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The aim of this work is to give a positiv answer to an old question of Arens and Kaplansky concerning the structure of commutativ algebraic regular algebras [1, pag.467].

If is expected that similar improved techniques would work in the noncommutative case too, or at least in some particular cases.

I have to express my gratitude to my collegues for the usefull discussions we had together and for the observations concerning the problem.

The problem is:

THEOREM 1.

Let A be a regular von Neumann commutative k-algebra with algebraic over k fibres, k-isomorphic to K.

Then $A \cong C(\mathrm{Spec}\, A, K)$.

We prove only that A is a K-algebra, and applying Theorem 2.3 [1] we are through.

N O T A T I O N S:

i) Spec A is the structure space of A and its points will be denoted by small letters p, q, eventually indexed.

ii) For X, Y topological spaces C(X,Y) is the space of continue mappings from X to Y. If no topology is specified on an amorph set it will be considered with discret topology.

iii) $\mathcal{V}_{oc}(p)$ is the set of all closed and open neighbourhoods of p.

iv) The canonic mappings $A \to A/_p \to K$ will be denote by φ_p.

v) For $p \in \mathrm{Spec}\, A$, $V \in \mathcal{V}_{oc}(p)$ e_V is the unic idempotent of A with the property:

$e_V = 1(q)$ if and only if $q \in V$

vi) The big letters L... and B... (eventually $B^{(p)}$ ...) denote subfields of K and respectively subalgebras of A, closely related (see the text).

vii) Other symbols and notations introduced are explained in the text.

a) <u>Some preliminary observations</u>

Proposition 1) For any B, a k-subalgebra of A, finite over k, and any $p \in \mathrm{Spec}\, A$, there exists $V \in \mathcal{V}_{oc}(p)$ so as to be $e_V B$ a field with unit element e_V and $\sigma^{e_V B}_{p,q} = \varphi_q \varphi_p /^{-1}_{e_V B} : L \to K$ are k-imbeddings of L in K for any $q \in V$.

Proposition 2). Conversely:

Let $L \subseteq K$ be finite over k and $p \in \mathrm{Spec}\, A$. There exists $B^{(p)}$ a k-subalgebra of A, finite over k thus $L = \varphi_p(B^{(p)})$, and this preimage of L in A is, in some sense, unic: if B' is an other finite over k preimage of L in A at p, there exists $V \in \mathcal{V}_{oc}^{(p)}$ so as to have $e_V B' = e_V B$.

Proposition 3) Let $L_1 \subseteq L_2 \subseteq K$ be finite extensions of k contained in K, B_i preimages, as above, of L_i at p; i=1,2.

Then there exists $V\in\mathcal{V}_{oc}(p)$ for which $B_1e_V\subseteq B_2e_V$.

The proofs of these observations consist in simple verifications and we omit them.

Proposition 4). Let $L\subseteq K$ be finite over k, $p\in\mathrm{Spec}\,A$, B be the preimage of L in discussion above, $V\in\mathcal{V}_{oc}(p)$ constructed at 1).

For any $W\subseteq V$, $W\in\mathcal{V}_{oc}(p)$ denote by $G_{W,B,p}$ the set of all imbeddings $\sigma^B_{p,q}$, q running over V.

Then there exists $W_o=W(B,p)\in\mathcal{V}_{oc}(p)$ for which $G_{W_o,B,p}=\bigcap_W G_{W,B,p}$ and so $G_{L,p}=\bigcap_W G_{W,B,p}$ depends not on B but on L.

<u>Proof.</u> Taking into account the unicity a preimage B of L in the sence of Proposition 2) and of the finiteness of the sets $G_{W,B,p}$, the fact above is a consequence of that that $(G_{W,B,p})_W$ is a decreasing inductiv system of finite sets.

Proposition 5). Let $L_1\subseteq L_2\subseteq K$ be finite subfields of K over k, and $p\in\mathrm{Spec}\,A$. Then $G_{L_1,p}=G_{L_2,p}|L_1$.

<u>Proof.</u> Indeed for B_i preimage of L_i at p, let $W\in\mathcal{V}_{oc}(p)$ be chosen in such a way that: $G_{L_i,p}=G_{W,B_i,p}$ and $B_1e_W\subseteq B_2e_W$. It follows that any $\sigma^{B_1}_{p,q}$ is a restriction of a $\sigma^{B_2}_{p,q}$ for any $q\in W$.

The following lemma is fundamental:

<u>Lemma:</u> Let $L\subseteq K$, L finite over k, $p\in\mathrm{Spec}\,A$.

Then $G_{L,p}\subseteq\mathrm{Aut}(K/k)|L$

<u>Proof:</u> Let $(G_{L_i,p},\psi_{ij})$ be the projectiv system, build up by taking ψ_{ij} to be the restrictions from $G_{L_i,p}$ to $G_{L_j,p}$ for $L_i\supseteq L_j$, $(L_i)_i$ being the family of subfields of K which are finite over k.

Applying 5) it follows that ψ_{ij} are surjectiv, thus by a wellknown result of Ore,

$G = \varprojlim_i G_{L_i,p} \neq \phi$, and the canonic projections $\psi_i : G \to G_{L_i,p}$ are surjectiv.

b) Galois-Krull nonnormal theory

Let $k \hookrightarrow K$ be an algebraic extension of k, G=Aut (K/k).

For any $L \subseteq K$, finite over k, consider the sets:

$$G^L = \{\sigma \in G \mid \sigma|_L = id_L\} \text{ and } G_L = G|_L$$

For $L_i \supseteq L_j$ canonic restriction $\psi_{ij} : G_{L_i} \to G_{L_j}$ is surjectiv and one obtains a surjectiv projectiv system of finite sets (G_{L_i}, ψ_{ij}).

THEOREM 2.

$G = \varprojlim_i G_{L_i}$, and the topology induced by $\prod_i G_{L_i}$ on G makes it into a topological, compact total disconected group. A fundamental system of neighbourhoods of $e = id_K$ is $\{G^{L_i}\}_{L_i}$ finite over k.

Proof is standard: the $\complement \varprojlim_i G_{L_i}$ is open in $\prod_i GL_i$ so $\varprojlim_i G_{L_i}$ is closed in the compact topological space $\prod_i G_{L_i}$, hence compact. Much more, for $(\sigma_i) \in \varprojlim_i G_{L_i}$ a fundamental system of neighbourhoods of (σ_i) is $\{V_F\}_F$ with $V_F = \prod_{L_i \notin F} G_{L_i} \times \{\sigma_i\}_{L_i \in F} \cap \varprojlim_i G_{L_i}$, F being an arbitrar finite subset of the set of all finite over k subfields of K.

If is easy to verify that $V_F = (\sigma_i) G^{L_F}$ with L_F the compositum of L_i; $L_i \in F$; over k which is finite over k because F is finite and every L_i is finite over k.

This implies that a fundamental system of neighbourhoods for e is $\{G^{L_i}\}_i$.

For the proof of the group topology one observes that for any $L_i \subseteq K$ the set of all elements of K which are conjugated to elements in L_i generates a field NL_i, finite over k and invariant for any $\sigma \in Aut(K/k)$ i.e. $\sigma(NL_i)=NL_i$.

Thus G^{L_i} is normal in G and the set $\{G^{NL_i}\}_{L_i}$ is a fundamental system of neighbourhoods of e. The proof is reduced now to simple verifications.

The total disconection of G is a consequence of that of $\prod_i G_{L_i}$ and so the proof of Theorem 2 is done.

<u>Remark</u>: The fundamental system of neighbourhoods involved above has only open-closed sets as members.

Let us introduce now some new objects:

For G=Aut(K/k) and Spec(A), and $L \subseteq K$ (not necessary finite over k) construct the following groups:

$$\mathbb{G} = \prod_{p \in Spec A} G$$
$$\mathbb{G}^L = \prod_p G^L .$$

Denote by (σ) the elements in $\mathbb{G}$ and observe that $(\sigma)\mathbb{G}^L$ is compact for any $(\sigma) \in \mathbb{G}$, and $L \subseteq K$ (not necessary finite over k)

Indeed: for L finite over k the assertion is obvious, and for L infinite let L be $\bigcup_i L_i$, L_i finite over k. Then $G^L = \bigcap_{L_i \subseteq L} G^{L_i}$, so G^L is compact in G, because G^{L_i} are compact, etc.

c) <u>The proof of the theorem 1:</u>

Let $B \subseteq A$ be a k-subalgebra of A, which is a field with unit element 1_A.

Then, for any $p,q \in Spec\ A$, $\sigma^B_{p,q} = \varphi_q \varphi_p^{-1}|_B$ is a k-imbedding of $L=\varphi_p(B)$ in K.

Let us fix a point $p \in Spec\ A$, and construct the family:

$$\mathcal{A}_p=\left\{ (B,(\sigma)\mathbb{G}^L) \;\middle|\; \begin{matrix} B \text{ and } L \text{ as above} \\ (\sigma)|_L=(\sigma^B_{p,q})_q \end{matrix} \right\}$$

We proceed now to an ordering of F_p:

$$(B',(\sigma)'\mathbb{G}^{L'})<(B'',(\sigma)''G^{L''}) \text{ if and only if } B'\subseteq B''.$$

Observe that from the definition it follows that:

i) $(k,\mathbb{G})\in\mathcal{A}_p$ thus $\mathcal{A}_p\neq\emptyset$

ii) $(B,(\sigma)\mathbb{G}^L)\in\mathcal{A}_p$ implies $(\sigma)\mathbb{G}^L\neq\emptyset$

iii) $(B',(\sigma)'\mathbb{G}^{L'})<(B'',(\sigma)''\mathbb{G}^{L''})$ implies $(\sigma)'\mathbb{G}^{L'}\supseteq(\sigma)''\mathbb{G}^{L''}$.

We prove now that any completely ordered subfamily $(B_i,(\sigma)_i G^{L_i})_i$ of $\mathcal{A}_p$ has a majorant:

j) $B=\bigcup_i B_i\subseteq A$ is a k-subalgebra of A, which is a field with a unit element 1_A.

jj) $L=\bigcup_i L_i$ is in fact $\bigcup_i\varphi_p(B_i)=\varphi_p(\bigcup_i B_i)=\varphi_p(B)$.

jjj) Let $\Sigma=\bigcap_i(\sigma)_i\mathbb{G}^{L_i}$ then $\Sigma\neq\emptyset$ because $(\sigma)_i G^{L_i}$ is a decreasing system of compact non-empty sets (see ii) above and final observations from b)).

Any $(\sigma)\in\Sigma$ has the property $(\sigma)|_{i}=(\sigma)_i$ for any i. We may conclude that $(B,(\sigma)\mathbb{G}^L)\in\mathcal{A}_p$ and it majorates any $(B_i,(\sigma)_i\mathbb{G}^{L_i})$.

Therefore, there exist maximal elements in $\mathcal{A}_p$. Let $(B,(\sigma)\mathbb{G}^L)$ be such an element. We prove that A is in a natural way an L-algebra, and L=K.

The proof of the first assertion:

Consider the modified canonic morphism $(\chi_q)_q=(\sigma)^{-1}(\varphi_q)_q$. Then $\chi_p=\varphi_p$ and $\chi_q\chi_p^{-1}|_B=id_L$:

$$\chi_q\chi_p^{-1}|_B(x)=\sigma_q^{-1}\varphi_q\varphi_p^{-1}|_B(x)=\sigma_q^{-1}\sigma^B_{q,p}(x)=x \text{ by definition of } (\sigma)\mathbb{G}^L.$$

Let us prove now that L=K.

From now on we are working vewing A as L-algebra by modified canonic morphisms $(\chi_q)_q$.

We prove that the presumtion $K \setminus L \neq \emptyset$ leeds to a contradiction about the maximality of

$$(B, (\sigma)G^L)$$

Indeed:

let $x \in K \setminus L$, thus $L_1 = L(x)$ is a finite extension of L, contented in K. For any $q \in$ Spec A let $a_q \in A$ be a local representativ for x at q, so $B_1^{(q)} = L(a_q)$ is a preimage for L_1 at q, finite over L.

In accordance with 1), 2), 3), 4), 5) from a) there exists $W_q \in \mathcal{V}_{oc}(q)$ for any $q \in$ Spec A, so that

$B^{(q)} e_{Wq}$ is a field with unit element e_{Wq} and

$$G_{L_1, q} = G_{W, B^{(q)} e_{Wq}, q}$$

Because of the compactness of Spec A there exists a finite set of points $\{q_o = p, q_1, \ldots, q_n\}$ so as to have $\bigcup_i W_{qi}$ =Spec A.

It may be presumed that $W_{qi} \neq \emptyset$ and disjoined, hence

$$e_{Wq_i} \cdot e_{Wq_j} = \delta_{ij} e_{Wq_i}$$

The preimage of L_1 we are looking for, is

$$B_1 = B[\alpha] \text{ where } \alpha = \sum_{i=0}^{n} e_{Wq_i} \cdot a_{q_i}$$

Indeed: $L_1 \simeq L[X]/(\mathrm{Irr}(x, L)) \simeq B[X]/(\mathrm{Irr}(\alpha, B)) \simeq$

$\simeq B[\alpha] = B_1$, so B_1 is a field and obviously a L-subalgebra of A (thus k-subalgebra) and $\chi_p(B_1)=L_1$.

Let us prove now that $\sigma^{B_1}_{p,q} = \chi_q \chi^{-1}_{p|B_1}$ can be prolongned to L-automorphisms of K:

$$\sigma^{B_1}_{p,q} = \chi_q \chi^{-1}_{p|B_1} = \chi_q \chi^{-1}_{q_i|B_1}(q_i) \circ \chi_{q_i|B_1}(q_i)\, \chi^{-1}_{p|B_1} = \chi_q \chi_{q_i|B_1}(q_i)$$

where q_i is the point for which $q \in W_{q_i}$ and the assertion follows from the choice of W_{p_i} and Lemma from a).

Let $(\tau)=(\tau_q)_q$ be the family of L-automorphisms which prolongue $\chi_q \chi^{-1}_{p|B_1}$ to L-automorphisms of K. It is a matter of simple verifications that

$(B_1(\sigma)(\tau)G^{L_1}) \in \mathcal{A}_p$ and the contradiction is achieved:

Indeed:

$$(\sigma)(\tau)|_{B_1} =$$

$$= (\sigma)(\chi_q \chi^{-1}_{p|B_1})_q = (\sigma)(\sigma)^{-1}(\varphi_q \varphi^{-1}_{p|B_1})_q = (\varphi_q \varphi^{-1}_{p|B_1})_q = (\sigma^{B_1}_{p,q})_q .$$

R e f e r e n c e s:

1 R.Arens and J.Kaplansky - Topological Representations of Algebras , T.A.M. Soc.63(1948), 457, 481.

2 N.Popescu and C.Vraciu - Sur la Structure des Anneaux Absoluments Plats Commutatifs. Journal of Algebra, vol. 40, No.2, June 1976.