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The purpose of this paper is to study H-cones of functions on a locally compact non-abelian group G.

In the first section we give two remarkable examples of standard H-cones of functions on G.

The section 2 is devoted to the study of natural and fine topology on G, with respect of the two standard H-cones of functions on G.

1. H-cones of functions on a locally compact group

We shall use in this section the definitions and the notations of [1], [2], [8].

1.1. Definition. A set $\mathcal{S}$ of positive, numerical functions on a set X is called H-cone of functions on X if:

F1) $\mathcal{S}$ endowed with the pointwise algebraic operations and order relation is an H-cone (see [2], section 1), with the convention $0\cdot\infty=0$.

F2) If $(s_i)_{i\in I}$ is an increasing net in $\mathcal{S}$ such that there exists its l.u.b. in $\mathcal{S}$, denoted by s, then for every $x\in X$ we have:

$$s(x)=\sup\{s_i(x) \mid i\in I\} \tag{1}$$

F3) For every $s,t\in\mathcal{S}$ and for every $x\in X$, we have:

$$(s\wedge t)(x)=\inf(s(x),t(x)) \tag{2}$$

where $s\wedge t$ is the l.u.b. of s and t in $\mathcal{S}$.

F4) The set $\mathcal{S}$ separates X and contains the positive constant functions.

1.2. Definition. If the H-cone of functions $\mathcal{S}$ on the set X is standard (see [2], section 2), then it is called a standard H-cone of functions on the set X. The set of universally continuous elements of $\mathcal{S}$ will be denoted by $\mathcal{S}_o$.

Throught this paper, G will be a locally compact non-abelian group, λ_s will be the left invariant Haar measure on G, λ_d will be the right invariant Haar measure on G, $\mathcal{B}$ will be the σ-algebra of Borel sets of G, $\mathcal{F}$ will be the set of positive numerical Borel functions on G, $\mathcal{C}(G)$ will be the real continuous functions on G, $M(G)$ will be the Radon measures on G, $\mathcal{K}(G)$ will be the functions of $\mathcal{C}(G)$ which have compact support and e will be the neutral element of G, and Δ will be the modular function of G, i.e. $\lambda_d(x)=\frac{1}{\Delta(x)}\cdot\lambda_s(x)=\lambda_s(x^{-1})$ for every $x\in G$.

We proved in [1] (corollary 3.8.) the following result:

1.3. Theorem. We assume that:

a) $k\in M^+(G)$ is a potential kernel, $k=\int\mu_t\,dt$;

b) $(k_\alpha)_{\alpha>0}$ is the resolvent of measures associated with k (i.e. $k_0=k$) by:

(3) $$k_\alpha(f)=:\int e^{-\alpha t}\mu_t(f)\,dt\text{, for } \alpha>0 \text{ and } f\in\mathcal{K}(G)$$

c) $\mathcal{N}=\{N_\alpha\}_{\alpha>0}$ is the submarkovian resolvent of operators associated with k by:

(4) $$N_0(g)=:k*g \qquad \text{for every } f\in\mathcal{F}$$

(5) $$N_\alpha(g)=:k_\alpha*g \qquad \text{for every } f\in\mathcal{F} \text{ and } \alpha>0$$

d) $\mu\in M^+(G)$ is a measure on G such there exists $\check{k}*\mu$ and such the kernel N_0 is basic with respect to μ.

Let us denote:

(6) $$\mathcal{S}(\mathcal{N})=:\{s\in\mathcal{F}\mid N_\alpha s\leq s \text{ for every } \alpha>0\}$$

(7) $$\mathcal{S}'(\mathcal{N})=:\{s\in\mathcal{S}(\mathcal{N})\mid \lim_{\alpha\to\infty}\alpha N_\alpha s=s\}$$

(7') $$\mathcal{E}(\mathcal{N})=:\{s\in\mathcal{S}'(\mathcal{N})\mid s<+\infty,\ \mathcal{N}\text{-a.e.}\}$$

Then we have:

1) k (and hence all k_α, $\alpha>0$) is absolutelly continuous with respect to λ_s (and with respect to λ_d, too), hence there exists $f\in L^1_{loc}(\lambda_s)$ and $f_\alpha\in L^1(\lambda_s)$ for every $\alpha>0$, such that:

(8) $$k=f\cdot\lambda_s \quad\text{and}\quad k_\alpha=f_\alpha\cdot\lambda_s\ ,\ \alpha>0$$

Moreover, we can assume that $f\in\mathcal{E}(\mathcal{N})$ and $f_\alpha\in\mathcal{E}(\mathcal{N}_\alpha)$, where $\mathcal{N}_\alpha=:\{N_{\alpha+\beta}\}_{\beta>0}$

2) $\mathcal{E}(\mathcal{N})$ is a standard H-cone if N_0 is propre.

Throughout this paper we shall assume that we have a), b), c), d) of the theorem 1.3. and that N_0 is propre; f and f_α will be fixed given by 1) of the above theorem.

1.6. Remark. For every $g\in\mathcal{K}(G)$ and for every $x\in G$ we can write:

(9) $$N_0(g)_{(x)}=(k*g)_{(x)}=(f*g)_{(x)}=\int g(y^{-1}x)f(y)\,d\lambda_s(y)=\int f(xz^{-1})g(z)\frac{1}{\Delta(z)}d\lambda_s(z)=$$
$$=\int f(xz^{-1})g(z)\,d\lambda_d(z)$$

(10) $$N_\alpha(g)_{(x)}=(k_\alpha*g)_{(x)}=\int g(y^{-1}x)f_\alpha(y)\,d\lambda_s(y)=\int f_\alpha(xz^{-1})g(z)\,d\lambda_d(z).$$

Using the following ~~remarks~~ notations (for every function h on G and for every measure $\mu\in M^+(G)$):

$$\check{h}(x)=:h(x^{-1})\ ,\ h^*(x)=:(\check{h\Delta})_{(x)}=h(x^{-1})\Delta(x^{-1})\quad\text{for every } x\in G$$

$$\check{\mu}(g)=:\mu(\check{g})\quad\text{for every } g\in\mathcal{K}(G)$$

(one can see that if $\mu=f\cdot\lambda_s$, then $\check{\mu}=f^*\cdot\lambda_s$)

we also can define (in the assumptions of the theorem 1.3.):

(9') $\check{N}_0(g)_{(x)}=:(\check{k}*g)_{(x)}=(f^* * g)_{(x)}=\frac{1}{\Delta(x)}\int f(yx^{-1})g(y)d\lambda_s(y)=$
$=\int f(yx^{-1})\Delta(yx^{-1})g(y)d\lambda_d(y)$

(10') $\check{N}_\alpha(g)_{(x)}=:(\check{k}_\alpha * g)_{(x)}=(f^* * g)_{(x)}=\frac{1}{\Delta(x)}\int f_\alpha(yx^{-1})g(y)d\lambda_s(y)=$
$=\int f_\alpha(yx^{-1})\Delta(yx^{-1})g(y)d\lambda_d(y)$, for every $\alpha>0$

One can easily see that $\check{\mathcal{N}}=:\{N_\alpha\}_{\alpha\geq 0}$ is ~~also a~~ the resolvent of operators associated with the potential kernel $\check{k}=:\int\check{\mu}_t dt$, and that the two resolvents $\mathcal{N}$ and $\check{\mathcal{N}}$ are in duality, i.e. for every $g,h\in\mathcal{K}(G)$ we have:

(11) $$\int g(x)\,N_\alpha(h)_{(x)}d\lambda_s(x)=\int\check{N}_\alpha(g)_{(x)}h(x)\,d\lambda_s(x),\ \text{for every } \alpha>0$$

Hence the dual of $\mathcal{E}(\mathcal{N})$, which is also a standard H-cone ~~by 2~~, coincides with $\mathcal{E}(\check{\mathcal{N}})$ (see [2] for proofs).

We saw in [1], the proof of the theorem 3.10., that $\alpha\int f_\alpha(x)d\lambda_s(x)\leq 1$ for every $\alpha>0$, hence the resolvent $\check{\mathcal{N}}$ is also sub-markovian, because:

$$\alpha\check{N}_\alpha(1)_{(x)}=\alpha(\check{k}_\alpha * 1)_{(x)}=\frac{\alpha}{\Delta(x)}\int f_\alpha(yx^{-1})d\lambda_s(y)=\alpha\int f_\alpha(z)d\lambda_s(z)\leq 1,(\forall)\ \alpha>0$$

<u>1.7.Remark</u>. For a function h on G and for any $a\in G$ we denote by h_a and ${}_ah$ the following functions on G;

$$h_a(x)=:h(xa)\quad,\quad {}_ah(x)=:h(ax)\qquad\text{for } x\in G.$$

One can easily verify that:

(12) $(N_\alpha g)_a=N_\alpha(g_a)$ for every $\alpha\geq 0$, $g\in\mathcal{F}$ and $a\in G$

(13) $(\check{N}_\alpha g)_a=\check{N}_\alpha(g_a)$ for every $\alpha\geq 0$, $g\in\mathcal{F}$ and $a\in G$,

using the following relations: $d\lambda_s(ya)=\Delta(a)d\lambda_s(y)$, $d\lambda_d(ay)=\frac{1}{\Delta(a)}d\lambda_d(y)$.

<u>1.8.Lemma</u>. Let $s\in\mathcal{E}(\mathcal{N})$ be a lower semi-continuous function. Then $s\in\mathcal{S}(\mathcal{N})$

<u>Proof</u>: We have already seen ~~that~~ in [1], section 2, (23), that for every $g\in\mathcal{K}(G)$ we have:

(14) $$\lim_{\alpha\to\infty}\alpha N_\alpha(g)_{(x)}=g_{(x)}\qquad\text{for every } x\in G$$

Let us consider now a function $g\in\mathcal{K}(G)$ such that $g\leq s$; we can write:

$$\lim_{\alpha\to\infty}\alpha N_\alpha s\geq\lim_{\alpha\to\infty}\alpha N_\alpha g=g$$

and because g is arbitrary, we obtain: $\lim_{\alpha\to\infty} \alpha N_\alpha s = s$, i.e. $s \in \mathcal{S}'(\mathcal{N})$.

<u>1.9. Remark</u>. Because $f \geq 0, f \not\equiv 0$, we have for every $x \in G$:

$$0 < N_0(1)_{(x)} = \check{N}_0(1)_{(x)} = \int f(y)\, d\lambda_s(y) \leq +\infty ,$$

hence $N_0(1)$ and $\check{N}_0(1)$ are strictely positive constants. It follows (see [8]) that:

$s \in \mathcal{E}(\mathcal{N})$ is a week unit for $\mathcal{E}(\mathcal{N}) \Longleftrightarrow s(x) > 0$ for every $x \in G$

Because the resolvant $\mathcal{N}$ is sub-markovian, we have that $1 \in \mathcal{S}(\mathcal{N})$; by lemma 1.8. we obtain that $1 \in \mathcal{E}(\mathcal{N})$. Hence 1 is a weak in $\mathcal{E}(\mathcal{N})$. Analoguous 1 is a weak unit in $\mathcal{E}(\check{\mathcal{N}})$, too.

<u>1.10. Remark</u>. Because $\mathcal{E}(\mathcal{N})$ is a standard H-cone, we can assert that:

a) If $(s_i)_{i\in I} \subseteq \mathcal{E}(\mathcal{N})$ is an increasing family, dominated in $\mathcal{E}(\mathcal{N})$, then there exists a sequence $\{i_n\}_{n\in N}$ of elements from I such that:

$$\bigvee_{i\in I} s_i = \bigvee_{n\in N} s_{i_n}$$

b) $(s_i)_{i\in I} \subseteq \mathcal{E}(\mathcal{N}) \Longrightarrow$ there exists $\{i_n\}_{n\in N} \subseteq I$ such that:

$$\bigwedge_{i\in I} s_i = \bigwedge_{n\in N} s_{i_n}$$

<u>1.11. Lemma</u>. If $g \in \mathcal{F}$ is bounded, with compact support K, then:

a) $N_0(g)$ is a continuous bounded function on G;

b) $N_0(g) \in (\mathcal{E}(\mathcal{N}))_0$.

<u>Proof</u>: a) follows from the fact that $N_0(g) = f * g$, knowing that the convolution between a function $f \in L^1_{loc}(\lambda_s)$ and a bounded function $g \in \mathcal{F}$ with compact support is real and continuous, and from the fact that the potential kernel k is a shift-bounded measure on G (see [4], 13.10.), i.e.:

$$h \in \mathcal{K}(G) \Longrightarrow N_0(h) =: k * h \in \mathcal{C}(G) \text{, bounded.}$$

b) By Hunt's theorem, $N_0(g) \in \mathcal{E}(\mathcal{N})$; let us consider a sequence of functions $\{s_n\}_{n\in N}$ from $\mathcal{E}(\mathcal{N})$ which is increasing to $N_0(g)$, a strictely positive real number ε and let $s \in \mathcal{E}(\mathcal{N})$ be an arbitrar weak unit (hence $s(x) > 0$ for every $x \in G$). Let us denote:

$$\alpha =: \inf \{ s(x) \mid x \in K \} > 0 .$$

By Dini's theorem, we have:

$$\{s_n\}_{n\in N} \xrightarrow{u} N_0(g) \text{ on } K,$$

hence there exists $n_\varepsilon \in N$ such that for every $n \geq n_\varepsilon$ we have:

$$s_n + \varepsilon s \geq s_n + \varepsilon \cdot \alpha \geq N_0(g) \text{ on } K.$$

By [1], proposition 2.6., N_o satisfies the complete maximum principle, hence we have $s_n+\varepsilon s \geqslant N_o(g)$ on G. Thus we proved that $N_o(g)\in(\mathcal{E}(\mathcal{N}))_o$.

1.12. Lemma. If we denote:

(15) $\qquad \mathcal{A} =: \{h\in\mathcal{A} \mid h$ is bounded, with compact support$\}$

and if we consider a bounded function $g\in\mathcal{F}$ such $N_o(g)$ is bounded, too, then we have:

a) $N_o(g)\in\mathcal{E}(\mathcal{N})$ (by Hunt's theorem)

b) Moreover, there exists an increasing sequence $\{g_n\}_{n\in N}$ of functions from $\mathcal{A}$, such that :

$$\{N_o(g_n)\}_{n\in N}\uparrow N_o(g)$$

Proof: b) Let $\{K_i\}_{i\in I}$ be an increasing net of compact subsets of G, such that $\bigcup_{i\in I} K_i=G$ and let $g_i=:g\cdot\chi_{K_i}$. Because $\{N_o(g_i)\}_{i\in I}$ is increasing to $N_o(g)$, we obtain from remark 1.10. the existence of the desired sequence (using the previous lemma, too).

1.13. Proposition. If $s\in\mathcal{E}(\mathcal{N})$, then there exists a sequence $\{g_n\}_{n\in N}\subseteq\mathcal{A}$ (where $\mathcal{A}$ was defined in (15)), suth that:

$$\{N_o(g_n)\}_{n\in N}\uparrow s$$

Proof; By Hunt's theorem, there exists a sequence $\{h_n\}_{n\in N}$ of positive bounded Borel functions, with $N_o(h_n)$ bounded for every $n\in N$, such that:

$$\{N_o(h_n)\}_{n\in N}\uparrow s .$$

The existence of the desired Sequence follows from the previous lemma applied to every h_n.

1.14. Corollary. a) Every $s\in\mathcal{E}(\mathcal{N})$ is lower semi-continuous on G.

b) Every $s\in(\mathcal{E}(\mathcal{N}))_o$ is continuous and bounded on G.

1.15. Proposition. a) $s\in\mathcal{E}(\mathcal{N})$, $a\in G \Longrightarrow s_a\in\mathcal{E}(\mathcal{N})$

b) $s\in\mathcal{E}(\check{\mathcal{N}})$, $a\in G \Longrightarrow s_a\in\mathcal{E}(\check{\mathcal{N}})$

Proof: a) Let us consider, by prvious proposition, a sequence $\{g_n\}_{n\in N}$ of functions from $\mathcal{A}$, such that:

(16) $\{N_o(g_n)\}_{n\in N}\uparrow s$ (hence we also have $\{(N_o(g_n))_a\}_{n\in N}\uparrow s_a$). Because of (12) and by Hunt's theorem, we have that:

$$(N_o(g_n))_a\in\mathcal{E}(\mathcal{N}), \text{for every } n\in N$$

By assumption, $s<+\infty$, $\mathcal{N}$-a.e., hence $s_a<+\infty$, $\mathcal{N}$-a.e.; using the second part of (16) and lemma 1.6., we obtain $s_a \in \mathcal{E}(\mathcal{N})$. b) can be proved analoguous.

1.16. Theorem. a) $\mathcal{E}(\mathcal{N})$ is a standard H-cone of functions on G (see definition 1.2.)

b) $\mathcal{E}(\check{\mathcal{N}})$ is a standard H-cone of functions on G.

Proof: a) We already have axiom F1) from the theorem 1.3. For F4) we ~~xlxxxx~~ already know that $\mathcal{E}(\mathcal{N})$ contains the positive constant functions (see the remark 1.9.); we will prove now that $\mathcal{E}(\mathcal{N})$ separates the points of G; by proposition 1.13. it is sufficient to consider $x \in G$ such that:

(17) $$(N_o g)_{(x)} = (N_o g)_{(e)} \qquad \text{for every} \quad g \in \mathcal{A}$$

and to show that $x=e$. Indeed, because N_α are bounded kernels, from:

(18) $$N_o(g) = N_\alpha(g) + \alpha N_\alpha N_o(g) \qquad \text{for every} \quad \alpha > 0$$

we obtain:

(19) $$(N_\alpha g)_{(x)} = (N_\alpha g)_{(e)} \qquad \text{for every } g \in \mathcal{K}(G) \text{ and } \alpha > 0$$

Hence, using (14) we have:

(20) $$g(x) = g(e) \qquad \text{for every} \quad g \in \mathcal{K}(G)$$

But relation (20) is possible iff $x=e$. Thus we proved F4).

F3) follows from the proposition 1.13. and lemma 1.8.; F2 follows from the corollary 1.14. and from the lemma 1.8., too.

b) can be proved analoguous.

Lemma 1.17. If U is an open subset of G, such that $U \neq \emptyset$ and $\overline{U}$ is compact then $N_o(\chi_U) \not\equiv 0$.

Proof: Indeed, otherwise, we would have for every $x \in G$:

$$N_o(\chi_U)_{(x)} = \int f(xy^{-1}) \chi_U(y) \, d\lambda_d(y) = 0$$

i.e. $f(xy^{-1}) = 0$ λ_d-a.e. $y \in U$. But because f is lower-semicontinuous on G, we have: $f(xy^{-1}) = 0$ for every $y \in U$;

~~Ifxxxxfixxxxx~~; from the last relation we obtain $f \equiv 0$ on G, which is not true. Hence we have the assertion of lemma 1.17.

2. The natural and the fine topologies on G with respect to $\mathcal{E}(\mathcal{N})$ and $\mathcal{E}(\check{\mathcal{N}})$.

Because we have just proved that $\mathcal{E}(\mathcal{N})$ and $\mathcal{E}(\check{\mathcal{N}})$ are H-cones of functions on G, we can define on G (see [2], section 3) the natural topology with respect to $\mathcal{E}(\mathcal{N})$, (resp. with respect to $\mathcal{E}(\check{\mathcal{N}})$), as being the coarsest topology on G for which the universally continuous elements of $\mathcal{E}(\mathcal{N})$, (resp. of $\mathcal{E}(\check{\mathcal{N}})$), are continuous function on G and it will be denoted by $\mathcal{T}_n$ (resp. $\check{\mathcal{T}}_n$), and the fine topology on G with respect to $\mathcal{E}(\mathcal{N})$, (resp. with resp. to $\mathcal{E}(\check{\mathcal{N}})$), as being the coarsest topology on G for which all the elements of $\mathcal{E}(\mathcal{N})$, (resp. of $\mathcal{E}(\check{\mathcal{N}})$), are continuous functions on G and it will be denoted by $\mathcal{T}_f$ (resp. $\check{\mathcal{T}}_f$); the initial given topology of the group G will be denoted by $\mathcal{T}_G$. G endowed with $\mathcal{T}_n$ (resp. with $\check{\mathcal{T}}_n$) is a metrisable and separable topological space and G endowed with $\mathcal{T}_f$ (resp. with $\check{\mathcal{T}}_f$) is a completely regular topological space (see [2])

Obviously $\mathcal{T}_n \subseteq \mathcal{T}_f$ and $\check{\mathcal{T}}_n \subseteq \check{\mathcal{T}}_f$.

By lemma 1.11. and proposition 1.13. it follows that $\mathcal{T}_n \subseteq \mathcal{T}_G$.

2.1. Lemma. 1) $V \in \mathcal{T}_f$ (resp. $V \in \check{\mathcal{T}}_f$), $a \in G \Longrightarrow V \cdot a \in \mathcal{T}_f$ (resp. $V \cdot a \in \check{\mathcal{T}}_f$)

2) $V \in \mathcal{T}_n$ (resp. $V \in \check{\mathcal{T}}_n$), $a \in G \Longrightarrow V \cdot a \in \mathcal{T}_n$ (resp. $V \cdot a \in \check{\mathcal{T}}_n$)

Proof: 1) If we consider the map $f: G \to G, f(x) =: xa^{-1}$, by proposition 1.15. we obtain $s \circ f = s_{a^{-1}} \in \mathcal{E}(\mathcal{N})$, (resp. $\in \mathcal{E}(\check{\mathcal{N}})$) is finely continuous for every $s \in \mathcal{E}(\mathcal{N})$, (resp. for every $s \in \mathcal{E}(\check{\mathcal{N}})$), hence (see [6], ch. I, § 2, n°3, prop. 4) it follows that f is finely continuous, which gives 1); 2) follows from 1).

2.2. Lemma. If $V \in \mathcal{T}_G$, $V \neq \phi$, there exists $x \in V$ such that V is a $\mathcal{T}_n$-neighbourhood for x.

Proof: Let us consider $U \in \mathcal{T}_G$, $U \neq \phi$, such that $\overline{U}$ is $\mathcal{T}_G$-compact and $\overline{U} \subseteq V$, and the compactification $\overline{G}$ of G with respect to the following family of bounded $\mathcal{T}_G$-continuous functions on G (see [9], th. 1.1.):

$$\mathcal{F}_0 = (\mathcal{E}(\mathcal{N}))_0 \cup \mathcal{K}(G)$$

We also consider the following cone of continuous functions on the compact space G (we'll also denote by s the extension by continuity on $\overline{G}$ of a function $s \in \mathcal{F}_0$):

$$\mathcal{S}_U = \{s+\alpha - N_o(\mathcal{X}_U) \mid s \in (\mathcal{E}(\mathcal{N}))_o ,\ \alpha \in R^+\}$$

By lemma 1.17. $N_o(\mathcal{X}_U) \neq 0$, hence there exists the Šilov's boundary $\overline{\partial G} \neq \emptyset$ of G with respect to $\mathcal{S}_U$. Because N_o satisfies the complete maximum principle and because every function from $\mathcal{S}(\mathcal{N})$ is N_o-dominating on G (see [8]), it follows that the closure $\overline{U}$ of U in $\overline{G}$ is a closed boundary set with respect to $\mathcal{S}_U$, i.e.:

$$f \in \mathcal{S}_U ,\ f \geqslant 0 \text{ on } \overline{U} \implies f \geqslant 0 \text{ on } \overline{G}$$

Thus we have $\overline{\partial G} \subseteq \overline{U}$

We assert now that:

(21) there exists $s \in \mathcal{S}_U$ such that $s \geqslant 0$ on C_V and $s(x) < 0$ for some $x \in U$ (where $C_V =: \overline{G} - V$). Indeed, otherwise, for every $s \in \mathcal{S}_U$ we would have:

(22) $s \geqslant 0$ on $C_V \implies s \geqslant 0$ on $U \implies s \geqslant 0$ on $\overline{G}$

and therefore we would obtain that C_V is a closed boundary set with respect to $\mathcal{S}_U$, which would imply that we also have $\overline{\partial G} \subseteq C_V$, hence:

$$\phi \neq \overline{\partial G} \subseteq \overline{U} \cap C_V = \phi$$

which is a contradiction. Thus we proved (21).

Let s given by (21) be of the form $s = u + \alpha - N_o(\mathcal{X}_U) \in \mathcal{S}_U$; if we take:

$$t =: N_o(\mathcal{X}_U) - (u+\alpha) \wedge N_o(\mathcal{X}_U)$$

which is $\mathcal{T}_n$-continuous, we have:

$$t \geqslant 0 \text{ on } \overline{G} ,\ t = 0 \text{ on } C_V ,\ t(x) > 0$$

$$x \in \{y \in G \mid t(x) > 0\} \subseteq V$$

and therefore V is a $\mathcal{T}_n$-neighbourhood of x.

<u>2.3. Theorem.</u> a) $\mathcal{T}_n = \mathcal{T}_G$; b) $\check{\mathcal{T}}_n = \mathcal{T}_G$.

<u>Proof</u>: 1) Let $V \in \mathcal{T}_G$ and let x be an arbitrary point of V; we will show that V is a $\mathcal{T}_n$-neighbourhood of x. Indeed. by the continuity of multiplicationinin the point $(e,x) \in G \times G$, there exists $U', W' \in \mathcal{T}_G$, such that:

(23) $e \in U'$, $e \in W'$, $U' \cdot (W' \cdot x) \subseteq V$

Let now $W \in \mathcal{T}_G$, such that $e \in W$, $W^{-1} = W$ and $W \subseteq U' \cap W'$. By lemma 2.2., there exists $y \in W \cdot x$ such that $W \cdot x$ is a $\mathcal{T}_n$-neighbourd of y, hence, by lemma 2.1. we can take $\Gamma \in \mathcal{T}_n$, such that:

$$e \in \Gamma ,\quad \Gamma \cdot y \subseteq W \cdot x$$

We have:

(24) $x \in \Gamma \cdot x \subseteq (W \cdot xy^{-1}) \cdot x = W \cdot (xy^{-1}) \cdot x \subseteq W \cdot W \cdot x \subseteq V$.

Hence, using again lemma 2.1., it follows that V is a $\mathcal{T}_n$-neighbourhood of x.

2) can be proved analoguous.

<u>2.4. Corollary.</u> 1) $\mathcal{T}_n = \check{\mathcal{T}}_n = \mathcal{T}_G$; 2) $(G, \mathcal{T}_G)$ is a locally compact topological group, whose topology is metrisable and separable.

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