}{V_T}^* \right)$$

(3) <u>Move-right and extends work-space</u>: $sa\# \to as'r\#$ is simulated by

$$X = \left[\bar{s} \rightarrow \binom{e}{a}, \binom{e}{a} \rightarrow s', \binom{e}{\#} \rightarrow \binom{e}{r}\binom{B}{\#} \;\middle|\; e \in V_T\right].$$

$$\cdot (h_o(X) \cap \binom{V_T}{V_T}^* \bar{s} \binom{V_T}{a}\binom{V_T}{\#}).$$

Similar equations are constructed for (4) Move-left and (5) Move-right and extends work space.

The π-functions of Definition 4 express the right-members of the above equations.

Let $\pi_o = (h_o, (Q \cup \bar{Q} \cup \{\binom{e}{d} \mid e \in V_T, d \in V_T \cup \bar{V}_T\})^*)$.

Each equation (1) - (5) contains a specific π-function $\pi_i = (h_i, R_i)$ such that equation (i) can be written

(i) $\qquad X = \pi_i(\pi_o(X)), \quad 1 \leq i \leq 5$.

To complete our construction, we need two π-functions for the "initial" and "final" steps.

Let us define h_{init} by $h_{init}(x_o) = \{\# y \#\}$, $h_{init}(y) = \{y\binom{a}{a}, s_o \mid a \in V_T - \{\#\}\}$ and $h_{init}(\binom{a}{a}) = \{\binom{a}{a}\}$ for $a \in V_T$. We put now

$\pi_{init} = (h_{init}, (\binom{V_T}{V_T} \cup \{x_o, y\})^*)$.

The function π_{fin} is given by

$$\pi_{fin} = (h_{fin}, \binom{V_T}{\#} \binom{V_T}{V_T}^* \underset{c \in \{\#, B\}}{Q_1} \binom{V_T}{V_T}^* \binom{V_T}{\#})$$

where $h_{fin}(\binom{c}{b}) = h_{fin}(s) = \lambda$ for $s \in Q$, and $h_{fin}(\binom{a}{b}) = a$ for $a \in V_T - \{\#, B\}$, $b \in V_T$.

Let F(i) be the subset of F consisting of all rules of type (i), for $1 \leq i \leq 5$. If $r \in F(i)$ and the equation "implementing" the work of r is $X = \pi_i(\pi_o(X))$, we shall write π_i^r instead of π_i. Also denote $\pi_i^r = (h_i^r, R_i^r)$.

The following system is associated to M, in order to simulate its work by equations:

$$S: \begin{cases} Y = \pi_{fin}(X) \\ X = \pi_{init}(X) \cup \{x_o\} \\ X = \pi_i^r(\pi_o(X)),\ 1 \leq i \leq 5,\ r \in F(i). \end{cases}$$

If (Y^{MiN}, X^{MiN}) is the minimal solution of the system (π-function are continuous) then $Y^{MiN} = L(M)$.

The function π_o implements the "nondeterministic choice of a "position" where a rule of F to be applied. It can be "separated" to work alone. Also we use "+" defined on π-functions by: $(\pi+\pi')$ $= \pi(L) \cup \pi'(L)$, in order to express the equivalent form of the system:

$$S': \begin{cases} Y = \pi_{fin}(X) \\ X = (\pi_{init} + \pi_o + \sum\limits_{\substack{r \in F(i) \\ 1 \leq i \leq 5}} \pi_i^r)(X) \cup \{x_o\} \end{cases}$$

Let be $K = V \cup Q \cup \bar{Q} \cup \{x_o, y\} \cup V_T$, i.e. the total alphabet used. Because all π-functions which occur in system S' are functions from K^* to 2^{K^*} we shall use for any π-function π its matrix associated M^{π} as well as the Translational Lemma.

Let be X_1, X_2 range in $CV(K^*)$, the characteristic vectors of subsets of K^*.

In the new unknowns, a matrix system is associated to S' as follows

$$S'': \begin{cases} X_1 = M^{\pi_{fin}}(X_2) \\ X_2 = (M^{\pi_{init}} + M^{\pi_o} + \sum\limits_{\substack{r \in F(i) \\ 1 \leq i \leq 5}} M^{\pi_i^r})(X_2) + X_{2(o)} \end{cases}$$

where $X_{2(o)} = \mathrm{Charv}(\{x_o\})$.

It is manifest now that S" is a linear-rational system of equations and

$M^{\pi_{fin}}, M^{\pi_{init}}, M^{\pi_o}, M^{\pi_i^r} \in \mathcal{M}\{ m^{h_{fin}}, m^{h_{init}}, m^{h_o}, m^{h_i^r} \mid$

$1 \leq i \leq 5, r \in F(i)\}$) for all $i, r, 1 \leq i \leq 5, r \in F(i)$. (Note that the finite matrices m^h are given by Lemma 1).

Applying the Translational lemma we have the desired result:

$$X_1^{MiN} = \text{Charv } (L\ (M)).$$

$\blacksquare$

As a consequence of Propositions 2 and 3 we obtain the following

THEOREM 1.

A language is recursive-enumerable iff its characteristic vector is a component of the minimal solution of a linear-rational system of equations.

A rational matrix is called <u>singular</u> if there are *ones* occuring in its first row. Otherwise, it is called <u>non-singular</u>. I.e. if $M = (e_{w,w'})_{w,w' \in K^*}$, then M singular means that there exists $w' \in K^*$ such that $e_{\lambda, w'} = 1$. This shows that erasure is permitted on some letters.

A linear-rational system is called <u>non-singular</u> if all its rational matrices are non-singular.

Non-singular linear-rational systems provides a similar characterization as the one in Theorem 1, at this time for context-sensitive sets.

THEOREM 2

A language is context-sensitive iff its characteristics vector is a component of the minimal solution of a non-singular linear-rational system of equations.

COROLLARY (The normal form for (non-singular) linear-rational systems)

A language is recursive-enumerable (context-sensitive) iff its characteristic vector is a component of the minimal solution of a 2-equations (non-singular) linear-rational system:

$$\begin{cases} X_1 = M_1 X_1 + X_{1(o)} \\ X_2 = M_2 X_1 + \hat{\theta} \end{cases}$$

3. SOME PICTORIAL CONSIDERATIONS

As in the classical theory of matrices, a picture showing the areas of non-zero elements, helps to understand the structure of the matrix (e.g. triangular, band, block-diagonal).

We present such pictures for our rational matrices. Note that hachured areas means, possible places of 1's.

1) A generic non-singular rational matrix is shown in Figure 5. It is an "exponential band"

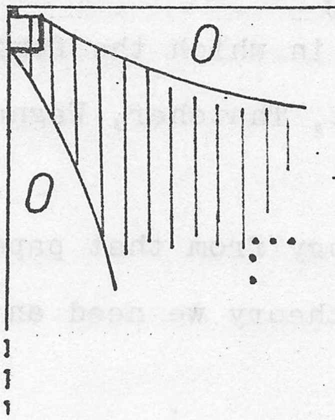

Figure 5:
A non-singular rational matrix

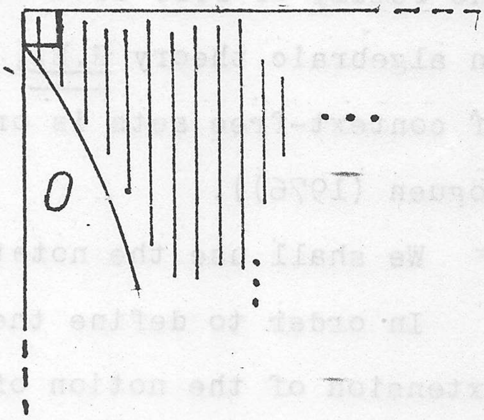

Figure 6: A singular rational matrix

2) A generic singular matrix is shown in Figure 6. It is an "ultimately upper curvilinear-triangular".

Let us remark also that if we use simple Turing machines (Stockmeyer (1974)) in the proof of Proposition 3, the picture of the matrix which essential simulates the work of the machine, i.e. $\sum\limits_{\substack{1\le i\le 5 \\ r\in F(i)}} M^{\pi_i^r}$, is exactly a block-diagonal. That is a "normal form" for the exponential band of Figure 5.

This picture is presented in Figure 7.

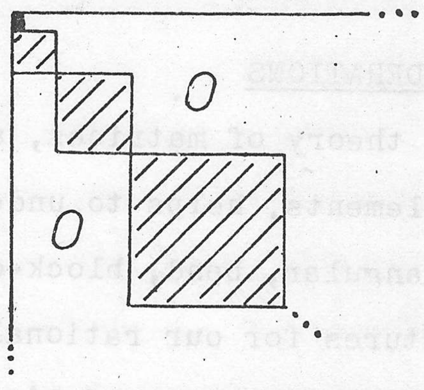

Figure 7.

4. AN APPLICATION: THE ALGEBRAIC THEORIES OF R.E. AND CS SETS

As a benefit of the characterization presented in Theorem 1, the family of r.e. sets over $K = \{a_1, \ldots, a_t\}$ can be organized as an algebraic theory R.E., in a similar way, in which the family of context-free sets is organized in (Wright, Thatcher, Wagner, Goguen (1976)).

We shall use the notations and terminology from that paper.

In order to define the above algebraic theory we need an extension of the notion of rational matrix.

To do that, we add in Definition 1:

(3) $\mathcal{M}\{m_1, \ldots, m_p\}$ is closed under matrix multiplication. The resulting notion is called extended-rational closure. This gives now a sense to the extended-rational matrices.

An extended-rational polynomial is a rational one with all matrices extended-rational.

A linear-extended-rational system is a linear-rational system will all right members extended-rational polynomials.

It is clear that Theorem 1 holds true when "rational" is replaced by "extended-rational". $\square$

Let be Var = $\{X_1, X_2, \ldots\}$ a set of variables.

We define R.E. (n,p) as the collection of all n-tuples $F=(F_1,\ldots,F_n)$, where for all i, $1 \leq i \leq n$, F_i is an extended-rational polinomial in p variables, i.e.

$$F_i(X_1,\ldots,X_p) = \sum_{j=1}^{p} M_{ij} X_j + X_{i(o)}$$

with $M_{ij} \in \mathcal{K}$ extended-rational and $X_{i(o)}$ rational, $1 \leq i \leq n$, $1 \leq j \leq p$.

R.E.(n,p) is a strict poset with "$\sqsubseteq$" defined as follows:

$F \sqsubseteq F'$ iff for all i and all $d_1,\ldots,d_p$ we have

$$F_i(d_1,\ldots,d_p) \sqsubseteq F'_j(d_1,\ldots,d_p).$$

For $d,d' \in CV(K^*)$: $d \sqsubseteq d'$ iff Lang (d) $\subseteq$ Lang (d').

The bottom is $\bot_{n,p} = (\theta_1,\ldots,\theta_n)$, where

$$\theta_i = \sum_{j=1}^{p} \bar{0}_{ij} X_{ij} + \hat{0}, \quad 1 \leq i \leq n, \text{ and } \bar{0}_{ij} \text{ is the}$$

everywhere-$\hat{0}$ matrix.

If $F = (F_1,\ldots,F_n) \in$ R.E. (n,p), $G=(G_1,\ldots,G_p) \in$ R.E.(p,q) then we define $F \circ G = H = (H_1,\ldots,H_n) \in$ R.E.(n,q) by

$$H_k(X_1,\ldots,X_q) = F_k(G_1(X_1,\ldots,X_q),\ldots,G_p(X_1,\ldots,X_q)),$$

$1 \leq k \leq n$.

The composition "o" is associative.

The identity $1_n \in$ R.E.(n,n) is given by $1_n = (X_1,\ldots,X_n)$ (where X_i can be written as $\sum_{j=1}^{i-1} \bar{0}_{ij} X_j + \mathbb{1}_{ii} X_i + \sum_{j=i+1}^{n} \bar{0}_{ij} X_j + \hat{0}$; of course $\mathbb{1}_{ii}$ is the unity matrix, which is rational).

Now any morphism $F=(F_1,\ldots,F_n): n \to n$ can be seen to correspond to a linear-extended-rational system.

We"solve" F by finding its least fixed-point: if $F^k = \underbrace{FoFo\ldots oF}_{k}$,

the sequence $\langle F^k o \perp_{n,o} \rangle_{k \geq o}$ is a chain in $\underline{\underline{R.E.}}(n,o)$ and

$F^{\nabla} = \bigsqcup F^k o \perp_{n,o}$ is the least fixed-point of F.

If x_i^n is the $i^{\underline{th}}$ projection for n-tuples, we can see a linear-extended-rational system which defines a r.e. set, as a pair $\langle x_i^n, F: n \longrightarrow n \rangle$, where x_i^n specifies the distinguished equation of the system defined by F.

The corresponding r.e. set, which this pair defines, is $x_j^n o F^{\nabla}$. $\square$

Similar considerations can be done for $\underline{\underline{CS}}$ in connexion with systems defined in terms of non-singular extended-rational matrices.

R E F E R E N C E S

ADJ: J. B. WRIGHT, J.W.THATCHER, E.G.WAGNER, J.A.GOGUEN "Rational algebraic theories and fixed-point solutions" Proceedings 17^{th} IEEE FOCS, Houston, Texas, (1976)

S. ISTRAIL - "A fixed-point theorem for recursive-enumerable sets and some considerations about fixed-point semantics of monadic programs" 6^{th} ICALP, Lecture Notes in Computer Science No. 71 (1979), p. 289-303.

S. ISTRAIL - "Chomsky-Schützenberger representations for families of languages and Grammatical Types". 8th ICALP, Haifa, (1981).

A. SALOMAA - "Formal Languages" Academic Press, New York (1973)

L. STOCKMEYER - "The Complexity of decision problems in automata theory and logic", Project Mac Tech. Report 133, 1974.

A. SALOMAA, M.SOITTOLA - "Automata-theoretic properties of formal power series" Texts and Monogr. in Comp. Sci., Springer-Verlag (1978).