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Abstract. A weak version of the maximum principle for the time optimal control problem in Banach spaces is obtained. An example involving linear retarded systems is derived.

1. INTRODUCTION

We are concerned here with the time optimal control problem for the equation

(1.1) $x'(t) = Ax(t) + Bu(t) \quad t > o$

$x(o) = x_o$

(1.2) $y(t) = \Lambda x(t)$

where $x(.)$ takes values in reflexive Banach Space E, $B \in L(U,E)$, $\Lambda \in L(U,F)$; U,F being other Banach Spaces, A is the infinitesimal generator of a strongly continuous semigroup $\{S(t); t \geq o\}$ on E.

We denote by L(X,Y) the algebra of linear continuous operator from X to Y endowed with the usual norm $\|\cdot\|_{L(X,Y)}$.

We shall consider the "mild" solution of (1.1), i.e.,

(1.3) $$x(t) = S(t)x_o + \int_0^t S(t-s)Bu(s)ds$$

and thus we may rewrite (1.2) such as

(1.4) $$y(t) = \Lambda S(t)x_o + \Lambda \int_0^t S(t-s)Bu(s)ds.$$

Let $x_o \in E$ and $y_1 \in F$ be fixed and for $M > o$ we denote

(1.5) $$L_M = \{u \in L^{\infty}(R^+, U);\ \|u(t)\| \leq M \text{ a.e. } t > 0\}$$

A trajectory $y(.)$ is admissible if $x(o) = x_o$, $y(t) = y_1$ for some

$t>0$. Since trajectories are always continuous, there exists a smallest t for which $y(t) = y_1$ holds; this number will be called the transition time of $y(\cdot)$. The infimum T of the transition times of all admissible controls $u \in L_M$ is called the optimal time. The time optimal control problem is the following:

(a) Does there exists a control $u_o \in L_M$ (optimal control) such that $y(T) = y_1$? (T is optimal time)

(b) Assuming u_o exists, how can it be characterized and what properties it has? The answer to (a) is generally affirmative. Using similar assumption to those of Lemma 2.1 in [4], the existence of optimal control for our problem is assured.

The question (b) was studied for some particular cases: ([4], [5], [6].) The basic technique used in these papers is the application one of the standard separation theorems for convex sets. The difficulty consists in the sets which are considered in a natural way (see Ω_T defined in (3.4)), have interior void in the state space. The separation theorem is applied then in other spaces, suitable chosen, which have stronger topologies. For example Fattorini, [4], uses $D(A)$ with the graph norm and obtains a version of the maximum principle without restrictions on $S(t)$ (Th. 2.1) and in the case when the semigroup is analytic and E is a Hilbert space obtains:

Corollary 5.2. [4]. If $0 \leq t < T$, there exists $x_t \in H$, with

$$(1.6) \qquad u_o(s) = \frac{S^*(t-s)x_t}{|S^*(t-s)x_t|}, \quad 0 \leq s \leq t$$

$U_o(\cdot)$ being the optimal control.

In the papers mentioned above $U = E = F$ and $B = \Lambda = I$ are considered (Here I is the identical operator in E) The same problem is studied by Henry too, [7], in the case of parabolic equations, using the analytic semigroups, in fact.

In this note we shall demonstrate a similar result with (1.6), without proposing the analyticity of the semigroup and in the framework generally indicated at the beginning. We also mention that the introduction of the operator Λ in this study is determined by the fact that in certain problems which are written in the form (1.1) (e.g. the equation with delay) the controllability is not studied in the whole space but on certain subspaces. In this case, Λ a projection operators appears; for example see [1], [10].

The space in which we shall apply the separation theorem is similar with that used by Schmidt [6], (he studies the bang-bang principle for parabolic equations with boundary control) and by Henry [7].

2. PRELIMINARIES

We introduce the operator $V(t)$: $L^{\infty}(o,t;U) \to E$, by

$$(2.1) \qquad V(t)u = \int_o^t S(t-s)Bu(s)ds \quad , t > 0$$

and so we may rewrite (1.4) as

$$(2.2) \qquad y(t) = \Lambda S(t)x_o + \Lambda V(t)u.$$

The basic assumption which will be in effect throughout this paper is the null controllability property of the system (1.1) (1.2), i.e.

$$(2.3) \qquad \Lambda S(t)E \subset \Lambda V(t)L^{\infty}(0,t;U) \text{ for every } t > 0.$$

Denote by

$$(2.4) \qquad R_t(x_o, L^{\infty}(o,t;U)) = \{ y \in F;\ \text{there exists } u \in L^{\infty}(o,t;U)$$

such that $y = \Lambda S(t)x_o + \Lambda V(t)u$

and so (2.3) can be written as

$$(2.5) \qquad \Lambda S(t)E \subset R_t(o, L^{\infty}(o,t;U)).$$

We introduce on $L^{\infty}(R^+;U)$ the following operators:

(2.6) $(J_s u)(t) = u(t+s)$ for $s \geq 0$

$$(J_s u)(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq |s| \\ u(t+s) & \text{if } t > |s| \end{cases} \quad \text{for } s < 0$$

(see [6]).

Lemma 2.1 The following identities hold:

(2.7) $\Lambda V(t_1 + t_2)u = \Lambda S(t_2)V(t_1)u + \Lambda V(t_2)J_{t_1} u$ for $t_1, t_2 > 0$

(2.8) If $t_1 < t_2$ then $\Lambda V(t_1)u = \Lambda V(t_2)J_{t_1 - t_2} u$.

Proof. The proof of this Lemma can be obtained through some rather standard manipulations involving (2.1) , (2.6) .

Using Lemma 2.1 it is easy to prove the following

Proposition 2.1 Assume that condition (2.3) holds. Then $R_t(x_0, L^\infty(0,t;U))$ is independent of $x_0 \in E$ and $t > 0$. See also [6] , [11] .

Denote by $R = R_t(0, L^\infty(0,t;U))$ and introduce on R the following norms:

(2.9) $\|y\|_t = \inf\{ \|u\|_\infty \; ; \; u \in L^\infty(0,t;U) \, , \, y = \Lambda V(t)u \}$,

$t > 0$, which define a Banach Space topology on R. For the proof see [3], [8].

Using (2.8) we may infer that for $s < t$ we have $\|y\|_t \leq \|y\|_s$. The closed graph theorem shows that the norms $\|\cdot\|_t$, $t > 0$, are equivalent.

In what follows we shall consider R as a Banach space with the norm $\|\cdot\|_1$.

We also observe that the inclusion mapping from R into F, $I: R \to F$ is continuous. We shall denote

(2.10) $X = \mathrm{Cl}(\bigcup_{t > 0} \Lambda S(t)E)$ (Here "Cl" denotes the closure in the topology of R).

Let $P(t): E \to X$, $t > 0$, be defined by

(2.11) $P(t) = \Lambda S(t)$

We summarize some properties of the operator $P(t)$ below.

Lemma 2.2 (a) $P(t)$ is a linear and bounded operator.

(b) $\lim_{t \to t_0} P(t)x = P(t_0)x$ for every $x \in E$ and $T_0 > 0$

(c) If $x^* \in X^*$, $x^* \neq 0$, then there exists $\varepsilon_0 > 0$,

Such that $P^*(\varepsilon)x^* \neq 0$ for every $\varepsilon < \varepsilon_0$.

Proof. (a) results from the closed graph theorem .

b) For $t > t_0$ we have $P(t)x - P(t_0)x = P(t_0)\left(S(t-t_0)x - x\right)$

and so $\lim_{\substack{t \to t_0 \\ t > t_0}} P(t)x - P(t_0)x = 0$.

If $t_1 < t < t_0$ we have $\|P(t)\|_{L(E,X)} = \|P(t_1)S(t-t_1)\|_{L(E,X)} \leq$

$\leq \|P(t_1)\|_{L(E,X)} \|S(t-t_1)\|_{L(E,E)} \leq C$ (C being a positive constant).

On the other hand we have

$\|P(t)x - P(t_0)x\|_X = \|P(t)\left(x - S(t_0 - t)x\right)\|_X \leq C\|x - S(t_0)x\|_E$

and we may conclude that $\lim_{\substack{t \to t_0 \\ t < t_0}} P(t)x - P(t_0)x = 0$.

(c) Let us assume that there exists $\varepsilon_n > 0$, $\varepsilon_n \to 0$, such that $P^*(\varepsilon_n)x^* = 0$. For $t > 0$, there exists $\varepsilon_n < t$ and from (2.11) we have $P(t) = P(\varepsilon_n)S(t - \varepsilon_n)$. Then $P^*(t)x^* = S^*(t - \varepsilon_n)P^*(\varepsilon_n)x^* = 0$. Thus we have $P^*(t)x^* = 0$ for all $t > 0$, and according to (2.10) we get $x^* = 0$. The proof is complete.

3. THE MAIN RESULTS

Let us denote $R_t(x_0, L_M) = \{y \in F;$ there exists $u \in L_M$ such that $y = \Lambda S(t)x_0 + \Lambda V(t)u\}$.

The main result of this paper is the following

Theorem 3.1 Let $x_0 \in E$ and $y_1 \in R_t(x_0, L_M)$ such that

(3.1) $M > \text{dist}_R(y_1, X)$.

Let u_0 be an optimal control, T its transition time. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ there exists $x^*_\varepsilon \in E^*$, x^*

(3.2) $(B^*S^*(T-\varepsilon-t)x^*_\varepsilon, u_o(t)) = \sup_{\|u\|\le M} (B^*S^*(T-\varepsilon-t)x^*_\varepsilon, u) =$

$\|B^*S^*(T-\varepsilon-t)x^*_\varepsilon\|_{U^*}$ a.e. $t\in[0,T-\varepsilon]$.

For the proof of this theorem we need two lemmas

Lemma 3.1 Let y_1 as in Theorem 3.1 .Then there exist $z_1 \in X$, $u_1 \in L_M$ with $\|u_1\|_{oo} < M$ such that

(3.3) $$y_1 = z_1 + \Lambda V(t)u_1 .$$

For the proof see Lemma 1 in [6] .

Let

(3.4) $$\Omega_T = \{y\in X;\ \text{there is } u\in L_M,\ y=\Lambda V(t)(u-u_1)\}$$

where u_1 is given by Lemma 3.1 Clearly Ω_T is convex.

Lemma 3.2 o is an interior point of Ω_T.

Proof: If $\|y\|_T \le C\|y\|_R$ then $\{y\in X;\ \|y\|_R \le \frac{1}{C}(M-\|u_1\|_{oo}\}\subset\Omega_T$

See also [6] .

Proof, of Theorem 3.1 First we shall prove that $z_1 - \Lambda S(t)x_o$ is a boundary point of Ω_T.Note that from

(3.5) $$y_1 = z_1 + \Lambda V(t)u_1 = \Lambda S(T)x_o + \Lambda V(t)\,u_o$$

we obtain $z_1 - \Lambda S(T)x_o = \Lambda V(T)(u_o - u_1)\in\Omega_T$.

Assume $z_1 - \Lambda S(T)x_o$ is an interior point of Ω_T.Using the continuity of $P(.)x_o$ in X and the convexity of Ω_T is not difficult to deduce the existence of $\delta > o$, $r<1$ such that

$$\frac{1}{2}(z_1 - \Lambda S(T')x_o)\in\Omega_T \text{ for } T-\varepsilon<T'<T .$$

Hence there exists $u\in L_M$ such that $z_1 = \Lambda S(T')x_o + \Lambda V(T)(ru - ru_1)$ and using (3.5) we get $y_1 = \Lambda S(T')x_o + \Lambda V(T)(ru + (1-r)u_1)$.Denoting $u_2 = ru + (1-r)u_1$, we have $\|u_2\|_{oo} < M$ and $y_1 = \Lambda S(T')x_o + \Lambda V(T)u_2$

On the other hand , using (2.7) we have

$$\Lambda V(T)u_2 = \Lambda V(T')(J_{T-T'}u_2) + \Lambda S(T')V(T-T')u_2 .$$

Obviously , $\lim_{T'\to T} V(T'-T)u_2 = o$ in E. As $\|\Lambda S(T')\|_{L(E,X)}$ is bounded,we may infer that if we take T' sufficiently near T we have $\|\Lambda S(T')V(T-$

$-T')u_2\|_X < \delta = M - \|u_2\|_\infty$. Next, by (2.9) there is $u_3 \in L_\delta$ such that

$$\Lambda S(T')V(T-T')u_2 = \Lambda V(T')u_3 \ .$$

Hence $y_1 = \Lambda S(T')x_o + \Lambda V(T')(J_{T-T'}u_2+u_3)$, with

$\|J_{T-T'}u_2 + u_3\|_\infty \leqslant M$, which contradicts the time optimality of u_o.

The proof ends by applying a standard separation theorem for convex sets: there exists $x^* \in X^*$, $x^* \neq o$, such that

(3.6) $\langle x^*, \Lambda V(T)(u_o - u_1) - y\rangle \geqslant o$ for every $y \in \Omega_T$.

(Here $\langle\ ,\ \rangle$ denotes the natural pairing between X and X*). For every $u \in L_M$ we denote $u_\varepsilon = (1-\chi_\varepsilon)u_o + \chi_\varepsilon u$, where χ_ε denotes the characteristic function of the interval $[o, T-\varepsilon]$. Involving (2.1), (2.6) we get

(3.7) $\Lambda V(T)(u_\varepsilon - u_1) \in \Omega_T$

and so, by (3.6) we obtain $\langle x^*, \Lambda V(T)\chi_\varepsilon(u_o - u)\rangle \geqslant o$ for every $u \in L_M$ and by (2.1) we have

(3.8) $\langle x^*, \Lambda S(\varepsilon) \int_o^{T-\varepsilon} S(T-\varepsilon-s)B(u_o(s) - u(s))ds\rangle \geqslant o$

for every $u \in L_M$, and therefore

(3.9) $(P^*(\varepsilon)x^*, \int_o^{T-\varepsilon} S(T-\varepsilon-s)B(u_o(s) - u(s))ds) \geqslant o$ for every $u \in L_M$ (Here $(\cdot,\cdot)$ denotes the natural pairing between E and E*). Denoting $P^*(\varepsilon)x^* = x^*_\varepsilon \in E^*$, for $\varepsilon < \varepsilon_o$ (ε_o given by Lemma 2.2) we obtain from (3.9)

(3.10) $\int_o^{T-\varepsilon} (B^*S^*(T-\varepsilon-s)x^*_\varepsilon\ , u_o(s) - u(s))ds) \geqslant o$

for every $u \in L_M$.

Using the argument of Fattorini (Theorem 3.1), [4]), (3.10) implies (3.2). This ends the proof of Theorem 3.1.

We end this section with some remarks. If F = E = U and B= Λ =I

<u>Proposition 3.1</u> (a) $\lim_{\varepsilon\to o} S(\varepsilon)x = x$ <u>for every</u> $x \in D(A)$ (<u>the limit is taken in the topology of</u> X).

(b) $X = Cl(D(A))$ (As in (2.10), "Cl" denotes the closure in the topology of X).

<u>Proof</u>. (a) If $x \in D(A)$ an integration by parts shows that $x = \int_0^1 S(1-s)(x - sAx)ds$. Hence

$$(3.11) \qquad S(\varepsilon)x = \int_0^1 S(1-s)(S(\varepsilon)x - s\, S(\varepsilon)Ax)ds \quad , \qquad \varepsilon > 0 \quad .$$

Denoting $u(s) = x - s\,Ax$ and $u_\varepsilon(s) = S(\varepsilon)x - sS(\varepsilon)Ax$, we obtain $\|u_\varepsilon - u\|_{\infty} \leq \|S(\varepsilon)x - x\|_E + \|S(\varepsilon)Ax - Ax\|_E$. This implies that $\lim_{\varepsilon \to 0} u_\varepsilon = u$ in $L^\infty(0,1;u)$ and therefore $\lim_{\varepsilon \to 0} S(\varepsilon)x = x$ in X.

(b) From (a) we obtain $Cl(D(A)) \subset X$. On the other hand we may write $S(t)x = \int_0^t S(t-s)\frac{1}{t}S(s)x$ for every $x \in E$, $t > 0$. Let $J_\lambda = (I - \lambda A)^{-1}$, $\lambda > 0$. We have $J_\lambda S(s)x \in D(A)$. Using the continuity of $J_\lambda S(\cdot)x$ and $S(\cdot)x$ we may infer that for every $\lambda > 0$ there exists $s_\lambda \in [0,t]$ such that

$$(3.12) \qquad \|J_\lambda S(s)x - S(s)x\|_{\infty} = \|J_\lambda S(s_\lambda)x - S(s_\lambda)x\|_E \; .$$

There exists a subsequence (again denoted s_λ) such that $\lim_{\lambda \to 0} s_\lambda = s_0$. Using (3.12) we have

$$\|J_\lambda S(s_\lambda)x - S(s_\lambda)x\|_E \leq \|J_\lambda S(s_\lambda)x - J_\lambda S(s_0)x\|_E + \|S(s_\lambda)x - S(s_0)x\|_E +$$

$$+ \|J_\lambda S(s_0)x - S(s_0)x\|_E$$ and thus we get $\lim_{\lambda \to 0} J_\lambda S(\cdot)x = S(\cdot)x$ in $L^\infty(0,t;E)$ which implies $\lim_{\lambda \to 0} J_\lambda S(t)x = S(t)x$ in X.

Hence $X \subset Cl(D(A))$ and the proof is complete.

4. APLICATIONS

1. If $F = E = U$ and $B = \Lambda = I$, the null controllability condition is verified because $S(t)x_0 = \int_0^t S(t-s)\frac{1}{t}S(s)x_0$, i.e. $u(s) = \frac{1}{t}S(s)x_0$ for $s \in [0,t]$. Much more, if $S(t)E = E$ for some $t > 0$ we have $K = X = E$ and from the closed graph theorem it follows that $\|\cdot\|_t$

is equivalent to $\|\cdot\|_E$. The assumption (3.1) is instantly verified and letting ε tend to zero in (3.2) we obtain Theorem 3.1 in [4].

2. The wave equation. Let H be a Hilbert space, A a self adjoint operator in H such that $(Au,u) \leq -\omega |u|^2$, $u \in D(A)$, for some $\omega > 0$, where $(\cdot,\cdot)$ denotes the scalar product in H and $|\cdot|$ stands for the norm in H. Let V be the domain of $(-A)^{\frac{1}{2}}$ (the square root of $(-A)$). It is well known that A is the infinitesimal generator of a strongly continuous cosine family in H and $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ is the infinitesimal generator of a strongly continuous group in $V \times H$. The norm in V is defined by $\|u\| = |(-A)^{\frac{1}{2}} u|$ and the space $V \times H$ is endowed with its Hilbert product norm. Let $B_0 \in L(U_0, H)$, U_0 being other Hilbert space. We define $B: U \to V \times H$ by

$$(4.1) \qquad Bu = \begin{pmatrix} 0 \\ B_0 u \end{pmatrix}.$$

Consider the equation

$$(4.2) \qquad x''(t) = Ax(t) + B_0 u(t), \quad t > 0,$$
$$x(0) = x_0 \in V; \; x'(0) = x_1 \in H.$$

This problem can be reformulated as

$$(4,3) \qquad y'(t) = \mathcal{A} y(t) + Bu(t), \quad t \geq 0$$

where $y(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}$; $y_0 = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$.

Let $\{C(t), t \in R\}$ be the cosine family generated by A and $S(t)x = \int_0^t C(s)x\,ds$, $x \in V, t \in R$. Let $V(t)$ be the group generated by $\mathcal{A}$.

It is easy to see that

$$(4.4) \qquad V(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}.$$

The mild solution of (4.3) is given by

$$(4.5) \qquad y(t) = V(t)y_0 + \int_0^t V(t-s)Bu(s)ds.$$

The definition of controllability for (4.2) is given for example in [5]. The invariance of (4.2) with respect to time reversal implies that the null controllability is equivalent to global controllability

Hence $K = X = V \times H$. Letting ε tend to zero in (3.2), after some simple calculations involving (4.1), (4.4) we obtain

Corollary 4.1 Assume the null controllability of (4.2). Then there are $x_1^* \in V'$, $x_2 \in H$, $x_1^* \neq o$ or $x_2^* \neq o$, such that

(4.6) $$(B_o^* S^*(T-t)x_1^* + B_o^* C^*(T-t)x_2^*, u_o(t)) =$$
$$= \sup_{\|u\| \leq M} (B_o^* S^*(T-t)x_1^* + B_o^* C^*(T-t)x_2^*, u) \text{ a.e. on } [o, T-\varepsilon].$$

If $U = H$ and $B_o = I$ then the null controllability is proved in [5]. In this paper is proved the maximum principle too. We also remark that the maximum principle (3.2) is obtained if we study the controllability in the spaces V or H. Obviously, Λ will be Π_1, Π_2, respectively (Π_i, $i = 1,2$, will denote the projections of $V \times H$ onto its component spaces).

3. Linear retarded systems We consider the linear retarded equation.

(4.7) $$z'(t) = \sum_{i=o}^{N} A_i z(t-h_i) + \int_{-h}^{o} A_{o1}(\theta) z(t+\theta) d\theta + B_o u(t),$$

(4.8) $$z(o) = \phi^o, \quad z(\theta) = \phi^1(\theta); \; \theta \in (-h, o), \; t \geq o.$$

where $z \in R^n$, $\phi^o \in R^n$, $\phi^1 \in L^2(-h, o, R^n)$; $A_i \in \mathcal{L}_{n,n}$, the $\mathcal{L}_{m,n}$-valued function $\theta \longrightarrow A_{o1}(\theta)$ is bounded measurable, and $o = h_o < h_1 < \dots < h_N = h$; $B_o \in \mathcal{L}_{m,n}$, $u \in R^m$. Here $\mathcal{L}_{p,q}$ denotes the space of all $p \times q$ real matrices endowed with a suitable norm. The framework adopted in this paper and the general results stated below are essentially due to Bernier and Manitius [2].

For $t \geq o$, x_t denotes the function on $(-h, o]$ defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-h, o]$.

The solutions of this equation will be treated as elements of the Hilbert space $M^2 = M^2(-h, o; R^n) = R^n \times L^2(-h, o; R^n)$ endowed with usual inner product. It is well known [2], [9] that equation (4.7) induces a strongly continuous semigroup $\{S(t), t \geq o\}$ on M^2. Let $z(t)$ be a solution of (4.7), (4.8), then $x(t) = (z(t), z_t)$ is the mild solution of the abstract differential equation

(4.9) $$x'(t) = Ax(t) + Bu(t) \quad t > o$$
$$x(o) = \phi .$$

where: $A: D(A) \subset M^2 \to M^2$ is the infinitesimal generator of $\{S(t), t \geq o\}$ and $B: R^m \to M^2$ is a bounded linear operator defined by

(4.10) $$Bu = (B_o u, o) .$$

More detaild about $S(t), S^*(t), A$ can be found in [2].

For (4.7),(4.8), there are a lot of types of controllability studied in literatùre. See e.g. [10], [1]. Thus, for M^2 - controllability Λ in (1.2) is the identity operator in M^2, for L^2 - controllability and Euclidian controllability, Λ will be the projection of M^2 onto $L^2(-h,o;R^n)$ and R^n, respectively. For the case of F-controllability studied by Manitius [9] we have $\Lambda = F = \begin{bmatrix} I & o \\ o & H \end{bmatrix}$ (see [9] for details about F).

We are in the situation of general problem (1.1) (1.2) described in Section 1, where $E = M^2; U = R^m$, Λ and F are described above.

Following [2] we shall describe the operator $B^* S^*(t)$. We have

(4.11)
$$S^*(t) = F^* G_t^* + s^*(t) \quad \text{where}$$
$$[G_t^* \psi]^o = [G_t^* \psi]^1(o)$$
$$[G_t^* \psi]^1(\theta) = X^\tau(t+\theta)\psi^o + \int_{-h}^{o} X^\tau(t+s+\theta)\psi^1(s)ds$$
$$[s^*(t)\psi]^o = o, \quad [s^*(t)\psi]^1(\theta) = \psi^1(\theta - t)\chi_{[o,\theta+h)}(t)$$

where $\chi_{(a,b)}$ denotes the characteristic function of the interval (a,b) superscript τ will denote transpozition of a vector in R^n; $X(t)$ denote the fundamental matrix of Eq(4.7), i.e. $X(t) \equiv o$ for $t < o$, $X(o) = I$, $X(t) = L(X_t)$a.e. where $X_t(.)$ denotes $X_t(\theta) = X(t+\theta)$, $\theta \in [-h,o]$.

The operator L is as follows: Extend $A_{o1}(.)$ to $(-oo,oo)$ by putting $A_{o1}(s) \equiv o$ for $s \notin [-h,o]$ and define

(4.12) $$G(\theta) = -\sum_{i=1}^{N} A_i \chi_{(-oo,-h_i]}(\theta) - \int_{\theta}^{o} A_{o1}(s)ds$$

(4.13) $$N(\theta) = -A_o\chi_{(-\infty,o)}(\theta) + G(\theta) .$$

Both G(·) and N(.) are bounded variation functions .For $\phi \in C([-h,o], R^n)$ we define, using Stieltjes integral notation , the operator

(4.14) $$L(\phi) = \int_{-h}^{o} dN(\theta)\phi(\theta)$$

and for $\phi^1 \in L^2(-h,o;R^n)$ we define $H: L^2(-h,o;R^n) \longrightarrow L^2(-h,o;R^n)$ by

(4.15) $$(H\phi^1)(\theta) = \int_{-h}^{\theta} dG(s)\phi^1(s-\theta) .$$

As results from [2] , Prop. 3.1 , for $\psi^1 \in L^2(-h,o;R^n)$ we have

(4.16) $$(H^*\psi^1)(\theta) = \int_{-h}^{\theta} dG^{\tau}(s)\psi^1(s-\theta) .$$

We have also

(4.17) $$F^* = \begin{bmatrix} I & o \\ o & H^* \end{bmatrix}$$

Using (4.10), (4.11) , (4.17) we obtain

(4.18) $$B^*S^*(t)\psi = B_o^*X^{\tau}(t)\psi^o + B_o^*\int_{-h}^{o} X^{\tau}(t+s)\psi^1(s)ds$$

where $\psi = \begin{pmatrix}\psi^o \\ \psi^1\end{pmatrix} \in M^2$.

Defining R, R_t and X as in Section 3, Theorem 3.1 can therefore be applied to the present situation .

<u>Corollary 4.2 Let the system</u> (4.7),(4.8) <u>be null controllable</u> . <u>Let</u> $\phi_o = (\phi_o^o, \phi_o^1) \in M^2(-h,o;R^n)$ <u>and</u> $\phi_1 \in R_t(\phi_o, L_M)$ <u>such that</u> $M > \mathrm{dist}_R(\phi_1, X)$.<u>Let</u> u_o <u>be an optimal control</u> , T <u>its transition time</u>. <u>Then there exists</u> $\varepsilon_o > o$ <u>such that for every</u> $\varepsilon < \varepsilon_o$, <u>there exists</u> $\psi_\varepsilon = (\psi_\varepsilon^o, \psi_\varepsilon^1) \in M^2(-h,o;R^n)$ <u>such that</u>

(4.19) $$(B_o^* X^{\tau}(t)\psi_\varepsilon^o + B_o^*\int_{-h}^{o} X^{\tau}(t+s)\psi_\varepsilon^1(s)ds , u_o(s)) =$$

$$= \sup_{\|u\| \le M} (B_o^* X^{\tau}(t)\psi_\varepsilon^o + B_o^*\int_{-h}^{o} X^{\tau}(t+s)\psi_\varepsilon^1(s)ds , u) \quad \text{on } [0,T-\varepsilon].$$

<u>Remark:</u> The null controllability of (4.7) , (4.8) is studied for example in [1] .

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